Proper Diameter of 2-connected Bipartite Graphs

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Abstract

A properly colored path is a path in which no two consecutive edges have the same color. A properly connected coloring of a graph is one in which there exists a properly colored path between every pair of vertices. Given a graph $G$ with a properly connected coloring $c$, the proper distance between any two vertices is the length of a shortest properly colored path between them. Furthermore, the proper diameter of $G$ is the largest proper distance between any pair of vertices in $G$ under the given coloring $c$. Since there can be many properly connected colorings of $G$, there are possibly many different values for the proper diameter of $G$. Here we explore 2-colorings and the associated proper diameter for bipartite graphs, classifying 2-connected bipartite graphs with maximum proper diameter.

1 Introduction

The work of Vizing popularized the idea of proper edge-colorings of graphs \cite{8}. Since then some researchers have shifted their focus and studied edge-colorings in which certain subgraphs, rather than the entire graph, are properly colored \cite{1}, \cite{5}, \cite{9}. In a similar fashion, we are interested in graphs which have been colored so that between every pair of vertices there exists a properly colored path, that is, a path in which no two consecutive edges have
the same color. A coloring with this property is called a properly connected coloring and a graph with such a coloring is referred to as a properly connected graph. The notion of proper connectedness debuted independently in [2] and [3]. We refer the reader to [6] for a dynamic survey on topics related to proper connectedness and to [7] for results concerning the minimum number of colors needed for a graph to have a properly connected $k$-coloring.

Recently, the notions of proper distance and proper diameter were introduced to study the range of possible lengths of properly colored paths in a properly connected graph [4]. Given a graph $G$ with a properly connected $k$-coloring $c$, the proper distance between any two vertices $u, v$ is the minimum length of a properly colored path between them and is denoted $\text{pdist}_k(u, v, c)$. This notation emphasizes that proper distance is a function not only of the given vertex pair but of the graph coloring as well. Furthermore, again fixing a properly connected $k$-coloring $c$ of a given graph $G$, the largest proper distance amongst all vertex pairs is called the proper diameter of the graph $G$ under the given coloring $c$ and is denoted as $\text{pdiam}_k(G, c)$. As before, this notation emphasizes that proper diameter is a function of both the graph and its coloring. Finally, we use $\text{pdiam}_k(G)$ to denote the maximum possible proper diameter of the graph $G$ across all properly connected $k$-colorings of $G$.

We exemplify these definitions by coloring the edges of complete bipartite graphs $K_{n,m}$. When restricted to 2 colors, except for some cases when either partition class is very small (size 1 or 2), the only possible proper diameter values of $K_{n,m}$ are 2 and 4, so $\text{pdiam}_2(K_{n,m}) = 4$ [4]. Figure 1 shows a non-properly connected 2-coloring of $K_{2,3}$ — the reader can satisfy themselves that there is no properly colored path between $u$ and $v$ in $c_1$ — as well as properly connected 2-colorings that attain proper diameter values of 2 and 4.

![Diagram](image)

(a) $c_1$ is not properly connected  
(b) $\text{pdiam}_2(K_{2,3}, c_2) = 2$  
(c) $\text{pdiam}_2(K_{2,3}, c_3) = 4$

Figure 1: $\text{pdiam}(K_{2,3}) = 4$

Here we point out a subtlety of our notation. For a given $k$, some graphs require more than $k$ colors to make the graph properly connected. For example, a properly connected coloring of a graph with connectivity 1 may require many colors, such as with trees, which require $\Delta(G)$ colors where $\Delta(G)$ is the maximum degree of $G$ [3]. Hence, when we make reference to $\text{pdiam}_k(G)$, we implicitly assume that some properly connected $k$-coloring of $G$ actually exists. However, for reasons we explain shortly, our results focus on 2-connected graphs, which require only a few colors to make them properly connected. According to [3], if a graph is 2-connected, then at most 3 colors are needed to make the graph properly connected. More specifically, non-complete 3-connected graphs require only 2 colors, as do 2-connected bipartite graphs, but there are examples of 2-connected non-bipartite graphs that require 3 colors. In summary, our main results focus on $\text{pdiam}_2(G)$ where $G$ is a 2-connected...
connected bipartite graph, so by [3] properly connected 2-colorings exist for all graphs we consider.

In order to introduce the main question of our investigation, we now discuss bounds related to proper diameter. The diameter of a given graph provides a trivial lower bound for its proper diameter, as coloring the edges of a graph cannot decrease the distances between its vertices. Likewise, no path in a graph on $n$ vertices can have length more than $n - 1$, so this yields a trivial upper bound for the proper diameter. Put simply, for a graph $G$ of order $n$ with properly connected $k$-colorings,

$$\text{diam}(G) \leq \text{pdiam}_k(G) \leq n - 1. \quad (1)$$

When $k = 2$, there exists a non-trivial upper bound that depends on $\kappa(G)$, the connectivity of $G$ [4]. In particular, for any properly connected 2-colored graph $G$ of order $n \geq 2$,

$$\text{pdiam}_2(G) \leq n - \kappa(G) + 1. \quad (2)$$

The bound above is tight and implies that as connectivity increases, the maximum proper diameter decreases, which is consistent with our intuition.

There are two major parts to this paper. First, we are interested in bipartite graphs on $n$ vertices which attain the upper bound of $n - 1$ in (1). Certainly, a Hamiltonian path within the graph is a necessary condition to obtain the upper bound, but the presence of a Hamiltonian path is not sufficient.

For example, when $m \geq 7$, a 2-colored fan graph on $m + 1$ vertices, $F_{1,m}$, can only attain a maximum proper diameter of $m - 1$ [4] (see Figure 2). As another observation, it follows from the upper bound in (2) that, when restricted to only 2 colors, any 3-connected graph has proper diameter at most $n - 2$. Hence, we see that $\text{pdiam}_2(G) = n - 1$ is possible only when $\kappa(G) \leq 2$. This fact motivates our efforts to analyze the upper bound of $n - 1$ as it applies to 2-colorings of 2-connected graphs. We do not examine when $\kappa(G) = 1$ since, as mentioned above, properly connected colorings on such graphs may require many more than 2 colors.

![Figure 2: pdiam$^2(F_{1,10}) = 9$ even though a Hamiltonian path exists](image)

Based upon this discussion, we now pose the question that drives this part of our current investigation: If a bipartite graph $G$ is 2-connected, what features must $G$ and its colorings have so that $\text{pdiam}_2(G) = n - 1$? In Section 2, we explore such features, introduce relevant terminology, and define a new family of 2-connected bipartite graphs which we call \textit{tau graphs} (see Definition 2.10), and we show that this family classifies all 2-connected bipartite graphs.
which attain a proper diameter of \( n - 1 \) with 2 colors. The main result of this section is given by Theorem 2.11.

**Theorem 2.11.** Let \( G \) be a 2-connected bipartite graph on \( n \) vertices. Then \( \text{pdiam}_2(G) = n - 1 \) if and only if \( G \) is a tau graph.

The second portion of the paper is devoted to exploring the proper diameter of specific bipartite graph families. We incorporate the ideas proved in Section 2 as a tool to help find the proper diameter for each family. Additionally, we provide a method for constructing a coloring on the graph to attain any possible proper diameter value between the minimum, given by the diameter of the uncolored graph, and the maximum, proved with the aid of the result in Section 2. We consider both ladder graphs and general grid graphs.

Finally, throughout our investigation, we use \( c(e) \) to denote the color of the edge \( e \) under a coloring \( c \), and because we concentrate on 2-colorings, our figures include dashed and solid edges to distinguish between red (color 1) and blue (color 2) edges, respectively.

### 2 Tau Graphs

Given a graph \( G \) on \( n \) vertices, clearly if \( \text{pdiam}_2(G) = n - 1 \), then for some coloring \( c \) of \( G \), there exist two vertices with proper distance \( n - 1 \). In this case, we say the proper distance between two such vertices is given by a properly colored Hamiltonian path. Hence, we start by considering the structure of 2-connected graphs with a Hamiltonian path. We develop terminology to discuss the relevant features of these graphs and observe several direct results regarding these characteristics.

**Definition 2.1.** *A link on a path* \( P = v_1v_2\ldots v_n \) *in a graph* \( G \) *is an edge* \( v_iv_j \in E(G) - E(P) \) *where* \( 1 \leq i < j \leq n \). The link \( v_iv_j \) *is said to be over* \( v_t \) *if* \( i < t < j \).

![Figure 3: Link \( v_3v_8 \) over vertices \( v_4, v_5, v_6, \) and \( v_7 \)](image)

When the proper distance between two vertices is given by a properly colored Hamiltonian path, it is a contradiction if the links on \( P \), together with \( P \), yield a shorter properly colored path between the endpoints of \( P \). This logic can be used to show that when a link creates an even cycle with the vertices of \( P \), such as \( v_3v_8 \) does in Figure 3, only one coloring of the link prevents a shorter properly colored path. Theorem 2.2 makes use of this idea.

**Theorem 2.2.** Let \( G \) be a bipartite graph on \( n \) vertices with a properly connected 2-coloring \( c \) and a Hamiltonian path \( H = v_1v_2\ldots v_n \) that is a shortest properly colored path between \( v_1 \) and \( v_n \). The color of \( v_1v_2 \) forces the color of all edges. Specific color details are given below.
• For any edge on $H$, $c(v_iv_{i+1}) = c(v_1v_2)$ if and only if $i$ is odd.
• For any link incident to $v_1$, $c(v_1v_i) = c(v_i v_{i+1})$ where $2 < i < n$.
• For any link incident to $v_n$, $c(v_nv_1) = c(v_{i-1}v_i)$ where $1 < i < n-1$.
• For any link creating an even cycle with the intermediate vertices of $H$, $c(v_iv_{i+(2h+1)}) = c(v_{i-1}v_i) = c(v_{i+(2h+1)}v_{i+(2h+2)})$ where $1 < i$, $0 < h$, and $i + 2h + 2 < n$.

**Proof.** The hypotheses force that $\text{pdist}_2(v_1, v_n, c) = \text{pdiag}_2(G, c) = n - 1$. Without loss of generality, assume $c(v_1v_2) = 1$. Since $H$ is properly colored, $c(v_1v_{i+1}) = 1$ if $i$ is odd and $c(v_1v_{i+1}) = 2$ if $i$ is even.

Consider $v_1v_i$. If $i = 2$, the color of $v_1v_2$ is given above. If $i = n$, then $\text{pdist}_2(v_1, v_n, c) = 1$, which is a contradiction. Thus, we consider only $2 < i < n$. If $c(v_1v_i) \neq c(v_iv_{i+1})$, then $v_1v_i v_{i+1} \ldots v_n$ is a properly colored path from $v_1$ to $v_n$ of length less than $n - 1$ since $v_2$ is omitted from the path. Hence, $\text{pdiag}_2(G, c) < n - 1$, which is a contradiction. Therefore, $c(v_1v_i) = c(v_iv_{i+1})$.

![Figure 4: Illustration of contradiction for $c(v_1v_i) \neq c(v_iv_{i+1})$](image)

Consider $v_iv_n$. If $i = n - 1$, the color of $v_{n-1}v_n$ is given above. If $i = 1$, then $\text{pdist}_2(v_1, v_n, c) = 1$, which is a contradiction. Thus, we now consider $1 < i < n - 1$. If $c(v_iv_n) \neq c(v_{i-1}v_i)$, then $v_1v_2 \ldots v_{i-1}v_iv_n$ is a properly colored path from $v_1$ to $v_n$ of length less than $n - 1$ since $v_{i+1}$ is omitted from the path. Hence, $\text{pdiag}_2(G, c) < n - 1$, which is a contradiction. Therefore, $c(v_iv_n) = c(v_{i-1}v_i)$.

![Figure 5: Illustration of contradiction for $c(v_1v_n) \neq c(v_{i-1}v_i)$](image)

Consider $v_iv_{i+2h+1}$. Note that $c(v_{i-1}v_i) = c(v_{i+2h+1}v_{i+2h+2})$. If $c(v_{i}v_{i+2h+1}) \neq c(v_{i-1}v_i)$, $v_1v_2 \ldots v_iv_{i+2h+1}v_{i+2h+2} \ldots v_n$ is a properly colored path from $v_1$ to $v_n$ of length less than $n - 1$ since $v_{i+1}$ is omitted. Hence, $\text{pdiag}_2(G, c) < n - 1$, which is a contradiction. Therefore, $c(v_iv_{i+2h+1}) = c(v_{i-1}v_i) = c(v_{i+2h+1}v_{i+2h+2})$.

$\square$
We include Corollary 2.3 as an interesting property for a coloring attaining maximum proper diameter on a 2-connected bipartite graph.

**Corollary 2.3.** Let $G$ be a bipartite graph with $n$ vertices. If $\text{pdiam}_2(G) = n - 1$, there are exactly 2 colorings $c$ on $G$ such that $\text{pdiam}_2(G, c) = n - 1$, and the colorings are equivalent up to relabeling.

If a bipartite graph with a Hamiltonian path is additionally 2-connected, its connectivity forces the existence of links over every vertex in the path, as we now show.

**Lemma 2.4.** Let $G$ be a 2-connected bipartite graph. If $G$ has a Hamiltonian path $H = v_1v_2 \ldots v_n$, then there exists a link incident to $v_1$, a link incident to $v_n$, and a link over every vertex $v_i$ where $1 < i < n$.

**Proof.** There must exist a link incident to $v_1$ because otherwise removing $v_2$ would disconnect the graph. Likewise, there must exist a link incident to $v_n$ because otherwise removing $v_{n-1}$ would disconnect the graph. For every index $t$ with $2 \leq t \leq n - 1$, there must exist indices $i$ and $j$ such that $i < t < j$ and $v_iv_j$ is a link over $v_t$. Otherwise $v_t$ would be a cut vertex and hence $G$ would not be 2-connected. \hfill \square

Figure 7: A 2-connected graph with link over every vertex except the ends of the path

If we refer to traversing a subpath of a path $P = v_1v_2 \ldots v_n$ so that the indices are increasing as *forward traversing* along $P$. Similarly, when the indices decrease, we say that we are *backward traversing* along $P$.

Figure 8: Forward Traversing  
Figure 9: Backward Traversing

The existence of the links given by Lemma 2.4 allows us to define a *chain* through a 2-connected bipartite graph with a Hamiltonian path. Consider a 2-connected bipartite graph...
Let $G$ be a 2-connected bipartite graph that contains a Hamiltonian path $H = v_1v_2 \ldots v_n$. Let $a_0 = 1$, $b_0 = \max \{i : v_{a_1}v_i \in E(G)\}$, and let $v_{a_\ell}v_{b_\ell}$ for $\ell \geq 1$ be the link over $v_{b_{\ell-1}}$ with $a_\ell < b_\ell$ which maximizes $b_\ell$. If there exist multiple links over $v_{b_{\ell-1}}$ with maximum $b_\ell$, then $v_{a_\ell}v_{b_\ell}$ is chosen such that $a_\ell$ is also maximized. Continue defining $a_\ell$ and $b_\ell$ until there exists $z$ such that $b_z = n$. Note that $a_\ell < b_{\ell-1} < b_\ell$. The chain $C$ is unique to $H$. Figures 10 and 11 exemplify that the chain $C$ may or may not be Hamiltonian.

**Definition 2.5.** Let $G$ be a 2-connected bipartite graph that contains a Hamiltonian path $H = v_1v_2 \ldots v_n$. Let $a_0 = 1$, $b_0 = \max \{i : v_{a_0}v_i \in E(G)\}$, and let $v_{a_\ell}v_{b_\ell}$ for $\ell \geq 1$ be the link over $v_{b_{\ell-1}}$ with $a_\ell < b_\ell$ which maximizes $b_\ell$. If there exist multiple links over $v_{b_{\ell-1}}$ with maximum $b_\ell$, then $v_{a_\ell}v_{b_\ell}$ is chosen such that $a_\ell$ is also maximized. Continue defining $a_\ell$ and $b_\ell$ until there exists $z$ such that $b_z = n$. Note that $a_\ell < b_{\ell-1} < b_\ell$. The chain $C$ with links $\{v_{a_\ell}v_{b_\ell}\}_{\ell=0}^{z}$ is the trail from $v_1$ to $v_n$ consisting of the alternating sequence of forward traversing links and backwards traversing subpaths of $H$ defined as follows: $v_{a_0}v_{b_0}v_{b_{0-1}}v_{a_{0-1}} \ldots v_{a_1+1}v_{a_1}v_{b_1}v_{b_1-1}v_{b_1-2} \ldots v_{a_z+2}v_{a_z+1}v_{b_z}$.

Figure 10: A non-Hamiltonian chain $C$ through $H$, $v_1v_4v_3v_8v_7v_6v_5v_{14}v_{13}v_{12}v_{11}v_{10}v_{17}$

Figure 11: A Hamiltonian chain $C$ through $H$, $v_1v_4v_3v_2v_9v_8v_7v_6v_5v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_{10}v_{17}$

We now consider properly connected 2-colorings of these 2-connected bipartite graphs. Specifically, we focus on when the proper diameter of such graphs is one less than the number of vertices, the upper bound of inequality (1). We observe immediately that this value of proper diameter is only attainable when the graph contains a properly colored Hamiltonian path $H$ and when no shorter properly colored path exists between the endpoints of $H$. In Theorem 2.6, we show that in this scenario, $H$ is not the only properly colored Hamiltonian path in the graph, as the chain $C$ through $H$ has these properties as well. The corollaries of Theorem 2.6 point out additional helpful properties regarding the links of $C$. 
Theorem 2.6. Let $G$ be a 2-connected bipartite graph on $n$ vertices with properly connected 2-coloring $c$. If $\text{pdim}_2(G,c) = n - 1$, i.e. the proper diameter of $G$ is given by the length of a properly colored Hamiltonian path $H = v_1 v_2 \ldots v_n$, then the chain $C$ through $H$ with links $\{v_a v_b\}_{i=0}^{n-2}$ is a second properly colored Hamiltonian path from $v_1$ to $v_n$.

Proof. There are two things we must prove. We must show that $C$ is both a path and that $C$ is properly colored.

Let us begin by proving $C$ is a path from $v_1$ to $v_n$. Recall that $C$ is an alternating sequence of forward traversing links and backwards traversing subpaths of $H$. The only way $C$ is not a path is if it is incident to the same vertex multiple times, but no edges are duplicated since $C$ is defined as a trail. The only way for a vertex to be reached twice is if the compound inequality $a_i < b_{i-1} < a_{i+1} < b_i$ does not hold. By Lemma 2.4 and Definition 2.5, the first and last inequalities must hold, so the only one remaining is $b_{i-1} < a_{i+1}$. By way of contradiction, assume there exists a well defined chain $C$ such that $b_{i-1} = a_{i+1}$ for at least one $i$ where $1 \leq i \leq z - 1$. Then $c(v_{b_{i-1}}v_{a_{i-1}}) \neq c(v_{a_{i+1}}v_{b_{i+1}})$, by Theorem 2.2. Moreover, $c(v_{a_{i+1}}v_{b_{i+1}}) = c(v_{a_{i-1}}v_{b_{i-1}})$ and $c(v_{b_{i-1}}v_{a_{i-1}}) = c(v_{a_{i+1}}v_{b_{i+1}})$, meaning $c(v_{a_{i-1}}v_{b_{i-1}}) \neq c(v_{a_{i+1}}v_{b_{i+1}})$. Then define $C^*$ to be the trail that uses $C$ except all sections of $C$ between and including $v_{a_{i-1}}$ and $v_{b_{i+1}}$ for all $1 \leq i \leq z - 1$ where $b_{i-1} = a_{i+1}$ are replaced by $v_{a_{i-1}}v_{b_{i-1}}v_{b_{i+1}}$ and then continuing along $C$. This gives that $C^*$ from $v_1$ to $v_n$ is a path since $C^*$ is not a path is if it is incident to the same vertex multiple times, but no edges are replaced.

It only remains to show that $C$ is properly colored. By way of contradiction, suppose the chain $C$ through $H$ is not properly colored. Recall that by definition, $C$ is an alternating sequence of forward traversing links and backwards traversing subpaths of $H$. Since $H$ is properly colored, for $C$ to not be a properly colored path, there must exist a link $v_{a_i} v_{b_i}$ in $C$ for which at least one of the equalities $c(v_{a_i}v_{b_i}) = c(v_{a_i}v_{a_i+1})$ and $c(v_{a_i}v_{b_i}) = c(v_{b_i-1}v_{b_i})$ holds. Since $G$ is bipartite, $c(v_{a_i}v_{a_i+1}) = c(v_{b_i-1}v_{b_i})$. Hence, for $C$ to not be properly colored, there must exist a link $v_{a_i} v_{b_i}$ in $C$ where $c(v_{a_i}v_{b_i}) = c(v_{a_i}v_{a_i+1}) = c(v_{b_i-1}v_{b_i})$. This contradicts the coloring for $v_{a_i} v_{b_i}$ required by Theorem 2.2.

\[\Box\]

Corollary 2.7. Consider a 2-connected bipartite graph $G$ on $n$ vertices with a properly connected 2-coloring $c$ and a Hamiltonian path $H = v_1 v_2 \ldots v_n$ that is a shortest properly colored path between $v_1$ and $v_n$. The chain $C$ through $H$ with links $\{v_a v_b\}_{i=0}^{n-2}$ has the following properties: $a_1 = a_0 + 1, b_z = b_{z-1} + 1$, and $a_{\ell} = b_{\ell-2} + 1$ when $2 \leq \ell \leq z$.

Proof. By Theorem 2.6, $C$ is Hamiltonian. If $a_1 \neq a_0 + 1$ or $b_z \neq b_{z-1} + 1$, then $v_{a_0+1}$ or $v_{b_{(z-1)+1}}$, respectively, is not on $C$, thus contradicting that $C$ is Hamiltonian. Similarly, for any $\ell$ where $2 \leq \ell \leq z$, if it is the case that $a_\ell \neq b_{\ell-2} + 1$, then the vertices between $a_\ell$ and $b_{\ell-2} + 1$ in $H$ are not visited by $C$, again a contradiction.

\[\Box\]
Figure 12: Hamiltonian $C$ through $H$ where $a_1 = a_0 + 1, b_z = b_{z-1} + 1$, and $a_\ell = b_{\ell-2} + 1$

Having considered many of the universal features of 2-colored bipartite graphs with proper diameter equal to one less than the number of vertices, we now introduce a new graph family to classify such graphs. We begin by giving several definitions to describe the structure of this new family, which we call tau graphs and denote by $\mathcal{T}_n$. We then prove Theorem 2.11, the main result of the section, in which we show a 2-connected bipartite graph $G$ can be 2-colored to attain a proper diameter of $n - 1$ if and only if $G$ is a tau graph.

**Definition 2.8.** Given an even cycle $u_1 u_2 \ldots u_{2m} u_1$, a band on the cycle is defined as a newly inserted path of any length between vertices $u_i$ and $u_j$ where $i < j$ and $i + j = 2m + 2$. Label a band from $u_i$ to $u_j$ as $B_i$ and its vertices $b_{i1}, b_{i2}, \ldots, b_{i\ell}$ where $b_{i1} = u_i$ and $b_{i\ell} = u_j$.

![Figure 13: Cycle $C_8$ with bands $B_2$ and $B_4$](image)

Definition 2.9. Consider an even cycle $u_1 u_2 \ldots u_{2m} u_1$. A tau skeleton, denoted $T^*_n$, is a bipartite graph on $n$ vertices that results from adding a single band between each pair of vertices $u_i, u_j$ where $i < j$ and $i + j = 2m + 2$, so that all bands have an odd number of vertices. We refer to $u_1$ and $u_{m+1}$, the two vertices not part of a band, as the ears. The collection of all $T^*_n$ for a specific $n$ will be denoted $\mathcal{T}^*_n$.

Figures 14 and 15 show examples of tau skeletons, the most basic type of tau graphs.
Definition 2.10. Let $T^*_n \in T^*_n$ be a tau skeleton on $n$ vertices with vertex labeling given by Definitions 2.8 and 2.9. A tau graph denoted by $T_n$ is constructed by adding any number of extra edges of the form $b_i x b_i y$ where $1 \leq x < y \leq \ell$ to $T^*_n$. The collection of all possible tau graphs for a specific natural number $n$ is given by $T_n$. Note that $T^*_n \subset T_n$.

Figure 16: Tau graphs on 13 vertices

Theorem 2.11. Let $G$ be a 2-connected bipartite graph on $n$ vertices. Then $pdiam_2(G) = n - 1$ if and only if $G$ is a tau graph.

Proof. ($\Leftarrow$) Let $G$ be a tau graph on $n$ vertices, labeled as in Definition 2.9. Since $pdiam_2(G) \leq n - 1$, it is only necessary to construct one 2-coloring that achieves this proper diameter value. Define the coloring $c$ by the following. Let $c(u_1 u_2) = c(u_2 u_3) = \cdots = c(u_m u_{m+1}) = 1$. Let $c(u_{m+1} u_{m+2}) = c(u_{m+2} u_{m+3}) = \cdots = c(u_{2m} u_1) = 2$. Alternate colors on the edges of each band $B_i$, starting with color 2 on $b_i b_{i+1}$ and ending with color 1 on $b_i (\ell - 1) b_i$. Color remaining edges of the form $b_i x b_i y$, $x < y$, color 1 if $x$ is odd and color 2 if $x$ is even. This coloring yields $pdiam_2(G, c) = n - 1$. The proper diameter is given by $pdist_2(u_1, u_{m+1}, c)$ (see Figure 17).

Figure 17: Tau graph on 13 vertices

($\Rightarrow$) Suppose there exists a coloring $c$ such that $pdiam_2(G, c) = n - 1$. Then there exists a properly colored Hamiltonian path $H = v_1 v_2 \ldots v_n$ in $G$ where $H$ is a shortest such path from $v_1$ to $v_n$. By Theorem 2.6, the chain $C$ through $H$ with links $\{v_{a_i} v_{b_i}\}_{i=0}^{z}$ is a second properly colored Hamiltonian path from $v_1$ to $v_n$.

Let $G'$ be the spanning subgraph $H \cup C$ of $G$. We now construct a tau skeleton that is isomorphic to $G'$. Let $T^*_n \in T^*_n$ be the tau skeleton consisting of an even cycle $u_1 u_2 \ldots u_{2m} u_1$ with $m - 1$ bands where $m = z + 1$. For $1 \leq j \leq m - 1$, the number of interior vertices on band $B_{j+1}$ corresponds to the number of interior vertices on the subpath $H_{a_j} = v_{a_j} v_{a_{j+1}} \ldots v_{b_{j-1}}$ of $H$. Since $G$ is bipartite, $G'$ must be as well, so all bands have an odd number of vertices,
thus satisfying the definition of a tau skeleton. The isomorphism \( \phi : V(G') \to V(T_n^*) \) is given as follows. Let \( \phi(v_{a0}) = u_1 \) and \( \phi(v_{b0}) = u_{2m} \). If \( i \) is odd for \( 1 \leq i \leq z \), then \( \phi(v_{ai}) = u_{i+1} \) and \( \phi(v_{bi}) = u_{i+2} \). If \( i \) is even for \( 1 \leq i \leq z \), then \( \phi(v_{ai}) = u_{2m-i+1} \) and \( \phi(v_{bi}) = u_{2m-i} \) (see Figure 18).

Figure 18: Illustration of the isomorphism with \( H \) and \( \phi(H) \) highlighted

While the mapping \( \phi \) shows that the tau skeleton \( T_n^* \) and \( G' = H \cup C \) are isomorphic, it remains to show that the original graph \( G \) is also isomorphic to some tau graph, \( T_n \). Since \( G' \) is a spanning subgraph of \( G \), both \( G \) and \( G' \) have the same vertex set but \( G \) potentially has more edges. Any edge \( v_i v_j \in E(G) - E(G') \) corresponds to an edge \( \phi(v_i)\phi(v_j) \notin E(T_n^*) \). We must show that any such edge \( v_i v_j \) maps to an edge \( \phi(v_i)\phi(v_j) \) that when added to \( T_n^* \) yields a tau graph. To show this, we partition all possible edges in \( E(G) - E(G') \).

(a) Additional edges from \( v_1 \). These correspond to additional edges from \( u_1 \) in \( T_n \).

(b) Additional edges from \( v_n \). These correspond to additional edges from \( u_{m+1} \) in \( T_n \).

(c) Edges between vertices of non-consecutive subpaths, that is, edges between \( H_{ai} \) and \( H_{a(i+j)} \) where \( j \geq 2 \). These edges correspond to edges between vertices of non-consecutive bands in \( T_n \).

(d) Edges between interior vertices of consecutive subpaths \( H_{ai} \) and \( H_{a(i+1)} \), which correspond to edges between interior vertices of consecutive bands in \( T_n \).

(e) Edges between endpoints of consecutive subpaths \( H_{ai} \) and \( H_{ai+1} \), that is, between \( a_i \) and \( a_{i+1} \) or between \( b_i \) and \( b_{i+1} \). (Note the edges \( a_i b_j \) and \( a_{i+1} b_{i+1} \) are in the chain \( C \) and so are already in \( G' \) and thus are excluded from this case.)

(f) Edges between vertices in the same subpath \( H_{ai} \). These correspond to edges between vertices on the same band.
The edges described in case (f) are allowed by the definition of a tau graph and so when added to a tau graph yield another tau graph.

We now show that any edges described in (a)-(e) violate the hypothesis that pdiam\(_2(G, c) = n - 1\). Let the edge under consideration in the following cases be denoted \(v_iv_j\).

Case 1: Here we examine the edges \(v_iv_j\) considered in (a), so suppose \(i = 1\). If \(j = n\), then pdist\(_2(v_1, v_n, c) = 1\). If \(j \neq n\), then proceed with the following. Let \(s \in \mathbb{N}\) such that \(1 \leq s \leq z\) and \(s\) is the largest integer such that \(a_s \leq j\). If \(c(v_1v_j) = c(v_jv_{j+1})\), then the proper distance from \(v_1\) to \(v_n\) is given by \(v_1v_jv_{j+1}...v_{a_s+1}v_{a_s}\) and continuing from \(v_{a_s}\) along \(C\). This path omits \(v_{a_s}\) so it has length less than \(n - 1\). If \(c(v_1v_j) = c(v_{j-1}v_j)\), then the proper distance from \(v_1\) to \(v_n\) is given by \(v_1v_jv_{j+1}...v_n\). This path omits \(v_2\) so it has length less than \(n - 1\). Therefore, adding an edge described in (a) yields pdist\(_2(v_1, v_n, c) < n - 1\), which is a contradiction. Thus, these edges can’t be added, \(i \neq 1\), and \(v_1\) has degree 2 (see Figure 19).

![Figure 19: Illustration of contradiction for \(i = 1\)](image)

Case 2: Here we examine the edges considered in (b). Suppose \(j = n\). Since \(i \neq 1\), proceed with the following. If \(c(v_iv_n) = c(v_{i-1}v_i)\), then the proper distance from \(v_1\) to \(v_n\) is given by traversing chain \(C\) until reaching \(v_i\) and then taking the edge \(v_iv_n\). Since \(i \neq a_z\), this path omits \(v_{a_z}\) so it has length less than \(n - 1\). If \(c(v_iv_n) = c(v_i+1v_i)\), then the proper distance from \(v_1\) to \(v_n\) is given by \(v_1v_2...v_{i-1}v_iv_n\). This omits \(v_{i+1}\) so it has length less than \(n - 1\). Therefore pdist\(_2(v_1, v_n, c) < n - 1\), which is a contradiction, so the edges in (b) can’t be added, \(j \neq n\), and \(v_n\) has degree 2 (see Figure 20).
Figure 20: Illustration of contradiction for $j = n$

Case 3: Here we examine the edges considered in (c) and (d). We will consolidate these cases by supposing $i \neq 1$ and $j \neq n$. If $c(v_i v_j) = c(v_i v_{i+1}) = c(v_{j-1} v_j)$, then proceed along path $H$ until reaching $v_i$, take $v_i v_j$, then continue along $H$ until $v_n$. This path omits $v_{i+1}$. If $c(v_i v_j) = c(v_{i-1} v_i) = c(v_j v_{j+1})$, then proceed along $C$ until reaching $v_i$, take $v_i v_j$, then continue along $C$ until $v_n$. This path omits the vertex immediately after $v_i$ when traversing the chain $C$ (see Figure 21). Therefore, $\text{pdist}_2(v_1, v_n, c) < n - 1$, which is a contradiction.

Figure 21: Illustration of contradiction for edges between different subpaths

Case 4: Here we examine the edges considered in (e). If $c(v_i v_j) = c(v_i v_{i+1})$, the shorter path is given by $v_1 v_2 \ldots v_i v_j v_{j+1} \ldots v_n$ and omits $v_{i+1}$. If $c(v_i v_j) = c(v_{i-1} v_i)$ the shorter path is given by traversing $C$ until reaching $v_i$, taking $v_i v_j$, then continuing along $C$ until reaching $v_n$. This path omits $v_{j+1}$. As a result, adding this edge contradicts $\text{pdist}_2(G, c) = n - 1$, so any edge in (e) cannot be added (see Figure 22).
Since all $v_iv_j$ in (a)-(e) lead to contradictions, all edges in $E(G) - E(G')$ must fit the form described in (f). Therefore, $G$ is a tau graph.

3 Bipartite Families

3.1 Ladder Graphs

**Theorem 3.1.** Given the Ladder Graph on $2n$ vertices, $L_n$, $n \leq \text{pdiam}_2(L_n, c) \leq 2n - 2$. These bounds are tight and every intermediate proper diameter value is attainable.

*Proof.* Consider the Ladder Graph on $2n$ vertices, $L_n$. A proper diameter less than $n$ is unattainable since $\text{diam}(L_n) = n \leq \text{pdiam}_2(L_n, c)$. By Theorem 2.11, Tau Graphs are the only 2-connected bipartite graphs that may have a proper diameter given by a Hamiltonian path. While ladder graphs are 2-connected and bipartite, every vertex of degree 2 in a ladder graph is adjacent to another vertex of degree 2. This is not true of the ears of Tau Graphs, so no Ladder Graph is isomorphic to a Tau Graph. Thus, $\text{pdiam}_2(L_n, c) < 2n - 1$. We’ve now proven the bounds on $\text{pdiam}_2(L_n, c)$.

![Figure 23: $L_6$ and a $T_{12}$ with the degree 2 vertices highlighted to depict no isomorphism](image)

We now work to show that all intermediate values can be attained. Let the vertices in the top path of $L_n$ be denoted $t_1, t_2, \ldots, t_n$ and let the vertices in the bottom path be labeled $b_1, b_2, \ldots, b_n$. We will now define colorings $c_0, c_1, \ldots, c_{n-2}$ where this gives $\text{pdiam}_2(L_n, c_i) = 2n - 2 - i$ for $0 \leq i \leq n - 2$. We will continue by separating into cases.

Case 1: $n$ is odd. Let $c_0$ be the following coloring on $L_n$. Let $t_1b_1$ be red and the remaining $t_mb_m$ be blue for $2 \leq m \leq n$. Let $t_ml_{m+1}$ be red for $1 \leq m \leq n - 1$. Let $b_1b_2$ be
blue and the remaining $b_mb_{m+1}$ be red for $2 \leq m \leq n - 1$. This gives $\text{pdim}_2(L_n, c_0) = 2n - 2$ because $\text{pdist}_2(t_1, t_n, c_0) = 2n - 2$ is the longest proper distance within $L_n$ (See Figure 24).

Let $c_1$ be exactly the same as $c_0$ except $t_{n-1}t_n$ becomes blue. This gives $\text{pdim}_2(L_n, c_1) = 2n - 2 - 1 = 2n - 3$ because $\text{pdist}_2(t_1, t_n, c_1) = 2n - 3$ is the longest proper distance within $L_n$ (See Figure 25).

![Figure 24: $\text{pdim}_2(L_7, c_0) = 2n - 2 = 12$](image)

![Figure 25: $\text{pdim}_2(L_7, c_1) = 2n - 3 = 11$](image)

Let $c_2$ be exactly the same as $c_1$ except $b_{n-2}b_{n-1}$ becomes blue. This gives $\text{pdim}_2(L_n, c_2) = 2n - 2 - 2 = 2n - 4$ because $\text{pdist}_2(t_1, t_n, c_2) = 2n - 4$ is the longest proper distance within $L_n$. In general, for $3 \leq i \leq n - 2$, $c_i$ will be the following. If $i$ is odd, then $c_i$ will be the same as $c_{i-1}$ except $t_{n-i}t_{n-i+1}$ becomes blue. This gives $\text{pdim}_2(L_n, c_i) = 2n - 2 - i$ because $\text{pdist}_2(t_1, t_n, c_i) = 2n - 2 - i$ is the longest proper distance within $L_n$ (See Figure 27). If $i$ is even, then $c_i$ will be the same as $c_{i-1}$ except $b_{n-i}b_{n-i+1}$ becomes blue. This gives $\text{pdim}_2(L_n, c_i) = 2n - 2 - i$ because $\text{pdist}_2(t_1, t_n, c_i) = 2n - 2 - i$ is the longest proper distance within $L_n$ (See Figure 26).

![Figure 26: $\text{pdim}_2(L_7, c_4) = 2n - 6 = 8$](image)

![Figure 27: $\text{pdim}_2(L_7, c_5) = 2n - 7 = 7$](image)

Case 2: $n$ is even. Let $c_0$ be the following coloring on $L_n$. Let $t_1b_1$ be red and the remaining $t_mb_m$ be blue for $2 \leq m \leq n$. Let $t_mb_{m+1}$ be red for $1 \leq m \leq n - 1$. Let $b_1b_2$ be blue and the remaining $b_mb_{m+1}$ be red for $2 \leq m \leq n - 1$. This gives $\text{pdim}_2(L_n, c_0) = 2n - 2$ because $\text{pdist}_2(t_1, b_n, c_0) = 2n - 2$ is the longest proper distance within $L_n$ (See Figure 28). Let $c_1$ be exactly the same as $c_0$ except $b_{n-1}b_n$ becomes blue. This gives $\text{pdim}_2(L_n, c_1) = 2n - 2 - 1 = 2n - 3$ because $\text{pdist}_2(t_1, t_n, c_1) = 2n - 3$ is the longest proper distance within $L_n$ (See Figure 29).

![Figure 28: $\text{pdim}_2(L_6, c_0) = 2n - 2 = 10$](image)

![Figure 29: $\text{pdim}_2(L_6, c_1) = 2n - 3 = 9$](image)
Let \( c_2 \) be exactly the same as \( c_1 \) except \( t_{n-2}t_{n-1} \) becomes blue. This gives \( \text{pdiam}_2(L_n, c_2) = 2n - 2 - 2 = 2n - 4 \) because \( \text{pdist}_2(t_1, b_n, c_2) = 2n - 4 \) is the longest proper distance within \( L_n \). For \( 3 \leq i \leq n - 2 \), \( c_i \) will be the following. If \( i \) is odd, then \( c_i \) will be the same as \( c_{i-1} \) except \( b_{n-i}b_{n-i+1} \) becomes blue. This gives \( \text{pdiam}_2(L_n, c_i) = 2n - 2 - i \) because \( \text{pdist}_2(t_1, t_n, c_i) = 2n - 2 - i \) is the longest proper distance within \( L_n \) (See Figure 30). If \( i \) is even, then \( c_i \) will be the same as \( c_{i-1} \) except \( t_{n-i}t_{n+i} \) becomes blue. This gives \( \text{pdiam}_2(L_n, c_i) = 2n - 2 - i \) because \( \text{pdist}_2(t_1, b_n, c_i) = 2n - 2 - i \) is the longest proper distance within \( L_n \) (See Figure 31).

![Figure 30: pdiam\(_2\)(\( L_6, c_3 \)) = 2n - 5 = 7](image)

![Figure 31: pdiam\(_2\)(\( L_6, c_4 \)) = 2n - 6 = 6](image)

### 3.2 Grids

**Lemma 3.2.** Let \( G \) be a bipartite graph, and let \( w_0 \) be a degree two vertex with two distinct paths of length 2 to vertex \( w_3 \). If \( w_3 \) has degree greater than 2, then \( w_0 \) is not isomorphic to the ear of some Tau Graph \( T_n \).

**Proof.** Let \( Q_1 = w_0w_1w_3 \) and \( Q_2 = w_0w_2w_3 \) be the two paths from \( w_0 \) to \( w_3 \) in \( G \). Assume by way of contradiction that there exists a Tau Graph \( T_n \) such that \( w_0 \) is isomorphic to an ear in \( T \) with isomorphism \( \phi : G \rightarrow T \). Use the vertex labeling on \( T \) that is outlined in Definition 2.9. Without loss of generality, let \( \phi(w_0) = u_1 \), \( \phi(w_1) = u_2 \), and \( \phi(w_2) = u_{2m} \). Since \( w_1 \) and \( w_2 \) have two distinct paths of length two between them, the corresponding band \( B_2 \) in \( T \) must be length two. Note that \( w_3 \) is adjacent to both \( w_1 \) and \( w_2 \) in \( G \), so \( \phi(w_3) = b_{22} \). This is a contradiction since \( w_3 \) has degree greater than 2, but \( b_{22} \) must have degree two, since \( B_2 \) is of length 2. Therefore, \( w_0 \) is not isomorphic to an ear of \( T_n \).

Consequently, a corner vertex of a grid cannot be mapped to an ear of a Tau graph.

![Figure 32: Illustration of the Isomorphism Contradiction](image)
Theorem 3.3. Let $G_{m,n}$ be a grid. A proper diameter of $nm - 1$ is not attainable.

Proof. By Theorem 2.11, since $G_{m,n}$ is 2-connected and bipartite, $\text{pdiam}_2(G_{m,n}) = nm - 1$ is only attainable if $G_{m,n}$ is isomorphic to a Tau Graph $T_{nm}$. We now show that $G_{m,n}$ is not isomorphic to $T_{nm}$.

Assume by way of contradiction that $G_{m,n}$ is isomorphic to $T_{nm}$ with isomorphism $\phi : G_{m,n} \rightarrow T_{nm}$. Use the vertex labeling on $T_{nm}$ that is outlined in Definition 2.9. Note that there are four vertices in $G_{m,n}$ with degree two. By Lemma 3.2, none of these vertices can be mapped by $\phi$ to an ear. Therefore, no vertex in $G_{m,n}$ can be mapped to an ear in $T_{nm}$, so $G_{m,n}$ is not isomorphic to $T_{nm}$ and a proper diameter of $nm - 1$ is not attainable.

Theorem 3.4. For some grid $G_{m,n}$, $\text{pdiam}_2(G_{m,n}) = nm - 2$ if and only if at least one of $m$ and $n$ is even.

Proof. $(\Rightarrow)$ We seek to prove the contrapositive, which says if $m$ and $n$ are both odd, then a proper diameter of $nm - 2$ is unattainable. Consider $G_{m,n}$ where $m$ and $n$ are both odd. Suppose there exists a coloring $c$ such that $\text{pdiam}(G_{m,n}, c) = nm - 2$. Therefore, there exist $s, t \in V(G_{m,n})$ such that $\text{pdist}_2(s, t, c) = nm - 2$. Let $P$ be a properly colored path between $s$ and $t$ of length $nm - 2$. So there exists vertex $z$ not on $P$. Consider graph $G' = G_{m,n} - \{z\}$. Note that $G'$ has $nm - 1$ vertices and $\text{pdiam}_2(G') = nm - 2$, so $P$ is Hamiltonian in $G'$. By Theorem 2.11, if $G'$ is 2-connected, then it is isomorphic to some tau graph, $T_{nm-1}$. First, suppose $G'$ is 2-connected. We have two cases.

Case 1: Suppose $m = n = 3$. If $z$ is the degree 4 vertex in $G_{m,n}$, then $G'$ is a cycle, which is not isomorphic to a tau graph. Additionally, $z$ cannot be a degree 3 vertex because $G'$ would not be 2-connected. If $z$ is a degree 2 vertex, then there do not exist two degree 2 vertices where neither is adjacent to another degree 2 vertex. Therefore, $G'$ is not isomorphic to a tau graph, so at least one of $n$ and $m$ is not equal to 3.

Case 2: Suppose at most one of $m$ and $n$ is equal to 3. Without loss of generality, let $m \leq n$. Place the grid on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1, 1)$, and the grid expands into the first quadrant. Label the vertices $w_{x,y}$ according to their $x$ and $y$ coordinates. Note that by Lemma 3.2, no corner vertex of a grid can be mapped to an ear in a tau graph.
Thus $z$ and $w$ and then $w$ still has two edge-disjoint paths to a vertex with degree greater than two. Thus, if $G$ is not two connected and this contradicts the prior supposition that $G'$ is two connected. Without loss of generality, assume $z = w_{1,1}$. Then $w_{1,2}$ and $w_{2,1}$ are the only degree 2 vertices in $G'$ that are not in a corner position of $G_{m,n}$. By Lemma 3.2, $w_{1,2}$ and $w_{2,1}$ are the only vertices in $G_{m,n}$ that can map to the ears of $T_{nm-1}$. This is a contradiction, because there is exactly one path of length 2 between $w_{1,2}$ and $w_{2,1}$, while the ears of a tau graph must have either 0 or 2 paths of length 2 between them. Therefore $z$ cannot be a corner vertex.

Case 2.2: Suppose $z$ is a vertex of degree three. If $z$ is adjacent to a corner vertex, then $G'$ is not two connected and this contradicts the prior supposition that $G'$ is two connected. Thus $z$ cannot be adjacent to a corner vertex. Without loss of generality, say $z = w_{q,1}$ with $3 \leq q \leq n - 2$. The proof from Lemma 3.2 still applies to all corners of $G'$, since each corner still has two edge-disjoint paths to a vertex with degree greater than two. Thus, if $G'$ is isomorphic to some tau graph $T_{nm-1}$ then the isomorphism maps both of $w_{q-1,1}$ and $w_{q+1,1}$ to ears since they are the only two remaining degree two vertices. Since $3 \leq q \leq n - 2$, this means $w_{q-2,2}$ is defined and has degree at least three (equal to three if $q = 3$ and four otherwise). Therefore $w_{q-1,1}w_{q-2,1}w_{q-2,2}$ and $w_{q-1,1}w_{q-1,2}w_{q-2,2}$ are two edge-disjoint paths of length two in $G'$ from $w_{q-1,1}$ to $w_{q-2,2}$. Since $w_{q-2,2}$ has degree greater than two, Lemma 3.2 shows $w_{q-1,1}$ cannot be mapped to an ear in $T_{nm-1}$, further contradicting that $G'$ is isomorphic to some tau graph $T_{nm-1}$. Therefore $z$ is not a vertex of degree three.

Case 2.3: Suppose $z$ is a vertex of degree 4 with distance 2 from a corner vertex. Without loss of generality, say $z = w_{2,2}$. As noted before, the corner vertices $w_{n,1}$ and $w_{n,m}$ cannot be mapped to ears in $T_{nm-1}$. If $m = 3$ then $w_{1,1}, w_{2,1}, w_{1,2}, w_{1,3},$ and $w_{2,3}$ are all adjacent to a degree 2 vertex and thus cannot be mapped to ears. If $m \neq 3$ then $w_{1,1}, w_{1,2},$ and $w_{1,2}$ are all adjacent to a degree 2 vertex and thus cannot be mapped to ears. The corner vertex $w_{1,m}$ cannot be an ear by Lemma 3.2. Therefore $z$ cannot be a vertex of degree 4 with distance 2 from a corner vertex.

Case 2.4: Suppose $z$ is an interior vertex of degree 4, with distance greater than 2 from any corner vertex. If $m > 3$ then the removal of $z$ cannot lower the degree of more than one vertex to 2. Then by Lemma 3.2, there cannot be enough vertices in $G'$ that correspond to ears in $T_{nm-1}$. If $m = 3$, then without loss of generality let $z = w_{q,2}$, $3 \leq q \leq n - 2$. Then $w_{q,1}$ and $w_{q,3}$ have degree 2 and therefore must be mapped to the ears of $T_{nm-1}$. This is a contradiction because $w_{q,1}$ and $w_{q,3}$ form a cut set, which can never be true of the ears of $T_{nm-1}$. Therefore $z$ cannot be an interior vertex of degree 4, with distance more than 2 from any corner vertex.

Thus, $z$ must be adjacent to a corner vertex in $G_{m,n}$, so $G'$ is not 2-connected. Note that
$G_{m,n}$ is bipartite and all vertices adjacent to a corner vertex are in the smaller partite set. So the smaller partite set in $G'$ has two vertices fewer than the larger partite set. So there cannot exist a Hamiltonian path in $G'$, which is a contradiction. Therefore, $\text{pdiam}_2(G_{n,m}) < nm - 2$ when $n$ and $m$ are both odd.

$(\Leftarrow)$ Place $G_{m,n}$ on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1, 1)$, and the grid expands into the first quadrant. Label the vertices $w_{x,y}$ according to their $x$ and $y$ coordinates.

Either $m$ or $n$ is even. Without loss of generality, let $m$ be even. Let $G' = G - \{w_{2,m}\}$. We will construct coloring $c'$ such that $\text{pdiam}_2(G', c') = mn - 2$ in the following way. Consider the Hamiltonian path

$$P = w_{1,m}w_{1,m-1} \ldots w_{1,1}w_{2,1} \ldots w_{n,1}w_{n,2}w_{n-1,2} \ldots w_{2,2}w_{2,3}w_{3,3} \ldots w_{n,3} \ldots w_{n,m-1}w_{n,m}w_{n-1,m} \ldots w_{4,m}w_{3,m}.$$ 

The penultimate ellipsis in this construction indicates that the path traverses all the vertices in the $i$-th row followed by moving up to the $(i + 1)$-th row until reaching the $m$-th row. Let $c'(w_{1,m}w_{1,m-1}) = 1$ and alternate between colors 1 and 2 for the remaining edges of $P$. By Theorem 2.2, the colors of $E(G') - E(P)$ are fixed (See Figure 36).

![Figure 36: Properly colored path in $G'$ with coloring $c'$](image1)

![Figure 37: Properly colored path in $G$ with coloring $c$](image2)

Let $c$ on $G$ be equal to $c'$ with the addition of $c(w_{2,m}w_{1,m}) = c(w_{2,m}w_{3,m}) = c(w_{2,m}w_{2,m-1}) = 2$. This gives $\text{pdiam}_2(G_{m,n}, c) = mn - 2$ because $\text{pdist}_2(w_{1,m},w_{3,m}, c) = mn - 2$ is the longest proper distance within $G_{m,n}$ (See Figure 37).

**Theorem 3.5.** Let $G_{m,n}$ be a grid. If $n$ and $m$ are odd, then $n + m - 2 \leq \text{pdiam}_2(G_{m,n}, c) \leq nm - 3$. These bounds are tight and every intermediate value is attainable.

**Proof.** By Theorem 3.3 and Theorem 3.4, $\text{pdiam}_2(G_{m,n}) \leq nm - 3$. Again, place $G_{m,n}$ on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1, 1)$, and the grid expands into the first quadrant. Without loss of generality, let $m \leq n$ and let the side of length $m$ extend parallel to the $y$-axis. Label the vertices $w_{x,y}$ according to their $x$ and $y$ coordinates.
To see that the upper bound is tight, define graph $G' = G - \{w_{2,m}, w_{2,m-1}\}$ and coloring $c'$ on $G'$ in the following way. Consider the Hamiltonian path

$$P = w_{1,1}w_{1,2} \ldots w_{1,n-1}w_{1,n} \ldots w_{2,1}w_{2,2}w_{2,3} \ldots w_{2,2}w_{2,3}w_{3,3} \ldots w_{n,3}w_{n,4} \ldots w_{2,m-3}w_{2,m-2} \ldots w_{n,m-2}w_{n,m-1}w_{n-1,m-1}w_{n-2,m-1}w_{n-2,m}w_{n-3,m}w_{n-3,m-1}w_{n-4,m-1} \ldots w_{3,m-1}w_{3,m}.$$ Alternate the colors of $P$ with 1 and 2, beginning with $c'(w_{1,m}w_{1,m-1}) = 1$ and continuing for the remaining edges. With this alternating path, $G'$ is properly connected. Per Theorem 2.2, for $P$ to be a shortest properly colored Hamiltonian path between its endpoints, there is only one possible coloring of the edges in $E(G') - E(P)$. Color these edges as such. (See Figure 38.)

![Figure 38: Properly colored path in $G'$ with coloring $c'$](image)

![Figure 39: Properly colored path in $G$ with coloring $c$](image)

Let $c$ on $G$ be equal to $c'$ with the addition of $c(w_{2,m-1}w_{2,m}) = c(w_{2,m-1}w_{3,m-1}) = c(w_{2,m-1}w_{2,m-2}) = c(w_{2,m-1}w_{1,m-1}) = 1$ and $c(w_{2,m}w_{1,m}) = c(w_{2,m}w_{3,m}) = 2$. This gives $\text{pdim}_2(G_{m,n}, c) = nm - 3$ because $\text{pdist}_2(w_{1,m}w_{3,m}, c) = \text{pdist}_2(c) = nm - 3$ is the longest proper distance within $G_{m,n}$ (See Figure 39).

Any intermediate proper diameter value, as well as the lower bound, can be attained with the following algorithm. We will define $c_a$ such that $\text{pdim}_2(G_{m,n}, c_a) = a$ for $n + m - 2 \leq a < nm - 3$. Let $b = a - (n + m - 2)$. For each $a$, $\text{pdim}_2(G_{m,n}, c_a)$ will be given by $\text{pdist}_2(w_{1,m}, w_{x_a}, y_a)$ where

$$y_a = \left\lfloor \frac{b}{n-1} \right\rfloor + 1, \quad x_a = \begin{cases} n - [(b - 1) \mod (n - 1)] & y_a \text{ even} \\ 2 + [(b - 1) \mod (n - 1)] & y_a \text{ odd} \end{cases}$$

if $a \leq mn - 2n + 1$, and

$$y_a = \begin{cases} m & \text{if } (nm - 3 - a) \mod (4) = 0, 3 \\ m - 1 & \text{if } (nm - 3 - a) \mod (4) = 1, 2 \end{cases}, \quad x_a = 3 + \left\lfloor \frac{nm - 3 - a}{2} \right\rfloor$$

if $a > mn - 2n + 1$. 

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Note that in the previous formulas, \( b \) counts the length of the \( w_{n,1} - w_{x_a,y_a} \) sub-path within the properly colored \( w_{1,m} - w_{x_a,y_a} \) path. In the first case, the restriction \( a \leq mn - 2n + 1 = (m - 2)n + 1 \) implies that \( y_a \leq m - 2 \). In this case, \( n - 1 \) counts the number of sub-paths of the form \( w_{n,j-1}w_{n,j}w_{n-1,j} \ldots w_{3,j}w_{2,j} \) or \( w_{2,j-1}w_{2,j}w_{3,j}w_{4,j} \ldots w_{n,j} \). So \( \left\lceil \frac{b}{n-1} \right\rceil \) is equivalent to the number of edges in the path of the form \( w_{i,j-1}w_{i,j}, 2 \leq j \leq y_a \), and \( y_a = \left\lceil \frac{b}{n-1} \right\rceil + 1 \). To see how to obtain \( x_a \), first note that \( b - 1 \) counts the length of the path \( w_{n,2}w_{n-1,2} \ldots w_{x_a,y_a} \), so \( n - 1 \) gives the number of sub-paths of the form \( w_{n,j}w_{n-1,j} \ldots w_{3,j}w_{2,j+1} \) or \( w_{2,j}w_{3,j} \ldots w_{n,j}w_{n,j+1} \). Thus, when \( y_a \) is even, \((b-1) \mod (n-1)\) counts the length of the path \( w_{n,y_a}w_{n-1,y_a} \ldots w_{x_a,y_a} \), so \( x_a = n - [(b-1) \mod (n-1)] \). When \( y_a \) is odd, \((b-1) \mod (n-1)\) counts the length of the path \( w_{2,y_a}w_{3,y_a} \ldots w_{x_a,y_a} \). Hence, \( x_a = 2 + [(b-1) \mod (n-1)] \).

In the second case, \( a > mn - 2n + 1 \) implies \( y_a \geq m - 1 \). Here, the first step in obtaining \( x_a \) is to count \( w_{2,m}, w_{2,m-1}, \) and the number of vertices in the \( w_{1,m} - w_{x_a,y_a} \) path of proper diameter \( a \), and subtract that quantity from the total number of vertices in the grid. This gives \( nm - 3 - a \), the number of vertices in the path \( w_{x_{a+1}y_{a+1}}w_{x_{a+2}y_{a+2}} \ldots w_{x_{nm-3}y_{nm-3}} \). Then \( \left\lceil \frac{nm-3-a}{2} \right\rceil = x_a - x_{nm-3} \). Since \( x_{nm-3} = 3 \), we have \( x_a = 3 + \left\lceil \frac{nm-3-a}{2} \right\rceil \). Additionally, since \( nm - 3 - a \) counts the length of path \( w_{x_{nm-3}y_{nm-3}}w_{x_{nm-2}y_{nm-2}} \ldots w_{x_{a+1}y_{a+2}}w_{x_{a+1}y_{a+1}}w_{x_{a}y_{a}} \) and \( y_{nm-3} = m, y_a = m - 1 \) when \((nm - 3 - a) \mod (4) \in \{1,2\} \) and \( y_a = m \) otherwise.

![Figure 40: Illustration of \( w_{x_a,y_a} \)](image)

Use the following process to form coloring \( c_a \) from \( c \) for each \( n + m - 2 \leq a \leq nm - 4 \).

1. Let \( c_a(w_{2,m}w_{3,m}) = 1 \).

2. If \( a = nm - 4 \), \( c_a(w_{2,m-1}w_{2,m}) = 1 \). If \( a < nm - 4 \), \( c_a(w_{2,m-1}w_{2,m}) = 2 \) and \( w_{x_a,y_a}w_{x_{a+1},y_{a+1}} \) changes color.

3. If \( a < \left\lceil \frac{nm}{2} \right\rceil - 1 \), change coloring such that \( w_{1,m}w_{2,m} \ldots w_{n,m} \) is a path of alternating color, beginning with \( c_a(w_{1,m}w_{2,m}) = 2 \) and ending with \( c_a(w_{n-1,m}w_{n,m}) = 1 \). Additionally, change colors as needed such that \( c_a(w_{i,j}w_{i,j+1}) = 1 \) if \( j \) is odd and \( c_a(w_{i,j}w_{i,j+1}) = 2 \) if \( j \) is even, for all \( 2 \leq i \leq n \) and \( y_a < j < m \).

4. If \( a = n + m - 2 \), change coloring such that \( c_a(w_{n,2}w_{n,3}) = 2 \).
Let \( C \) be a cycle in \( G_{m,n} \) containing path \( P \) and path \( w_{1,m}w_{2,m}w_{3,m} \). Let \( C' \) denote the path \( C \setminus \{w_{x,a,y},w_{x,a+1,y+1}\} \). In this way, \( C' \) is a properly colored path in \( G_{m,n} \) containing \( V(G_{m,n}) \setminus \{w_{2,m-2}\} \). The above adaptations from coloring \( c \) to form \( c_a \) ensures that \( (G_{m,n}, c_a) \) is properly connected. Furthermore, similar to the coloring \( c \), the proper distance between \( w_{1,m} \) and \( w_{x,a,y} \) is \( a \). It remains to show that no longer proper distance exists in \( G_{m,n} \) with coloring \( c_a \). Clearly no two vertices of the segment of \( C' \) between \( w_{1,m} \) and \( w_{x,a,y} \) have proper distance exceeding \( a \). Additionally, step 3 above ensures that no two vertices outside the segment of \( C' \) between \( w_{1,m} \) and \( w_{x,a,y} \) have proper distance exceeding \( a \), noting that if \( a \geq \left\lceil \frac{nm}{2} \right\rceil - 1 \), then \( C' \) gives a properly colored path of length at most \( a \) (even if one of the two vertices considered is \( w_{2,m-1} \)). It only remains to consider a vertex of the segment of \( C' \) between \( w_{1,m} \) and \( w_{x,a,y} \) and another vertex off this path. The properly colored path between two such vertices can be constructed by traversing horizontal segments opposite the direction implied by path \( P \) and vertical segments of \( G_{m,n} \). A simple check of the coloring construction shows that paths of this form exist.

Therefore, each \( c_a \) gives \( \text{pdist}_2(G_{m,n}, c_a) = a \) for all \( a \in [n + m - 2 \leq a \leq nm - 2 \) \], showing that, in addition to the upper bound, the lower bound and all intermediate proper diameter values are attainable.

**Theorem 3.6.** Let \( G_{m,n} \) be a grid. If either \( n \) or \( m \) is even, then \( n + m - 2 \leq \text{pdist}_2(G_{m,n}, c) \leq nm - 2 \). These bounds are tight and every intermediate value is attainable.

**Proof.** Without loss of generality, let \( m \) be even. Let \( a \) be the desired proper diameter, with \( n + m - 2 \leq a \leq nm - 2 \). Place \( G_{m,n} \) on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at \((1, 1)\), and the grid expands into the first quadrant. Let the side of length \( m \) extend parallel to the \( y \)-axis. Label the vertices \( w_{x,y} \) according to their \( x \) and \( y \) coordinates.

We will define \( c_a \) such that \( \text{pdist}_2(G_{m,n}, c_a) = a \) for \( n + m - 2 \leq a \leq nm - 2 \). Let \( b = a - (n + m - 2) \). For each \( a \), \( \text{pdist}_2(G_{m,n}, c_a) \) will be given by \( \text{pdist}_2(w_{1,m}, w_{x,a,y}) \), where

\[
y_a = \left\lceil \frac{b}{n - 1} \right\rceil + 1, \quad x_a = \begin{cases} n - (b - 1) \mod (n - 1) & \text{if } y_a \text{ even} \\ 2 + (b - 1) \mod (n - 1) & \text{if } y_a \text{ odd} \end{cases}
\]

Note that these formulas are the same as those in the first case of Theorem 3.5 and in this case they apply to the entire grid.

![Illustration of \( w_{x,a,y} \)](image-url)
Let $c_{nm-2}$ be identical to the coloring of the grid in the backward direction of the proof in Theorem 3.4. Define $c_a$ to be identical to $c_{nm-2}$ except for the following.

If $y_a = m$, switch the colors of the edges $w_{i,y_a-1}w_{i,y_a}$ for $3 \leq i \leq x_a - 1$. Otherwise, follow these steps.

1. If $y_a$ is even, switch the color of edges $w_{i,y_a-1}w_{i,y_a}$ for $2 \leq i \leq x_a - 1$.
   If $y_a$ is odd, switch the color of edges $w_{i,y_a-1}w_{i,y_a}$ for $x_a + 1 \leq i \leq n$.

2. For $y > y_a$, preserve the coloring of $w_{1,y}w_{2,y}$ and alternate the colors along the path $w_{1,y}w_{2,y} \ldots w_{n,y}$.

3. For $x > 1$, color $w_{x,m}w_{x,m-1}$ such that $c_a(w_{x,m}w_{x,m-1}) \neq c_a(w_{x-1,m}w_{x,m})$ and then alternate colors along the vertical path $w_{x,m}w_{x,m-1} \ldots w_{x,y_a+1}$.

4. For $x > 1$, color each edge $w_{x,y_a+1}w_{x,y_a}$ the same color as $w_{x-1,y_a+1}w_{x,y_a+1}$. However, if $y_a \leq 2$, then color $w_{n,y_a+1}w_{n,y_a}$ the opposite color as $w_{n-1,y_a+1}w_{n,y_a+1}$.

Similar to the description provided after the construction for $c_a$ in Theorem 3.5, this yields $\text{pdiam}_2(G_{m,n}, c_a) = a$ for all $a \in [n+m-2, nm-2]$, so the bounds are tight and all intermediate proper diameter values are attainable.

4 Conclusion

The work presented here is strictly related to bipartite graphs. One clear question asks about extending the results to general 2-connected graphs. The following definitions extend the notion of Tau graphs in this paper and motivate a conjecture about general 2-connected graphs on $n$ vertices with proper diameter $n - 1$.

**Definition 4.1.** Consider a tau skeleton $S$ on a total of $n$ vertices consisting of an even cycle $u_1 u_2 \ldots u_{2m} u_1$ and bands all having an odd number of vertices (called an odd skeleton) or an even number of vertices (called an even skeleton) labeled as in Definition 2.8. A tau graph $T_n$ is a family of graphs on $n$ vertices that results from adding any or none of the following edges to $S$.

- Edges of the form $b_i b_y$ where $1 \leq x < y \leq \ell$ may be added on any band $B_i$.
- In an even skeleton, edges of the form $u_i u_{2m-(i+1)}$ for $1 \leq i \leq m-2$ and $u_i u_{2m-(i-1)}$ for $2 \leq i \leq m-1$ may be added to $S$, and we call such an edge a lace.

We call a tau graph with an odd skeleton an **odd tau graph** or an **odd $T_n$**. Similarly, a tau graph with an even skeleton is an **even tau graph** or an **even $T_n$**.
Conjecture 4.2. A 2-connected graph on $n$ vertices has proper diameter $n - 1$ iff $G$ is a tau graph as defined in Definition 4.1.

Between the work of this paper and the above conjecture, the authors consider all 2-connected graphs which attain the upper bound given by the result of Coll et al. [4]. We ask the same question of graphs on $n$ vertices which are $k$-connected for $3 \leq k \leq n - 2$. Given a graph $k$-connected graph $G$ which attains the maximum proper diameter given by the result of Coll et al. [4], is the subgraph of $G$ induced by the path that realizes the proper diameter isomorphic to some tau graph presented earlier?

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