Mechanized Verification of q-Pulse Rectifier Voltage Equations in ACL2(r)

Certified Results and a Refined Roadmap

Pablo O. Cayuela

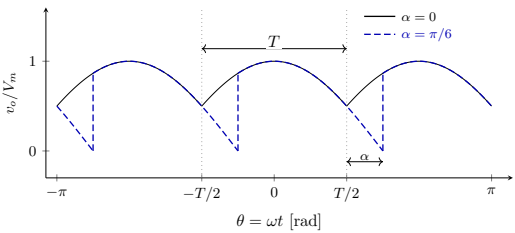
Universidad Tecnológica Nacional, Facultad Regional Córdoba, Argentina

ACL2 Workshop 2026

Two numbers per rectifier

- Every power-electronics course characterizes a rectifier by
  - the average output voltage V_{DC}, the useful DC component, and
  - the RMS voltage V_{RMS}, from which ripple and the figures of merit follow.
- Textbooks give them topology by topology: single-phase, three-phase midpoint, bridge, ...
- A companion paper (CVM 2026, in Spanish) derives one master equation, in the form of the sampling function Sa, for uncontrolled, controlled and semi-controlled rectifiers, for any q (single-phase half-wave excepted).
- This paper: how far ACL2(r) gets in checking it exactly.

The model: everything is the sampling function



Pulses of peak V_m and width T; firing delay α. Midpoint: $T = 2\pi/q$. Bridge: $T = \pi/q$. Single-phase half-wave ($T = 2\pi$): outside the model, no dead time.

$$Sa(x) = \frac{\sin x}{x}$$

Integrating one pulse, and its square:

$$V_{DC}(\alpha) = V_m\, Sa(T/2)\cos\alpha$$

$$V_{RMS}(\alpha) = V_{mRMS}\sqrt{1 + Sa(T)\cos 2\alpha}$$

with $V_{mRMS} = V_m/\sqrt{2}$.

Note: Sa is the unnormalized sinc; in DSP notation, $Sa(x) = \mathrm{sinc}(x/\pi)$ with $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$.

Every topology is an instance of these two master equations.

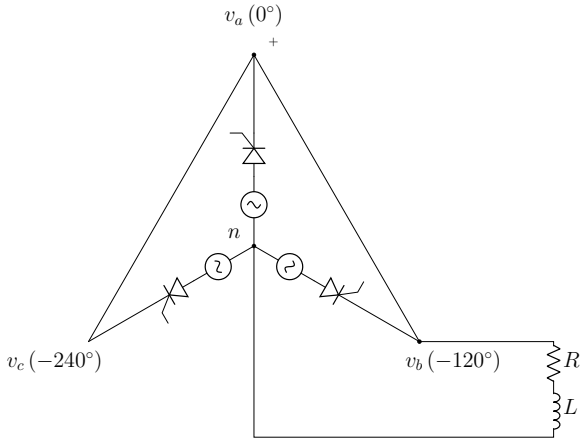
Generalizing: a family and a regime

Family	V_m	T	V_{mRMS}
Midpoint	V_{max}	$2\pi/q$	$V_{max}/\sqrt{2}$
Bridge	$\sqrt{3}\,V_{max}$ *	π/q	$V_m/\sqrt{2}$

Regime	$f_{AV}(\alpha)$	$f_{RMS}(\alpha)$
Uncontrolled	1	1
Semi-controlled, $\alpha \le \pi/3$	$(1+\cos\alpha)/2$	$\cos^2\alpha$
Semi-controlled, $\alpha > \pi/3$	$(1+\cos\alpha)/2$	$(\pi/3 - \alpha + \frac{1}{2}\sin 2\alpha)/\sqrt{3}$
Controlled	$\cos\alpha$	$\cos 2\alpha$

- $V_{DC} = V_m\, Sa(T/2)\, f_{AV}(\alpha)$, $V_{RMS} = V_{mRMS}\sqrt{1 + Sa(T)\, f_{RMS}(\alpha)}$.
- * $q = 3$: line-to-line peak; $V_m = V_{max}$ for $q = 1$. q: secondary phase voltages (a center tap doubles it).
- Semi-controlled rows: three-phase semi-converter; its RMS changes form at $\alpha = \pi/3$ (later slide).

Midpoint rectifier, $q = 3$



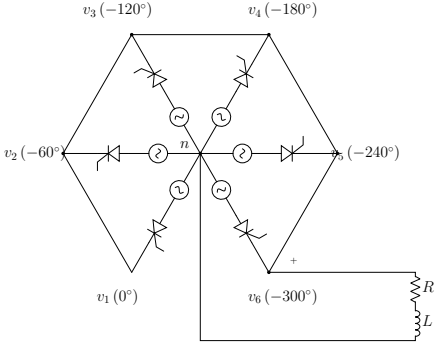
$T = 2\pi/3$, $V_m = V_{max}$,
$V_{mRMS} = V_{max}/\sqrt{2}$.
Sources $v_k = V_{max}\cos(\omega t + \varphi_k)$; n is the midpoint.

$$V_{DC} = V_m\, Sa(\pi/3)\cos\alpha$$
$$= \tfrac{3\sqrt{3}}{2\pi} V_{max}\cos\alpha$$

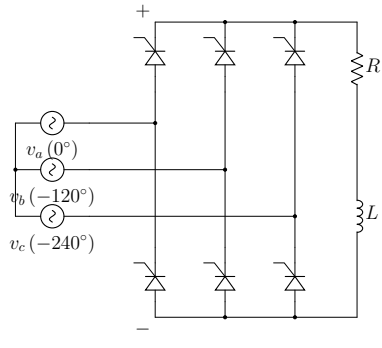
$$V_{RMS} = V_{mRMS}\sqrt{1 + Sa(2\pi/3)\cos 2\alpha}$$
$$= V_{mRMS}\sqrt{1 + \tfrac{3\sqrt{3}}{4\pi}\cos 2\alpha}$$

Classical at $\alpha = 0$ (Rashid 3.25–3.26).

Six pulses per cycle: two circuits, one pair of equations



six-phase midpoint ($q = 6$)



three-phase bridge ($q = 3$)

$T = \pi/3$ in both; V_m is the peak of each source (six-phase) or the line-to-line peak $\sqrt{3}\,V_{max}$ (bridge).

$$V_{DC} = V_m\,Sa(\pi/6)\cos\alpha = \tfrac{3}{\pi}V_m\cos\alpha, \qquad V_{RMS} = V_{mRMS}\sqrt{1+\tfrac{3\sqrt{3}}{2\pi}\cos 2\alpha}$$

$3\sqrt{3}/2\pi = Sa(\pi/3)$: our early formula had $3\sqrt{3}/4\pi$, the midpoint coefficient.

First check: numerical, with Claude

- Given the closed forms, Claude rebuilt each output waveform from the circuit definition and integrated it directly (trapezoidal rule, 2×10^5 points).
- Independent of the formula under test.
- Agreement to five significant figures everywhere, except the bridge RMS with the midpoint coefficient: $V_{RMS} \approx 1.6554\, V_{max}$ (Rashid, 4th ed., Eq. 3.34) matches only $3\sqrt{3}/2\pi$.
- But a numerical check samples a few values of α; it is not a proof for every α in $[0, \pi]$.

Next step: exact statements, certified in ACL2(r).

Exact values of *Sa* in ACL2(r)

```
(defun sa (x) (if (equal x 0) 1 (/ (acl2-sine x) x)))

(defthm sa-pi/3
  (equal (* (sa (* 1/3 (acl2-pi))) 2 (acl2-pi)) (* 3 (acl2-sqrt 3)))
  :hints (("Goal" :use (sine-pi/3 pi-between-2-4) :in-theory (enable sa))))

(defthm sa-pi-val (equal (sa (acl2-pi)) 0)
  :hints (("Goal" :use (sine-pi) :in-theory (enable sa))))
```

- Five values: $Sa(\pi/6)$, $Sa(\pi/3)$, $Sa(2\pi/3)$, $Sa(\pi/2)$, $Sa(\pi)$ — every coefficient of the table.
- $\sin(2\pi/3)$ and $\cos(2\pi/3)$ are not in `nonstd/nsa/trig`: derived from $\sin(\pi - x)$, $\cos(\pi - x)$.
- `arithmetic-5` is needed to simplify divisions by rational multiples of `(acl2-pi)`.
- The multiplied-through form is a presentation choice; the direct form also proves.

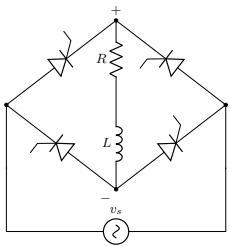
Finding 1: the bridge coefficient, confirmed exactly

```
(defthm eq12-VCC-bridge-3ph-coefficient
  (equal (* (acl2-sqrt 3) (sa (* 1/6 (acl2-pi))) (acl2-pi))
         (* 3 (acl2-sqrt 3)))
  :hints (("Goal" :use sa-pi/6)))

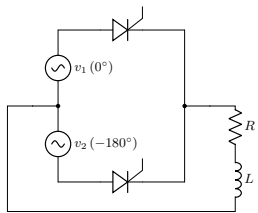
(defthm bridge-vs-midpoint-Sa-differ
  (not (equal (sa (* 1/3 (acl2-pi))) (sa (* 2/3 (acl2-pi)))))
  :hints (("Goal" :use (sa-pi/3 sa-2pi/3 pi-between-2-4
                        (:instance sqrt-sqrt (x 3)))
                  :nonlinearp t)))
```

- The two coefficients are different exact irrational numbers, not a rounding artifact.

The single-phase / two-phase identity



(a) single-phase bridge ($q = 1$)



(b) two-phase midpoint ($q = 2$)

Both: $T = \pi$, $V_m = V_{max}$,

$$V_{DC} = \frac{2}{\pi} V_{max} \cos\alpha, \quad V_{RMS} = V_{mRMS}.$$

A center-tapped rectifier *is* a two-phase midpoint: two sources 180° apart. Not the same transformer: for a secondary of peak V_s, the bridge has $V_m = V_s$; the center-tapped, $V_s/2$.

```
(defthm bifasico-T-coincides
  (equal (/ (acl2-pi) 1) (/ (* 2 (acl2-pi)) 2)))
```

So `sa-pi/2` and `sa-pi-val` are shared by both parametrizations.

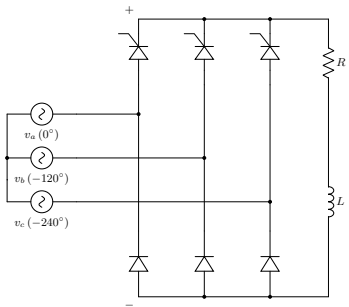
Finding 2: an inconsistency the mechanization found

- $Sa(\pi) = 0$, so for $T = \pi$ the master equation gives $V_{RMS}/V_{mRMS} = 1$ for every α.
- That is the textbook fact: the waveform only shifts in time, and RMS is shift-invariant.
- The companion paper's table printed $\sqrt{1 + \cos 2\alpha}$, which is not constant.

```
(defthm q1-table-VRMS-row-inconsistent-with-master-eq
  (and (equal (sa (acl2-pi)) 0)
       (not (equal (acl2-sqrt (+ 1 (acl2-cosine (* 2/3 (acl2-pi))))) 1)))
  :hints (("Goal" :use (sa-pi-val cosine-2pi/3 sqrt-1/2-not-1))))
```

Witness $\alpha = \pi/3$: $\sqrt{1 + \cos(2\pi/3)} = \sqrt{1/2} \neq 1$. An erratum, now corrected in the table.

The semi-converter: a case split at $\alpha = \pi/3$



three-phase semi-converter: thyristors (fired with delay α) and diodes

$$V_{RMS} = \sqrt{3}V_{max}\sqrt{\tfrac{3}{4\pi}\left(\tfrac{2\pi}{3} + \sqrt{3}\cos^2\alpha\right)}, \quad \alpha \le \pi/3$$

$$V_{RMS} = \sqrt{3}V_{max}\sqrt{\tfrac{3}{4\pi}\left(\pi - \alpha + \tfrac{1}{2}\sin 2\alpha\right)}, \quad \alpha > \pi/3$$

Three exact cross-checks:

1. continuity at $\pi/3$: there $2\pi/3 = \pi - \alpha$, so it reduces to $\sqrt{3}\cos^2\alpha = \frac{1}{2}\sin 2\alpha$;
2. at $\alpha = 0$, the first branch gives the uncontrolled bridge;
3. at $\alpha = \pi$, the second gives $V_{RMS} = 0$, as $V_{DC} = 0$.

```
(defthm semiconv-continuity-at-pi/3
  (equal (* (acl2-sqrt 3)
            (expt (acl2-cosine (* 1/3 (acl2-pi))) 2))
         (* 1/2 (acl2-sine (* 2/3 (acl2-pi))))))
```

What these theorems are, and are not

- Thirteen theorems (plus two auxiliary lemmas), in three books, all certified.
- Most are instances of facts `nonstd/nsa/trig` already proves: ACL2(r) as an exact calculator.
- What the mechanization adds to the numerical check: exactness, and the two findings.
- No hypothesis on $\sqrt{3}$: it is `(acl2-sqrt 3)`, whose square the library proves to be 3.
- The circuits are ideal: ideal switches, ideal source, continuous load current. The theorems say nothing about real diodes.

The integral-based master theorem: done after the camera-ready

- The deepest check: *define* V_{DC}, V_{RMS} as integrals of the pulse and prove the closed forms.
- Done after the camera-ready, in ACL2(r): `vcc-master-theorem` (eq. 7) and `vrms-master-theorem` (eq. 8), certified from clean sources, no `skip-proofs`.

$$V_m \tfrac{1}{T}\int_{-T/2}^{T/2} \cos(\theta+\alpha)\,d\theta = V_m\,Sa(T/2)\cos\alpha$$

$$V_m^2 \tfrac{1}{T}\int_{-T/2}^{T/2} \cos^2(\theta+\alpha)\,d\theta = \tfrac{1}{2}V_m^2(1+Sa(T)\cos 2\alpha)$$

- How: sine and cosine registered with the Reid–Gamboa differentiator (`sin-cos-minimal`); a macro `def-ftc2-instance` builds each integral and its `ftc-2` value.
- Limits: α and $T > 0$ are standard reals (encapsulated); one pulse, not a whole period; the RMS theorem states the square; semi-converters, 6- and 12-pulse rows not covered.
- Not in the paper's books: branch `ftc2-integral-master-theorems`, tag `v1.1-ftc2`.

Continuation: a duty-factor generalization (to be verified)

One pulse of width W, centered at c, in an output period T:

$$V_{DC} = V_m \frac{W}{T} Sa(W/2)\cos c$$

$$V_{RMS} = V_{mRMS}\sqrt{\frac{W}{T}\left(1 + Sa(W)\cos 2c\right)}$$

Case	W	c	V_{DC}/V_m
Uncontrolled	T	0	$Sa(T/2)$
Controlled, continuous	T	α	$Sa(T/2)\cos\alpha$
Controlled, resistive, $\alpha \geq \frac{\pi - T}{2}$	$\frac{T+\pi}{2} - \alpha$	$\frac{\alpha}{2} + \frac{\pi - T}{4}$	$\frac{1+\sin(T/2-\alpha)}{T}$
Single-phase half-wave ($T = 2\pi$)	$\pi - \alpha$	$\alpha/2$	$\frac{1+\cos\alpha}{2\pi}$

- The master equations are the case $W = T$, $c = \alpha$; T and V_m come from the family.
- Same form as before, with $f_{AV} = \frac{W}{T}\frac{Sa(W/2)}{Sa(T/2)}\cos c$ and $f_{RMS} = \frac{(W/T)(1+Sa(W)\cos 2c)-1}{Sa(T)}$, except for $T = 2\pi$, where $Sa(T/2) = 0$.
- Checked by hand and numerically. Not mechanized.

Can the paper alone reproduce the results?

The chairs plan an experiment: reproduce the results from the paper only. Before the final version we tried it.

- A fresh AI agent, given only the PDF of the accepted paper, offline ACL2(r) in Docker.
- Everything certified on the first attempt, in about three minutes.
- Eight results needed its own wording: the paper named them without statements.
- It found that the direct form of $Sa(\pi/3)$ *does* prove — the paper had said it did not.
- In the final version: a table of all fifteen `defthm` forms, with statements, and the direct-form remark corrected.

Related work

- Textbooks, one topology at a time: the q-pulse mean and RMS at $\alpha = 0$ (Rashid Eqs. 3.25–3.26, 3.30), the bridge (10.15, 10.17), the semi-converter (14.25); Mohan et al., Sec. 5-6, Eq. (5-68).
- Not found there: one pair of equations in the form of Sa, for all regimes and any q (search not exhaustive).
- ACL2(r): the FFT proof (Gamboa 1999) and Fourier series (Chau, Kaufmann, Hunt 2015). No power-electronics equations before this work.
- Closest for the integral side: automatic differentiation (Reid, Gamboa 2011), used in the circle-geometry book (Bapanapally, Gamboa 2021).

Summary

- One model, *Sa*-based, for uncontrolled, controlled and semi-controlled *q*-pulse rectifiers.
- Thirteen certified ACL2(r) theorems: exact *Sa* values, the bridge coefficient, an erratum, the semi-converter cross-checks.
- Numerical check first, mechanized check second: each caught something.
- The integral-based master theorem (eqs. 7, 8) was proved after the camera-ready, outside the paper's books (tag `v1.1-ftc2`).

Books: `[books]/workshops/2026/cayuela-rectifiers/`
Development version: `github.com/pcayuela/ACL2-rectifiers-check`
Developed with Claude Code driving ACL2(r) through an MCP server; ACL2(r) is the arbiter.

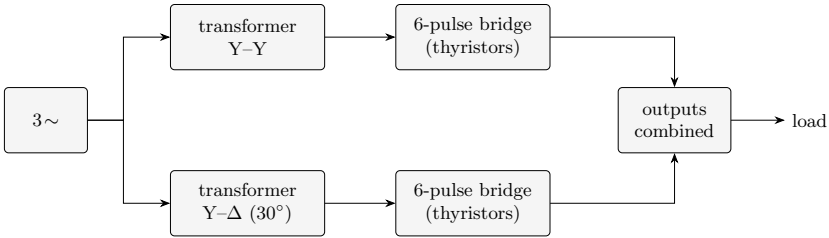
Thank you — questions?

Backup: all instantiations

Topology	q	$V_{DC}(\alpha)$	$V_{RMS}(\alpha)/V_{mRMS}$
Single-phase bridge	1	$(2/\pi)V_{max}\cos\alpha$	1
Two-phase midpoint (center-tap)	2	$(2/\pi)V_{max}\cos\alpha$	1
Three-phase midpoint	3	$(3\sqrt{3}/2\pi)V_{max}\cos\alpha$	$\sqrt{1+(3\sqrt{3}/4\pi)\cos 2\alpha}$
Three-phase bridge	3	$(3\sqrt{3}/\pi)V_{max}\cos\alpha$	$\sqrt{1+(3\sqrt{3}/2\pi)\cos 2\alpha}$
Six-phase midpoint	6	$(3\sqrt{3}/\pi)V_{max}\cos\alpha$	$\sqrt{1+(3\sqrt{3}/2\pi)\cos 2\alpha}$
Twelve-pulse	6	$(6/\pi)\sin(\pi/12)\sqrt{3}V_{max}\cos\alpha$	$\sqrt{1+Sa(\pi/6)\cos 2\alpha}$

- Classical at $\alpha = 0$: Rashid 3.25–3.26, 10.15, 10.17, 14.25. New: one pair for all.
- q: secondary phase voltages feeding the rectifier (a center tap doubles it). Pulses per cycle: q (midpoint), $2q$ (bridge); same T for a q-phase bridge and a $2q$-phase midpoint. Rows: controlled regime.
- Early bridge RMS formula: $3\sqrt{3}/4\pi$ (midpoint). Bridge $Sa(\pi/3) = 3\sqrt{3}/2\pi$; midpoint $Sa(2\pi/3) = 3\sqrt{3}/4\pi$.

Backup: twelve-pulse arrangement



(b) twelve-pulse: two bridges fed with a 30° shift

(schematic)

- $V_{DC} = \frac{6}{\pi} \sin \frac{\pi}{12} \sqrt{3}\, V_{max} \cos \alpha$, $V_{RMS} = V_{mRMS} \sqrt{1 + Sa(\pi/6) \cos 2\alpha}$.
- Schematic: how the two outputs are combined is not part of the model.

Backup: the fifteen certified `defthm` forms

Book	Event	Statement
00	`sine-2pi/3`, `cosine-2pi/3`†	$\sin(2\pi/3) = \sqrt{3}/2$, $\cos(2\pi/3) = -1/2$
	`sa-pi/3`, `sa-2pi/3`	$Sa(\pi/3) \cdot 2\pi = 3\sqrt{3}$, $Sa(2\pi/3) \cdot 4\pi = 3\sqrt{3}$
	`sa-pi/6`, `sa-pi/2`, `sa-pi-val`	$Sa(\pi/6) \cdot \pi = 3$, $Sa(\pi/2) \cdot \pi = 2$, $Sa(\pi) = 0$
01	`eq12-VCC-bridge-3ph-coefficient`	$\sqrt{3}\, Sa(\pi/6) \cdot \pi = 3\sqrt{3}$
	`bridge-vs-midpoint-Sa-differ`	$Sa(\pi/3) \neq Sa(2\pi/3)$
	`bifasico-T-coincides`, `sqrt-1/2-not-1`†	$\pi/1 = 2\pi/2$, $\sqrt{1/2} \neq 1$
	`q1-table-VRMS-row-inconsistent-...`	$Sa(\pi) = 0$ and $\sqrt{1 + \cos(2\pi/3)} \neq 1$
02	`semiconv-continuity-at-pi/3`	$\sqrt{3}\cos^2(\pi/3) = \frac{1}{2}\sin(2\pi/3)$
	`semiconv-alpha-0-matches-bridge`	first branch at 0 = uncontrolled bridge
	`semiconv-alpha-pi-zero`	$\pi - \pi + \frac{1}{2}\sin(2\pi) = 0$

† auxiliary lemmas, used only in hints.