An Alternative Proof in ACL2(r) of Euler's Solution to the Basel Problem

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Outline

- The Basel Problem
- Bounds on the Basel Sum
- The Key Lemma
- Navigating the Subtleties
- The End Game
- Questions?



Some History

- Pietro Mengoli (1644): What is $\sum_{n=1}^{\infty} \frac{1}{n^2}$?



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- Help: Johann Bernoulli, Daniel Bernoulli, Leibniz, Stirling, de Moivre, ...
- Leonhard Euler (1734): $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$



Some History

- The Bernoulli dynasty held the chair of mathematics at the University of Basel for over 100 years (1687–1790)
- Euler was a student of Johann Bernoulli's in Basel
- He actually solved the Basel problem in 1734 when he was a young professor at the University of St. Petersburg, invited by another of the Bernoullis!
- This would have been enough to make Euler famous, though he would go on to perform other miracles



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- This would have been enough to make Euler famous, though he would go on to perform other miracles
- Did you know e (as in e^x) is known as Euler's number?
- Or that $e^{i\pi} = -1$?
- Have you heard about Eulerian cycles in graph theory?



Euler's Proof (Sketch)

- Consider a polynomial $p(x)$ with nonzero roots $a_1, a_2, \ldots, a_n$, and such that $p(0) = 1$
- It can be written as

$$p(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\cdots\left(1 - \frac{x}{a_n}\right)$$



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- Surely this also holds for an infinite polynomial, like

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

so it can also be written as

$$p(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\left(1 - \frac{x}{a_3}\right)\cdots$$



Euler's Proof (Sketch)

- Consider again

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

- From the Taylor series of $\sin(x)$, we know that $p(x) = \frac{\sin(x)}{x}$



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$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$
- From the Taylor series of $\sin(x)$, we know that $p(x) = \frac{\sin(x)}{x}$
- So we have
$$\frac{\sin(x)}{x} = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\left(1 - \frac{x}{a_3}\right)\left(1 - \frac{x}{a_4}\right)\cdots$$
- What are the a_i? They are the roots of $\sin(x)/x$, which are $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
- Note that 0 is not a root!



Euler's Proof (Sketch)

- What we know so far:

$$p(x) = \frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\cdots$$

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- So let's expand the right-hand side to get

$$p(x) = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)x^2 + \cdots$$



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- So let's expand the right-hand side to get

$$p(x) = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)x^2 + \cdots$$

- But we also know that $p(x) = 1 - \frac{x^2}{3!} + \cdots$, so we can conclude that

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots$$

- And rearranging, we get that

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$



The Problems with Euler's Proof in ACL2

- The roots of $p(x)$ are an infinite set
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- Euler's argument plays fast and loose with infinite polynomials
- This wasn't resolved until Weierstrass's Factorization Theorem (1876)
- This theorem is not formalized in ACL2, and it would be a huge undertaking to do so



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- The roots of $p(x)$ are an infinite set
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- So I've been hunting for an alternative proof for years!
- The breakthrough came this year, when I came across a beautiful proof by ...
- ... Eric Moorhouse from UW Math, whose office is close to mine
- He presented this proof at the math colloquium, which I inexplicably missed
- This is an ACL2 project that was meant to be!



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The Basel Sum Is Bounded

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$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$



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$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

- All terms are positive, so the sum is bounded below by 0
- For an upper bound, observe that

$$\frac{1}{(i+1)^2} \leq \int_i^{i+1} \frac{1}{x^2}\, dx \leq \frac{1}{i^2}$$

- Consequently, for $i \geq 2$,

$$\int_i^{i+1} \frac{1}{x^2}\, dx \leq \frac{1}{i^2} \leq \int_{i-1}^{i} \frac{1}{x^2}\, dx$$



The Basel Sum Is Bounded

- Adding up the inequalities gives

$$\sum_{i=2}^{n-1} \int_{i}^{i+1} \frac{1}{x^2}\, dx \leq \sum_{i=2}^{n-1} \frac{1}{i^2} \leq \sum_{i=2}^{n-1} \int_{i-1}^{i} \frac{1}{x^2}\, dx$$



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- The term on the right is easy to evaluate as

$$\sum_{i=2}^{n-1} \int_{i-1}^{i} \frac{1}{x^2}\, dx = \int_{1}^{n-1} \frac{1}{x^2}\, dx = 1 - \frac{1}{n-1}$$



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- And from that, it's obvious that

$$\sum_{i=1}^{n-1} \frac{1}{i^2} = 1 + \sum_{i=2}^{n-1} \frac{1}{i^2} \le 1 + 1 - \frac{1}{n-1} \le 2$$



The Bound in ACL2

```
(defthmd basel-series-gross-bounds
    (implies (and (posp i)
                  (posp n)
                  (<= i n))
             (and (<= 0 (basel-series i n))
                  (<= (basel-series i n) 2)))
    :hints ...)
```



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- Important consequence: **The Basel Sum is limited** (in the sense of nonstandard analysis)



The Basel Sum Converges

- In only remains to show that for large m and n, the partial sums are close

$$\sum_{i=1}^{m} \frac{1}{i^2} \approx \sum_{i=1}^{n} \frac{1}{i^2}$$

The Basel Sum Converges

- In only remains to show that for large m and n, the partial sums are close

$$\sum_{i=1}^{m} \frac{1}{i^2} \approx \sum_{i=1}^{n} \frac{1}{i^2}$$

- But this follows easily
- Assuming that $m \leq n$, we have that

$$\sum_{i=1}^{n} \frac{1}{i^2} - \sum_{i=1}^{m} \frac{1}{i^2} = \sum_{i=m+1}^{n} \frac{1}{i^2}$$

- And as before, the last sum is bounded by the corresponding integral

$$\sum_{i=m+1}^{n} \frac{1}{i^2} \leq \int_{m}^{n} \frac{1}{x^2}\, dx = \frac{1}{m} - \frac{1}{n}$$

- As m and n are (infinitely) large, the difference $\frac{1}{m} - \frac{1}{n}$ is (infinitely) close, so the two partial sums are (infinitely) close



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The Key Players

- The “Lighthouse Proof” that we are following relies on the properties of the powers of the complex root of unity
- Let ζ_m be the m^{th} complex root of unity, and ζ_m^k its k^{th} power

$$\zeta_m^k = (e^{2\pi i/m})^k$$



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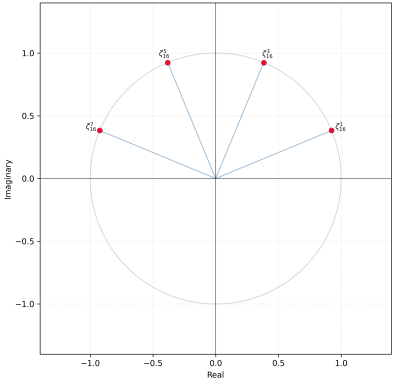
- For a fixed value of n, we will consider the following sum

$$\omega_n \triangleq \sum_{k=1}^{2^{n-1}} \frac{1}{\left|1 - \zeta_{2^{n+1}}^{2k-1}\right|^2}$$



The Key Players

- For $n = 3$, the sum involves ζ_{16}^{1}, ζ_{16}^{3}, ζ_{16}^{5}, and ζ_{16}^{7}.
- These are all powers of ζ_{16}
- These are all **odd** powers
- And these are all in the first two quadrants of the complex plane



The Key Lemma

- The **key lemma** is that $\omega_n = 2^{2n-3}$
- We proved this by induction on n
- The base case is trivial: $\omega_1 = \frac{1}{\left|1-\zeta_4^1\right|^2} = \frac{1}{2} = 2^{2-3}$



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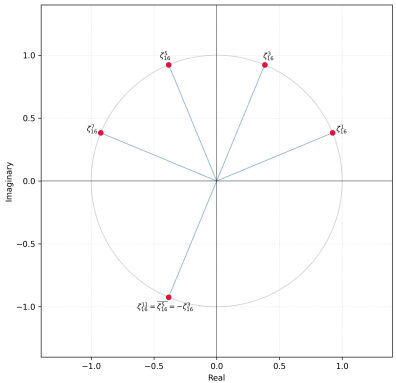
- For the induction case, we need a lemma that $\omega_{n+1} = 4\omega_n$
- Then,

$$\omega_{n+1} = 4\omega_n = 4 \cdot 2^{2n-3} = 2^{2n-1} = 2^{2(n+1)-3}$$



But Why Is $\omega_{n+1} = 4\omega_n$?

- The first observation is that the powers of ζ_m below the real line are obviously related to the powers above the real line
- This is seen in the following diagram



- Algebraically, it follows from the fact that $\zeta_{2^{n+1}}^{2^n} = \left(e^{2\pi i/2^{n+1}}\right)^{2^n} = e^{i\pi} = -1$

But Why Is $\omega_{n+1} = 4\omega_n$?

- The second observation follows from the first: The sum of the odd powers below the real line is equal to the sum of the odd powers above the real line
- So we can write

$$\omega_n = \sum_{k=1}^{2^{n-1}} \frac{1}{\left|1 - \zeta_{2^{n+1}}^{2k-1}\right|^2} = \frac{1}{2}\left(\sum_{k=1}^{2^{n-1}} \frac{1}{\left|1 - \zeta_{2^{n+1}}^{2k-1}\right|^2} + \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{\left|1 - \zeta_{2^{n+1}}^{2k-1}\right|^2}\right)$$

$$= \dots$$

$$= \frac{1}{2}\sum_{k=1}^{2^{n-1}} \left(\frac{1}{\left|1 - \zeta_{2^{n+1}}^{2k-1}\right|^2} + \frac{1}{\left|1 + \zeta_{2^{n+1}}^{2k-1}\right|^2}\right)$$



But Why Is $\omega_{n+1} = 4\omega_n$?

- We're almost there!
- Using a little algebra, we have that

$$\frac{1}{\left|1-\zeta_{2^{n+1}}^{2k-1}\right|^2}+\frac{1}{\left|1+\zeta_{2^{n+1}}^{2k-1}\right|^2}=\frac{\left|1-\zeta_{2^{n+1}}^{2k-1}\right|^2+\left|1+\zeta_{2^{n+1}}^{2k-1}\right|^2}{\left|1-\zeta_{2^{n+1}}^{2k-1}\right|^2\times\left|1+\zeta_{2^{n+1}}^{2k-1}\right|^2}$$

- More algebraic manipulations and the facts that $|z|^2 = z\bar{z}$ and $|\zeta_m^k| = 1$, the numerator can be reduced to 4



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- More algebraic manipulations and the facts that $|z|^2 = z\bar{z}$ and $|\zeta_m^k| = 1$, the numerator can be reduced to 4
- In turn, the denominator can be reduced to $\left|1-\left(\zeta_{2^{n+1}}^{2k-1}\right)^2\right|^2$
- But $\zeta_{2u}^{2v} = \zeta_u^v$, so the denominator is actually $\left|1-\zeta_{2^n}^{2k-1}\right|^2$
- And this all reduces to $\omega_{n+1} = 4\omega_n$, as desired!



The Key Lemma in ACL2

```
(defthmd omega-series-closed-form
    (implies (posp n)
             (equal (omega-series 1 (expt 2 (1- n)) n)
                    (expt 2 (- (* 2 n) 3))))
    :hints ...)
```



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From the Complex Plane to the Real Line

- The Key Lemma shows that a given series of complex numbers has a closed form that is real
- The next step in the proof is to show that the complex series actually resolves to a real series
- The key fact is that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$



From the Complex Plane to the Real Line

- The Key Lemma shows that a given series of complex numbers has a closed form that is real
- The next step in the proof is to show that the complex series actually resolves to a real series
- The key fact is that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

- Some more algebraic manipulation shows that

$$\left|1 - \zeta_n^k\right|^2 = 2 - 2\mathrm{Re}(\zeta_n^k)$$

- $\mathrm{Re}(\zeta_n^k) = \cos(2\pi k/n)$, so some more algebra and trig identities gives

$$\left|1 - \zeta_n^k\right|^2 = \left|2 - 2\left(1 - 2\sin^2\left(\frac{\pi k}{n}\right)\right)\right| = \left|4\sin^2\left(\frac{\pi k}{n}\right)\right| = 4\sin^2\left(\frac{\pi k}{n}\right)$$



A Real Series

- So consider

$$\sum_{k=1}^{\infty} \frac{1}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)}$$

- Recall the Taylor series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- From this, it follows that for $0 < x \leq 2$,

$$1 - \frac{x^2}{6} \leq \frac{\sin(x)}{x} \leq 1$$



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- Even more algebra shows that for $0 < x < \sqrt{3}$

$$\frac{1}{\left(1 - \frac{x^2}{6}\right)^2} \leq 1 + x^2$$



Lower Bound for the Real Series

- With some prior work approximating π, ACL2 can verify by computation that $0 < \frac{\pi}{2} < \sqrt{3}$
- So, for $0 < x < \pi/2$, we have that

$$1 \leq \left(\frac{x}{\sin(x)}\right)^2 \leq 1 + x^2$$

- Recall again the series $\sum_{k=1}^{2^{n-1}} \frac{1}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)}$
- The angles are all in the range $(0, \pi/2)$, so we can apply the above inequality to get, for each k in the range of the sum,

$$1 \leq \frac{\left(\frac{2k-1}{2^{n+1}}\pi\right)^2}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)}$$



Lower Bound for the Real Series

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- Rearranging terms, we find that

$$\frac{2^{2n+2}}{\pi^2}\frac{1}{(2k-1)^2} \leq \frac{1}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)}.$$



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- Taking the entire sum and recalling our prior result for the sum on the right, we have that

$$\frac{2^{2n+2}}{\pi^2}\sum_{k=1}^{2^{n-1}}\frac{1}{(2k-1)^2} \leq \sum_{k=1}^{2^{n-1}}\frac{1}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)} = 2^{2n-1}.$$



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- Rearranging once more, we find that

$$\sum_{k=1}^{2^{n-1}} \frac{1}{(2k-1)^2} \leq \frac{\pi^2}{2^{2n+2}} \cdot 2^{2n-1} = \frac{\pi^2}{8}$$



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- Note that the left-hand side is like the Basel sum, but only including the odd terms



Upper Bound for the Real Series

- From before,

$$\frac{\left(\frac{2k-1}{2^{n+1}}\pi\right)^2}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)} \leq 1 + \left(\frac{2k-1}{2^{n+1}}\pi\right)^2$$

- Dividing by the numerator,

$$\frac{1}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)} \leq \frac{2^{2n+2}}{\pi^2}\frac{1}{(2k-1)^2} + 1$$

- And adding all the terms yields

$$2^{2n-1} = \sum_{k=1}^{2^{n-1}} \frac{1}{\sin^2\left(\frac{2k-1}{2^{n+1}}\pi\right)} \leq \frac{2^{2n+2}}{\pi^2}\sum_{k=1}^{2^{n-1}}\frac{1}{(2k-1)^2} + 2^{n-1}$$



Complete Bounds for the Real Series

- Rearranging terms again, we find that

$$\sum_{k=1}^{2^{n-1}} \frac{1}{(2k-1)^2} \geq \frac{\pi^2}{8} - \frac{\pi^2}{2^{n+3}}$$

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- Combining the lower and upper bounds, we have that

$$\frac{\pi^2}{8} - \frac{\pi^2}{2^{n+3}} \leq \sum_{k=1}^{2^{n-1}} \frac{1}{(2k-1)^2} \leq \frac{\pi^2}{8}$$



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- And taking the standard part of the inequality

$$\frac{\pi^2}{8} - {}^*\left(\frac{\pi^2}{2^{n+3}}\right) \leq {}^*\left(\sum_{k=1}^{2^{n-1}} \frac{1}{(2k-1)^2}\right) \leq \frac{\pi^2}{8}$$



The Limit of the Real Series

- In other words,

$$^{*}\left(\sum_{k=1}^{2^{n-1}} \frac{1}{(2k-1)^2}\right) = \frac{\pi^2}{8}$$

- Thus, we've found the limit of the sum of the odd terms of the Basel series



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```
(defthmd basel-odds-closed-form
  (implies (and (posp n)
                (i-large n))
           (equal (standard-part (basel-odds 1 (1+ (expt 2 (1- n))))
                  (/ (expt (acl2-pi) 2) 8)))
  :hints ...)
```



The Limit of the Real Series

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- Hooray! We've found the limit of the sum of the odd terms of the Basel series



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- Hooray! We've found the limit of the sum of the odd terms of the Basel series
- Well, sort of
- The upper limit is a power of 2, so we've really only shown it for partial sums that end in a power of 2!
- And we still have to find the limit for the entire Basel series
- Next, we'll show that we can take care of those details with just a little care



Outline

- The Basel Problem
- Bounds on the Basel Sum
- The Key Lemma
- Navigating the Subtleties
- The End Game
- Questions?



The Sum of the Odd Terms Is All We Need

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^2} &= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k)^2} \\ &= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{4}{3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

The Sum of the Odd Terms Is All We Need

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Note: This argument is Euler's, too! He showed the connection to the primes

$$\zeta(2) = \sum_{i=1}^{\infty}\frac{1}{i^2} = \prod_{p \text{ prime}}\frac{1}{1-\frac{1}{p^2}}$$



Tying Up the Loose Ends

In ACL2, we start by adding the odd and even terms up to some arbitrary limit

```
(defthmd basel-series-split-evens-odds
  (implies (and (natp i)
                (natp n))
           (equal (+ (basel-odds i n)
                     (basel-evens i n))
                  (basel-series (1- (* 2 i)) (1- (* 2 n))))))
  :hints ...)
```



Tying Up the Loose Ends

Using the prior results, this shows that

```
(defthmd standard-part-basel-series-split-just-odds-complete-2
  (implies (posp n)
           (equal (+ (standard-part (basel-odds 1 (1+ (expt 2 (1- n))))
                     (* 1/4 (standard-part
                             (basel-series 1 (1+ (expt 2 (1- n)))))))
                  (standard-part
                   (basel-series 1 (1- (* 2 (1+ (expt 2 (1- n)))))))))
  :hints ...)
```

Note: Here we need the fact that all the partial sums are limited!



Tying Up the Loose Ends

The upper bounds of the partial sums are different!
But since we showed that those partial sums are always close to each other, the standard parts must be the same

```
(defthmd standard-part-basel-series-split-just-odds-complete-4
  (implies (and (posp n)
                (i-large n))
           (equal (standard-part (basel-series 1 (1+ (expt 2 (1- n)))))
                  (* 4/3 (standard-part (basel-odds 1 (1+ (expt 2 (1- n)
  :hints ...)
```



The Final Theorem

Combining the above with the theorem about the limit of the odd terms completes the proof of the Basel Problem in ACL2(r)

```
(defthmd basel-series-closed-form
  (implies (and (posp n)
                (i-large n))
           (equal (standard-part (basel-series 1 n))
                  (/ (expt (acl2-pi) 2) 6)))
  :hints ...)
```



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