A Mechanical Proof of Stirling's Approximation in ACL2(r)

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Outline

- Why Stirling's Approximation?
- First Results
- Estimating the Error Term
- The Wallis Product Formula for π
- Finally, Stirling's Formula
- Questions?



Some History

- Abraham de Moivre (1721): Approximated the factorial function
  - Working on probability theory, binomial distribution, normal distribution
  - Discovered $n! \approx K \cdot n^{n+1/2} \cdot e^{-n}$
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  - Tried to discover the value of the constant K but failed
- James Stirling (1730): Discovered the value of the constant K
  - $K = \sqrt{2\pi}$



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- Logarithmic version:

$$\ln(n!) \sim n\ln n - n + \ln(\sqrt{2\pi n})$$



Applications of the Approximation

- Statistical Mechanics
  - Estimating entropy of a system $S = k \ln W$
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  - Can also be used in the other direction!
- Computer Science
  - Prove lower bound on the complexity of sorting
  - The bound is $O(n!)$, which becomes the familiar $O(n \log n)$



The Beauty of Stirling's Formula

- Naturally, we're drawn to problems that require ACL2(r)
- Stirling's formula is a good example of such a problem

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

- The left-hand side is just factorial, which is a trivial function—often the first function you learn to define in ACL2
- The right-hand side involves π and e and $\sqrt{\cdot}$
- The proof uses integrals, infinite series, and the function $ln(\cdot)$
- So it's a great example of why you need ACL2(r)!



Where ACL2(r) Comes In

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- ACL2(r) is based on nonstandard analysis, which is a non-traditional way of formalizing calculus
- In non-standard analysis, some numbers are called **standard**
- All the numbers you know (including e and π) are standard
- But there are nonstandard numbers, such as small positive numbers that are smaller than **all** standard positive numbers
- A good name for these numbers is **infinitesimals**, and you can think of them as the ϵ's from calculus



Nonstandard Analysis in 2 1/2 Slides

- Some numbers are **limited**, meaning that they are bounded by some standard number
- For example, 1 is limited, but $1/\epsilon$ is not
- If a number is not limited, it is called **(infinitely) large**



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- All limited numbers can be written as $x = {}^*x + \epsilon$, where ${}^*x$ is the **standard part** of x, and ϵ is infinitesimal

- Standard part has some nice properties:
  1. ${}^*(x + y) = {}^*x + {}^*y$
  2. ${}^*(x \cdot y) = {}^*x \cdot {}^*y$ but only when x and y are limited
  3. $x \leq y \Rightarrow {}^*x \leq {}^*y$



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Nonstandard Analysis in 2 1/2 Slides

- Two numbers are **(infinitely) close** if their difference is infinitesimal
- A sequence $a_1, a_2, a_3, \ldots$ **converges** to a limit L if ${}^*a_n = L$ for all infinitely large n
- Equivalently, a_i is limited and a_i is close to a_j for all infinitely large i and j
- Notice that in the second definition, we do not need to know what the limit L is, reminiscent of the Cauchy criterion in standard analysis



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- We start by considering $\ln(n!)$, which we can write as a sum:

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- To translate this into ACL2(r)
  1. Define a function that computes the sum of the logs

$$\texttt{(sum-ln i0 n)} \triangleq \sum_{i=i_0}^{n} \ln(i)$$

  2. Then proceed by induction on n

```
(defthmd ln-factorial
    (implies (posp n)
             (equal (acl2-ln (factorial n))
                    (sum-ln 1 n))))
```



The Sum of the Logs as an Integral Approximation

- The sum is obviously related to an integral:

$$\ln(n!) = \sum_{i=1}^{n} \ln(i) \approx \int_{1}^{n} \ln(x)dx$$



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- A Riemann sum would be missing one of the endpoints
- But the trapezoid rule is very close to this:

$$T(1,n) = \frac{1}{2}\ln(1) + \sum_{i=2}^{n-1} \ln(i) + \frac{1}{2}\ln(n)$$



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$$T(1,n) = \frac{1}{2}\ln(1) + \sum_{i=2}^{n-1} \ln(i) + \frac{1}{2}\ln(n)$$

- Rearranging, we can write the sum in terms of the trapezoid rule:

$$\sum_{i=1}^{n} \ln(i) = T(1,n) + \frac{1}{2}\ln(n)$$



The Integral Itself

- ACL2(r) knows about integrals, but it's rather tedious to work with them
- With a bit of work and the Fundamental Theorem of Calculus, we can show that

$$\int_a^b \ln(x)\,dx = b\ln(b) - a\ln(a) - (b-a)$$

```
(defthmd int-ln-computation
  (implies (and (inside-interval-p a (positive-domain))
                (inside-interval-p b (positive-domain)))
           (equal (int-ln a b)
                  (+ (- (* b (acl2-ln b))
                        (* a (acl2-ln a)))
                     (- a b)))))
```

- We found it useful to specialize this result for the case $a = 1$ and the case $a = i, b = i + 1$.



What We Have So Far

- We know that
$$\sum_{i=1}^{n} \ln(i) = T(1,n) + \frac{1}{2}\ln(n)$$
- And we know that $T(1,n)$ is close to $n\ln(n) - n + 1$
- So we can expect that
$$\ln(n!) = \sum_{i=1}^{n} \ln(i) \approx n\ln(n) - n + 1 + \frac{1}{2}\ln(n)$$
- This is in the right direction, but we want to replace $1 + \frac{1}{2}\ln(n)$ with $\ln(\sqrt{2\pi n})$
- We'll need to be more precise about the error term in $T(1,n)$



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The Trapezoid Error Term

- The trapezoid rule has a well-known error term for each trapezoid:

$$\int_a^b f(x)dx - T(a,b) = -\frac{(b-a)^3}{12} f''(c)$$

for some $c \in (a,b)$

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for some $c \in (a,b)$

- When n trapezoids are used, this leads to the following error term:

$$\left| \int_a^b f(x)dx - T(a,b) \right| \leq \frac{M \cdot (b-a)^3}{12n^2}$$

where M is an upper bound on $|f''(x)|$ for $x \in [a,b]$



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- But for this project, we need an alternative approach

- What we did was to prove this result, but only for the very specific case that
  - $f(x) = \ln(x)$
  - $a = 1$
  - $b = n$
  - n is a positive integer
  - we use $n - 1$ trapezoids, so the width of each trapezoid is 1



The Trapezoid Error Term Proof (by Hand)

- First integrate by parts (on each trapezoid):

$$\int_i^{i+1} f(x)\,dx = [(x+A)f(x)]_i^{i+1} - \int_i^{i+1} (x+A)f'(x)\,dx$$

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- Then choose A so that $x + A$ looks like the trapezoidal approximation:

$$\int_i^{i+1} f(x)\,dx = T(i, i+1) - \int_i^{i+1} (x - (i+1/2))f'(x)\,dx$$



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$$\int_i^{i+1} f(x)\,dx = T(i, i+1) - \int_i^{i+1} (x-(i+1/2))f'(x)\,dx$$

- The error term **is** the second integral, so let's focus on that:

$$\int_i^{i+1} (x-(i+1/2))f'(x)\,dx =$$

$$\left[\left(\frac{1}{2}(x-(i+1/2))^2 + B\right) f'(x)\right]_i^{i+1} - \int_i^{i+1} \left(\frac{1}{2}(x-(i+1/2))^2 + B\right) f''(x)\,dx$$



The Trapezoid Error Term Proof (by Hand)

- Now we choose $B = 1/8$ to get rid of the first term:

$$\int_i^{i+1} f(x)\,dx = T(i, i+1) + \int_i^{i+1} \left(\frac{1}{2}(x - (i + 1/2))^2 - \frac{1}{8}\right) f''(x)\,dx$$



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- Finally, let M_i be an upper bound on $|f''(x)|$ for $x \in [i, i+1]$, and we get:

$$\int_i^{i+1} f(x)\,dx \leq T(i, i+1) + M_i \cdot \int_i^{i+1} \left(\frac{1}{2}(x - (i+1/2))^2 - \frac{1}{8}\right) dx$$



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- Note that the above is scratch work
- It is not formalized in ACL2(r)
- What we'll do is to use that scratch work to motivate some definitions (out of thin air) and manipulate them to get the ACL2(r) proof



Step 1

```
(defun trapezoidal-error-integrand (i x)
    (let ((i (realfix i))
          (x (realposfix x)))
        (- (/ (expt (- x (+ i 1/2)) 2) 2) 1/8)))
```



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(defthmd int-trapezoidal-error-integrand-i-to-i+1
    (implies (and (integerp i)
                  (< 0 i))
             (equal (int-trapezoidal-error-integrand i i (1+ i))
                    -1/12)))
```



Step 2a

We specialize

$$\int_i^{i+1} \left(\frac{1}{2}(x - (i + 1/2))^2 - \frac{1}{8} \right) f''(x)\, dx$$

when $f(x) = \ln(x)$ and $f''(x) = -1/x^2$

```
(defun trapezoidal-error-integrand-for-ln (i x)
  (let ((i (realfix i))
        (x (realposfix x)))
    (* (trapezoidal-error-integrand i x)
       (- (/ (expt x 2))))))
```

Next we define the antiderivative of the integrand and invoke the FTC



Step 2b

```
(defun trapezoidal-error-integrand-for-ln-antiderivative (i x)
  (let ((i (realfix i))
        (x (realposfix x)))
    (* -1/2
       (+ x
          (- (* (+ (* 2 i) 1)
                (acl2-ln x)))
          (- (/ (* i (1+ i)) x ))))))

(defthmd int-trapezoidal-error-integrand-for-ln-computation
  (implies (and (inside-interval-p a (positive-domain))
                (inside-interval-p b (positive-domain))
                (realp i))
           (equal (int-trapezoidal-error-integrand-for-ln i a b)
                  (- (trapezoidal-error-integrand-for-ln-antiderivative
                       i b)
                     (trapezoidal-error-integrand-for-ln-antiderivative
                       i a)))))
```



Step 3

Recall that

$$\int_i^{i+1} f(x)\,dx = T(i, i+1) + \int_i^{i+1} \left(\frac{1}{2}(x - (i + 1/2))^2 - \frac{1}{8}\right) f''(x)\,dx$$

We have the pieces to concentrate on that right-hand side

```
(defthmd int-trapezoidal-error-integrand-for-ln-computationa-rhs
  (implies (and (realp i)
                (< 0 i))
           (equal (+ (int-trapezoidal-error-integrand-for-ln i i (1+ i))
                     (/ (+ (acl2-ln (1+ i)) (acl2-ln i)) 2))
                  (+ (* (1+ i) (acl2-ln (1+ i)))
                     (- (* i (acl2-ln i)))
                     -1))))
```

The right-hand side is exactly equal to $\int_i^{i+1} \ln(x)\,dx$!
Coincidence?



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```

The right-hand side is exactly equal to $\int_i^{i+1} \ln(x)\,dx$!
Coincidence? Of course not! We rigged the game



Step 4

Introduce new names to refer to the pieces of the approximation:

```
(defthmd int-ln-trapezoid-error-step
 (implies (and (realp i)
               (< 0 i))
          (equal (int-ln i (1+ i))
                 (+ (ln-trapezoid-step i)
                    (ln-trapezoid-step-error i)))))
```



Step 5a

Next, we find a lower bound for the error term of a single trapezoid

```
(defthm-std int-trapezoidal-error-integrand-for-ln-lower-bound-lemma
  (implies (and (realp i)
                (< 0 i))
           (<= (* (expt (1+ i) -2)
                  -1
                  (int-trapezoidal-error-integrand i i (1+ i)))
               (int-trapezoidal-error-integrand-for-ln i i (1+ i)))))
```

Recall: the integral in the left-hand side evaluates to $-1/12$

```
(defthm int-trapezoidal-error-integrand-for-ln-lower-bound
  (implies (posp i)
           (<= (* 1/12 (expt (1+ i) -2))
               (int-trapezoidal-error-integrand-for-ln i i (1+ i)))))
```



Step 5b

And now we find an upper bound for the error term of a single trapezoid

```
(defthm-std int-trapezoidal-error-integrand-for-ln-upper-bound-lemma
    (implies (and (realp i)
                  (< 0 i))
             (<= (int-trapezoidal-error-integrand-for-ln i i (1+ i))
                 (* (expt i -2)
                    -1
                    (int-trapezoidal-error-integrand i i (1+ i))))))

(defthm int-trapezoidal-error-integrand-for-ln-upper-bound
  (implies (posp i)
           (<= (int-trapezoidal-error-integrand-for-ln i i (1+ i))
               (* 1/12 (expt i -2)))))
```



Step 6

We have bounds and error terms for each trapezoid.
Now we define functions that sum up these error terms and their bounds.

```
(defthmd int-ln-trapezoid-approximation-lemma
  (implies (and (posp i)
                (posp n))
           (equal (sum-int-ln i n)
                  (+ (ln-trapezoid-approximation i n)
                     (ln-trapezoid-approximation-error i n))))
  :hints ...)
```

Now we simplify each sum in the equation above



Step 7a

The left-hand side is easy

$$\texttt{(sum-int-ln i n)} = \sum_{1}^{n} \int_{i}^{i+1} \ln(x)\,dx = \int_{1}^{n} \ln(x)\,dx = n\ln(n) - n + 1$$



Step 7b

For the trapezoid approximation, we have

$$\sum_{i=1}^{n} \frac{1}{2}(\ln(i) + \ln(i+1)) = \frac{1}{2}\ln(1) + \sum_{i=2}^{n-1} \ln(i) + \frac{1}{2}\ln(n) \qquad \text{(by induction)}$$

$$= \sum_{i=1}^{n} \ln(i) - \frac{1}{2}\ln(n)$$



Step 7c

The last sum is the sum of the logs, which is just $\ln(n!)$, so we have

```
(defthm ln-factorial-as-stirling-term-lemma-2
  (implies (and (integerp n)
                (< 1 n))
           (equal (acl2-ln (factorial n))
                  (+ (* n (acl2-ln n))
                     (- n)
                     1
                     (* 1/2 (acl2-ln n))
                     (- (ln-trapezoid-approximation-error 1 n)))))
  :hints ...)
```



Step 8

Take the exponential of both sides and regroup to get

```
(defthm factorial-as-stirling-term-lemma
  (implies (and (integerp n)
                (< 1 n))
           (equal (/ (factorial n)
                     (* (expt n n)
                        (/ (acl2-exp n))
                        (acl2-sqrt n)))
                  (acl2-exp
                   (- 1 (ln-trapezoid-approximation-error 1 n)))))
  :hint ...)
```

We can see the shape of Stirling's approximation!
The last term, $\exp\{1 - E(1, n)\}$, must be the famous constant that Stirling found!



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```

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Awkward: That depends on n. It's not constant at all!



Step 9

Recall that `(ln-trapezoid-approximation-error 1 n)` is the sum of the error terms in the trapezoid approximation

We already have good error bounds for the error term of a single trapezoid and adding those we get

$$\frac{1}{12}\sum_{i=1}^{n-1}\frac{1}{(i+1)^2} \leq E(1,n) \leq \frac{1}{12}\sum_{i=1}^{n-1}\frac{1}{i^2}$$



Panic!

Recognize that series? $\sum_{i=1}^{n} \frac{1}{i^2}$

It is the famous series from the Basel problem, which converges to $\pi^2/6$

Luckily, we **did not** have to prove that in ACL2(r) to finish this proof



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All we needed was

$$0 \leq \sum_{i=1}^{n} \frac{1}{i^2} \leq 2$$

and

$$0 \leq \sum_{i=m}^{n} \frac{1}{i^2} \text{ is infinitesimal when } m \text{ and } n \text{ are large}$$



Step 10

```
(defthmd ln-trapezoid-approximation-error-gross-bounds
  (implies (and (posp m)
                (posp n)
                (<= m n))
           (and (<= 0 (ln-trapezoid-approximation-error m n))
                (<= (ln-trapezoid-approximation-error m n) 1/6))))

(defthmd ln-trapezoid-approximation-error-small-for-large-values
  (implies (and (posp m)
                (posp n)
                (<= m n)
                (i-large m)
                (i-large n))
           (i-small (ln-trapezoid-approximation-error m n))))
```



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The Wallis Produce



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Wrapping Up Stirling's Formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}$$

To finish the proof, we start with the Wallis Product (which has a known left-hand side) and convert the right-hand side into factorials

Then we replace the factorials with Stirling's approximation (including the unknown constant)

And finally, we can solve for the constant



Step 1

Restate the numerator and denominator as factorials:

Numerator: $4^n(n!)^2$

Denominator: $(2n-1)!!\,(2n+1)!!$

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Denominator: $(2n-1)!!\,(2n+1)!!$

Useful fact: $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$

Step 2

This is a useful restatement of our earlier result for the trapezoidal error:

```
(defthm ln-factorial-as-stirling-term
  (implies (and (integerp n)
                (< 1 n))
           (equal (acl2-ln (factorial n))
                  (+ (acl2-ln (expt n n))
                     (- n)
                     (acl2-ln (acl2-sqrt n))
                     (ln-stirling-constant-n n)))))
```

This just names the error terms suggestively!
We introduced the names

```
ln-stirling-constant-n  ln-stirling-constant
stirling-constant-n     stirling-constant
```



Step 3

Scale the numerator and denominator to make the Stirling terms appear:

```
(defun scaled-wallis-numerator (n)
  (/ (* (expt 4 n) (expt (factorial n) 2))
     (expt (* (expt n n)
              (/ (acl2-exp n))
              (acl2-sqrt n))
           4)))

(defun scaled-wallis-denominator (n)
  (/ (* (!! (1- (* 2 n)))
        (!! (1+ (* 2 n))))
     (expt (* (expt n n)
              (/ (acl2-exp n))
              (acl2-sqrt n))
           4)))
```



Step 4

```
(defthmd scaled-wallis-numerator-as-stirling-constant
  (implies (and (natp n) (< 1 n))
           (equal (scaled-wallis-numerator n)
                  (* (expt 4 n)
                     (expt (stirling-constant-n n) 4)
                     (/ (expt (factorial n) 2))))))

(defthmd scaled-wallis-denominator-as-stirling-constant
  (implies (and (integerp n) (< 1 n))
           (equal (scaled-wallis-denominator n)
                  (* (expt 4 n)
                     (expt (stirling-constant-n (* 2 n)) 2)
                     4
                     (1+ (* 2 n))
                     (/ (* 2 n))
                     (/ (expt (factorial n) 2))))))
```



Step 4

```
(defthmd scaled-wallis-numerator-as-stirling-constant
  (implies (and (natp n) (< 1 n))
           (equal (scaled-wallis-numerator n)
                  (* (expt 4 n)
                     (expt (stirling-constant-n n) 4)
                     (/ (expt (factorial n) 2))))))

(defthmd scaled-wallis-denominator-as-stirling-constant
  (implies (and (integerp n) (< 1 n))
           (equal (scaled-wallis-denominator n)
                  (* (expt 4 n)
                     (expt (stirling-constant-n (* 2 n)) 2)
                     4
                     (1+ (* 2 n))
                     (/ (* 2 n))
                     (/ (expt (factorial n) 2))))))
```

Everything almost cancels out, but the “constant” terms are problematic



Step 5

```
(defthm lemma-17
  (implies (and (posp n)
                (i-large n))
           (equal (* (standard-part (expt (stirling-constant-n n)
                                          4))
                     (standard-part (expt (stirling-constant-n (* 2 n))
                                          -2))
                     1/4)
                  (/ (acl2-pi) 2))))
```



Step 6

Now we can solve for Stirling's constant:

```
(defthmd stirling-constant-value
  (implies (and (posp n)
                (i-large n))
           (equal (stirling-constant)
                  (acl2-sqrt (* 2 (acl2-pi))))))
```



Step 7

And that leads to Stirling's approximation:

```
(defthm stirling-approximation-to-factorial
  (implies (and (natp n)
                (i-large n))
           (i-close (/ (factorial n)
                       (* (acl2-sqrt (* 2 (acl2-pi)))
                          (expt n n)
                          (/ (acl2-exp n))
                          (acl2-sqrt n)))
                    1)))
(defthm stirling-approximation-to-ln-factorial
  (implies (and (posp n)
                (i-large n))
           (i-close (acl2-ln (factorial n))
                    (+ (acl2-ln (expt n n))
                       (- n)
                       (acl2-ln (acl2-sqrt (* 2 (acl2-pi) n)))))))
```



Outline

- Why Stirling's Approximation?
- First Results
- Estimating the Error Term
- The Wallis Product Formula for π
- Finally, Stirling's Formula
- Questions?

