Proper Diameter of 2-connected Bipartite Graphs

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Abstract

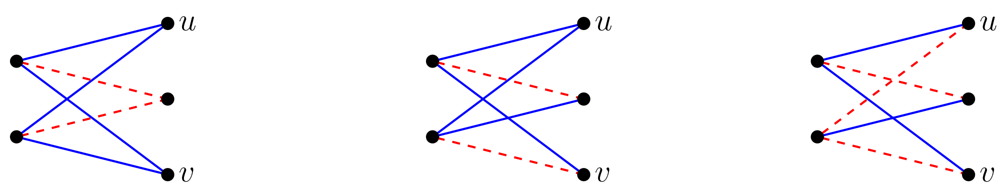
A properly colored path is a path in which no two consecutive edges have the same color. A properly connected coloring of a graph is one in which there exists a properly colored path between every pair of vertices. Given a graph G with a properly connected coloring c, the proper distance between any two vertices is the length of a shortest properly colored path between them. Furthermore, the proper diameter of G is the largest proper distance between any pair of vertices in G under the given coloring c. Since there can be many properly connected colorings of G, there are possibly many different values for the proper diameter of G. Here we explore 2-colorings and the associated proper diameter for bipartite graphs, classifying 2-connected bipartite graphs with maximum proper diameter.

1 Introduction

The work of Vizing popularized the idea of proper edge-colorings of graphs [8]. Since then some researchers have shifted their focus and studied edge-colorings in which certain subgraphs, rather than the entire graph, are properly colored [1], [5], [9]. In a similar fashion, we are interested in graphs which have been colored so that between every pair of vertices there exists a *properly colored* path, that is, a path in which no two consecutive edges have the same color. A coloring with this property is called a *properly connected coloring* and a graph with such a coloring is referred to as a *properly connected graph*. The notion of proper connectedness debuted independently in [2] and [3]. We refer the reader to [6] for a dynamic survey on topics related to proper connectedness and to [7] for results concerning the minimum number of colors needed for a graph to have a properly connected k-coloring.

Recently, the notions of proper distance and proper diameter were introduced to study the range of possible lengths of properly colored paths in a properly connected graph [4]. Given a graph G with a properly connected k-coloring c, the *proper distance* between any two vertices u, v is the minimum length of a properly colored path between them and is denoted $\mathrm{pdist}_k(u, v, c)$. This notation emphasizes that proper distance is a function not only of the given vertex pair but of the graph coloring as well. Furthermore, again fixing a properly connected k-coloring c of a given graph G, the largest proper distance amongst all vertex pairs is called the *proper diameter* of the graph G under the given coloring c and is denoted as $\mathrm{pdiam}_k(G, c)$. As before, this notation emphasizes that proper diameter is a function of both the graph and its coloring. Finally, we use $\mathrm{pdiam}_k(G)$ to denote the maximum possible proper diameter of the graph G across *all* properly connected k-colorings of G.

We exemplify these definitions by coloring the edges of complete bipartite graphs $K_{n,m}$. When restricted to 2 colors, except for some cases when either partition class is very small (size 1 or 2), the only possible proper diameter values of $K_{n,m}$ are 2 and 4, so $\mathrm{pdiam}_2(K_{n,m}) = 4$ [4]. Figure 1 shows a non-properly connected 2-coloring of $K_{2,3}$ — the reader can satisfy themselves that there is no properly colored path between u and v in c_1 — as well as properly connected 2-colorings that attain proper diameter values of 2 and 4.



(a) c_1 is not properly connected (b) $\mathrm{pdiam}_2(K_{2,3}, c_2) = 2$ (c) $\mathrm{pdiam}_2(K_{2,3}, c_3) = 4$

Figure 1: $\mathrm{pdiam}(K_{2,3}) = 4$

Here we point out a subtlety of our notation. For a given k, some graphs require more than k colors to make the graph properly connected. For example, a properly connected coloring of a graph with connectivity 1 may require many colors, such as with trees, which require $\Delta(G)$ colors where $\Delta(G)$ is the maximum degree of G [3]. Hence, when we make reference to $\mathrm{pdiam}_k(G)$, we implicitly assume that some properly connected k-coloring of G actually exists. However, for reasons we explain shortly, our results focus on 2-connected graphs, which require only a few colors to make them properly connected. According to [3], if a graph is 2-connected, then at most 3 colors are needed to make the graph properly connected. More specifically, non-complete 3-connected graphs require only 2 colors, as do 2-connected bipartite graphs, but there are examples of 2-connected non-bipartite graphs that require 3 colors. In summary, our main results focus on $\mathrm{pdiam}_2(G)$ where G is a 2-

connected bipartite graph, so by [3] properly connected 2-colorings exist for all graphs we consider.

In order to introduce the main question of our investigation, we now discuss bounds related to proper diameter. The diameter of a given graph provides a trivial lower bound for its proper diameter, as coloring the edges of a graph cannot decrease the distances between its vertices. Likewise, no path in a graph on n vertices can have length more than $n-1$, so this yields a trivial upper bound for the proper diameter. Put simply, for a graph G of order n with properly connected k-colorings,

$$\operatorname{diam}(G) \leq \operatorname{pdiam}_k(G) \leq n-1. \tag{1}$$

When $k = 2$, there exists a non-trivial upper bound that depends on $\kappa(G)$, the connectivity of G [4]. In particular, for any properly connected 2-colored graph G of order $n \geq 2$,

$$\operatorname{pdiam}_2(G) \leq n - \kappa(G) + 1. \tag{2}$$

The bound above is tight and implies that as connectivity increases, the maximum proper diameter decreases, which is consistent with our intuition.

There are two major parts to this paper. First, we are interested in bipartite graphs on n vertices which attain the upper bound of $n-1$ in (1). Certainly, a Hamiltonian path within the graph is a necessary condition to obtain the upper bound, but the presence of a Hamiltonian path is not sufficient.

For example, when $m \geq 7$, a 2-colored fan graph on $m+1$ vertices, $F_{1,m}$, can only attain a maximum proper diameter of $m-1$ [4] (see Figure 2). As another observation, it follows from the upper bound in (2) that, when restricted to only 2 colors, any 3-connected graph has proper diameter at most $n-2$. Hence, we see that $\operatorname{pdiam}_2(G) = n-1$ is possible only when $\kappa(G) \leq 2$. This fact motivates our efforts to analyze the upper bound of $n-1$ as it applies to 2-colorings of 2-connected graphs. We do not examine when $\kappa(G) = 1$ since, as mentioned above, properly connected colorings on such graphs may require many more than 2 colors.

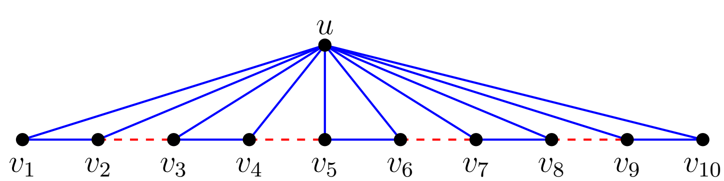


Figure 2: $\operatorname{pdiam}_2(F_{1,10}) = 9$ even though a Hamiltonian path exists

Based upon this discussion, we now pose the question that drives this part of our current investigation: If a bipartite graph G is 2-connected, what features must G and its colorings have so that $\operatorname{pdiam}_2(G) = n-1$? In Section 2, we explore such features, introduce relevant terminology, and define a new family of 2-connected bipartite graphs which we call *tau graphs* (see Definition 2.10), and we show that this family classifies all 2-connected bipartite graphs which attain a proper diameter of $n-1$ with 2 colors. The main result of this section is given by Theorem 2.11.

Theorem 2.11. *Let G be a 2-connected bipartite graph on n vertices. Then* $\mathrm{pdiam}_2(G) = n-1$ *if and only if G is a tau graph.*

The second portion of the paper is devoted to exploring the proper diameter of specific bipartite graph families. We incorporate the ideas proved in Section 2 as a tool to help find the proper diameter for each family. Additionally, we provide a method for constructing a coloring on the graph to attain any possible proper diameter value between the minimum, given by the diameter of the uncolored graph, and the maximum, proved with the aid of the result in Section 2. We consider both ladder graphs and general grid graphs.

Finally, throughout our investigation, we use $c(e)$ to denote the color of the edge e under a coloring c, and because we concentrate on 2-colorings, our figures include dashed and solid edges to distinguish between red (color 1) and blue (color 2) edges, respectively.

2 Tau Graphs

Given a graph G on n vertices, clearly if $\mathrm{pdiam}_2(G) = n-1$, then for some coloring c of G, there exist two vertices with proper distance $n-1$. In this case, we say the proper distance between two such vertices is given by a properly colored Hamiltonian path. Hence, we start by considering the structure of 2-connected graphs with a Hamiltonian path. We develop terminology to discuss the relevant features of these graphs and observe several direct results regarding these characteristics.

Definition 2.1. *A* ***link on a path*** $P = v_1v_2\dots v_n$ *in a graph G is an edge $v_iv_j \in E(G) - E(P)$ where $1 \le i < j \le n$. The link v_iv_j is said to be* ***over vertex*** v_t *if $i < t < j$.*

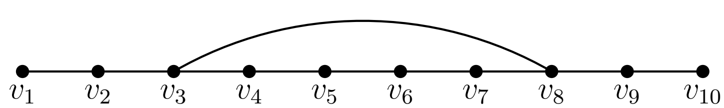


Figure 3: Link v_3v_8 over vertices v_4, v_5, v_6, and v_7

When the proper distance between two vertices is given by a properly colored Hamiltonian path, it is a contradiction if the links on P, together with P, yield a shorter properly colored path between the endpoints of P. This logic can be used to show that when a link creates an even cycle with the vertices of P, such as v_3v_8 does in Figure 3, only one coloring of the link prevents a shorter properly colored path. Theorem 2.2 makes use of this idea.

Theorem 2.2. *Let G be a bipartite graph on n vertices with a properly connected 2-coloring c and a Hamiltonian path $H = v_1v_2\dots v_n$ that is a shortest properly colored path between v_1 and v_n. The color of v_1v_2 forces the color of all edges. Specific color details are given below.*

- *For any edge on H, $c(v_iv_{i+1}) = c(v_1v_2)$ if and only if i is odd.*
- *For any link incident to v_1, $c(v_1v_i) = c(v_iv_{i+1})$ where $2 < i < n$.*
- *For any link incident to v_n, $c(v_iv_n) = c(v_{i-1}v_i)$ where $1 < i < n-1$.*
- *For any link creating an even cycle with the intermediate vertices of H, $c(v_iv_{i+(2h+1)}) = c(v_{i-1}v_i) = c(v_{i+(2h+1)}v_{i+(2h+2)})$ where $1 < i$, $0 < h$, and $i+2h+2 < n$*

Proof. The hypotheses force that $\text{pdist}_2(v_1, v_n, c) = \text{pdiam}_2(G, c) = n-1$. Without loss of generality, assume $c(v_1v_2) = 1$. Since H is properly colored, $c(v_iv_{i+1}) = 1$ if i is odd and $c(v_iv_{i+1}) = 2$ if i is even.

Consider v_1v_i. If $i = 2$, the color of v_1v_2 is given above. If $i = n$, then $\text{pdist}_2(v_1, v_n, c) = 1$, which is a contradiction. Thus, we consider only $2 < i < n$. If $c(v_1v_i) \neq c(v_iv_{i+1})$, then $v_1v_iv_{i+1}\dots v_n$ is a properly colored path from v_1 to v_n of length less than $n-1$ since v_2 is omitted from the path. Hence, $\text{pdiam}_2(G, c) < n-1$, which is a contradiction. Therefore, $c(v_1v_i) = c(v_iv_{i+1})$.

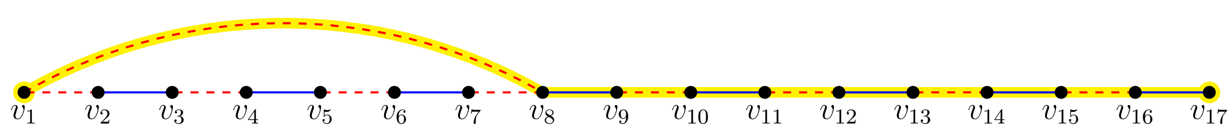


Figure 4: Illustration of contradiction for $c(v_1v_i) \neq c(v_iv_{i+1})$

Consider v_iv_n. If $i = n-1$, the color of $v_{n-1}v_n$ is given above. If $i = 1$, then $\text{pdist}_2(v_1, v_n, c) = 1$, which is a contradiction. Thus, we now consider $1 < i < n-1$. If $c(v_iv_n) \neq c(v_{i-1}v_i)$, then $v_1v_2\dots v_{i-1}v_iv_n$ is a properly colored path from v_1 to v_n of length less than $n-1$ since v_{i+1} is omitted from the path. Hence, $\text{pdiam}_2(G, c) < n-1$, which is a contradiction. Therefore, $c(v_iv_n) = c(v_{i-1}v_i)$.

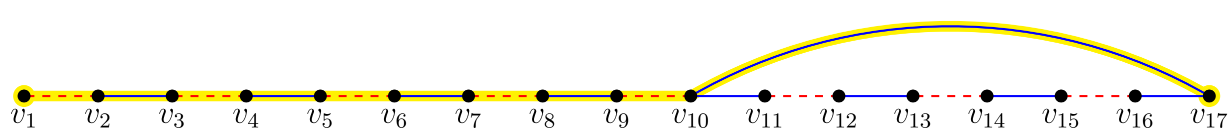


Figure 5: Illustration of contradiction for $c(v_iv_n) \neq c(v_{i-1}v_i)$

Consider v_iv_{i+2h+1}. Note that $c(v_{i-1}v_i) = c(v_{i+2h+1}v_{i+2h+2})$. If $c(v_iv_{i+2h+1}) \neq c(v_{i-1}v_i)$, $v_1v_2\dots v_iv_{i+2h+1}v_{i+2h+2}\dots v_n$ is a properly colored path from v_1 to v_n of length less than $n-1$ since v_{i+1} is omitted. Hence, $\text{pdiam}_2(G, c) < n-1$, which is a contradiction. Therefore, $c(v_iv_{i+2h+1}) = c(v_{i-1}v_i) = c(v_{i+2h+1}v_{i+2h+2})$. □

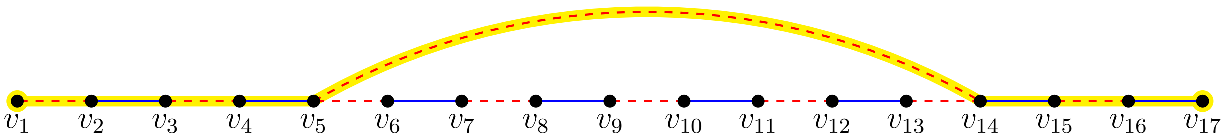


Figure 6: Illustration of contradiction for $c(v_i v_{i+2h+1}) \neq c(v_{i-1}v_i)$

We include Corollary 2.3 as an interesting property for a coloring attaining maximum proper diameter on a 2-connected bipartite graph.

Corollary 2.3. *Let G be a bipartite graph with n vertices. If $\mathrm{pdiam}_2(G) = n-1$, there are exactly 2 colorings c on G such that $\mathrm{pdiam}_2(G, c) = n-1$, and the colorings are equivalent up to relabeling.*

If a bipartite graph with a Hamiltonian path is additionally 2-connected, its connectivity forces the existence of links over every vertex in the path, as we now show.

Lemma 2.4. *Let G be a 2-connected bipartite graph. If G has a Hamiltonian path $H = v_1v_2\dots v_n$, then there exists a link incident to v_1, a link incident to v_n, and a link over every vertex v_i where $1 < i < n$.*

Proof. There must exist a link incident to v_1 because otherwise removing v_2 would disconnect the graph Likewise, there must exist a link incident to v_n because otherwise removing v_{n-1} would disconnect the graph. For every index t with $2 \leq t \leq n-1$, there must exist indices i and j such that $i < t < j$ and v_iv_j is a link over v_t. Otherwise v_t would be a cut vertex and hence G would not be 2-connected. □

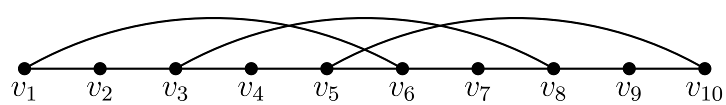


Figure 7: A 2-connected graph with link over every vertex except the ends of the path

We refer to traversing a subpath of a path $P = v_1v_2\dots v_n$ so that the indices are increasing as *forward traversing* along P. Similarly, when the indices decrease, we say that we are *backward traversing* along P.

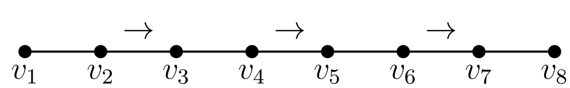


Figure 8: Forward Traversing

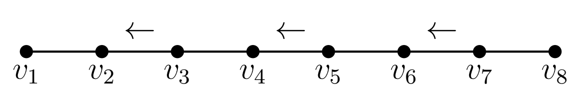


Figure 9: Backward Traversing

The existence of the links given by Lemma 2.4 allows us to define a *chain* through a 2-connected bipartite graph with a Hamiltonian path. Consider a 2-connected bipartite graph

G with a Hamiltonian path H and links on H. We will see in the next definition that the chain C through H is a walk from v_1 to v_n in G that alternates between traversing forward on a link and traversing backwards to the next link on subpaths of H. Theorem 2.6 shows the chain C through H is actually a path. Because the chain arises from a unique subset of links, the chain C is unique to H. Figures 10 and 11 exemplify that the chain C may or may not be Hamiltonian.

Definition 2.5. *Let G be a 2-connected bipartite graph that contains a Hamiltonian path $H = v_1v_2 \dots v_n$. Let $a_0 = 1$, $b_0 = \max\{i : v_{a_0}v_i \in E(G)\}$, and let $v_{a_\ell}v_{b_\ell}$ for $\ell \geq 1$ be the link over $v_{b_{(\ell-1)}}$ with $a_\ell < b_\ell$ which maximizes b_ℓ. If there exist multiple links over $v_{b_{(\ell-1)}}$ with maximum b_ℓ, then $v_{a_\ell}v_{b_\ell}$ is chosen such that a_ℓ is also maximized. Continue defining a_ℓ and b_ℓ until there exists z such that $b_z = n$. Note that $a_\ell < b_{\ell-1} < b_\ell$. The* ***chain*** *$C$* ***through*** *$H$* ***with links*** *$\{v_{a_i}v_{b_i}\}_{i=0}^{i=z}$ is the trail from v_1 to v_n consisting of the alternating sequence of forward traversing links and backwards traversing subpaths of H defined as follows: $v_{a_0}v_{b_0}v_{b_0-1}v_{b_0-2} \dots v_{a_1+1}v_{a_1}v_{b_1}v_{b_1-1}v_{b_1-2} \dots v_{a_z+2}v_{a_z+1}v_{a_z}v_{b_z}$.*

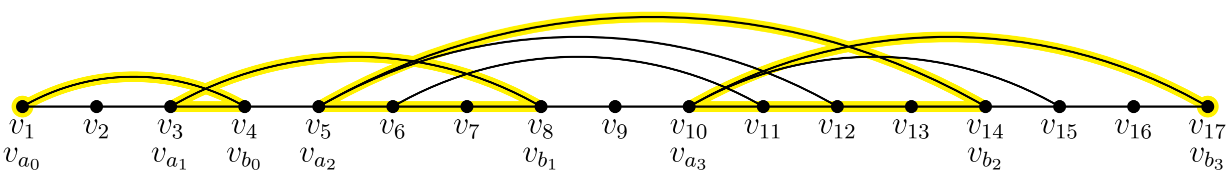


Figure 10: A non-Hamiltonian chain C through H, $v_1v_4v_3v_8v_7v_6v_5v_{14}v_{13}v_{12}v_{11}v_{10}v_{17}$

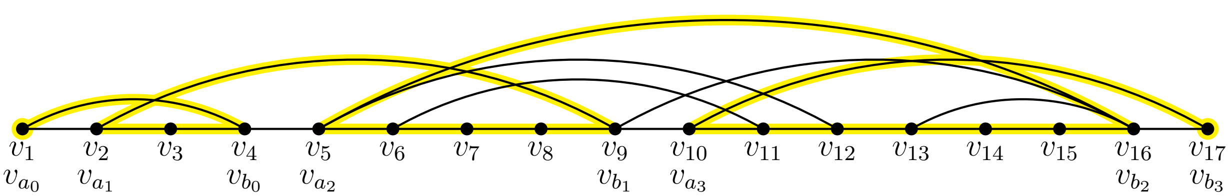


Figure 11: A Hamiltonian chain C through H, $v_1v_4v_3v_2v_9v_8v_7v_6v_5v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_{10}v_{17}$

We now consider properly connected 2-colorings of these 2-connected bipartite graphs. Specifically, we focus on when the proper diameter of such graphs is one less than the number of vertices, the upper bound of inequality (1). We observe immediately that this value of proper diameter is only attainable when the graph contains a properly colored Hamiltonian path H and when no shorter properly colored path exists between the endpoints of H. In Theorem 2.6, we show that in this scenario, H is not the only properly colored Hamiltonian path in the graph, as the chain C through H has these properties as well. The corollaries of Theorem 2.6 point out additional helpful properties regarding the links of C.

Theorem 2.6. *Let G be a 2-connected bipartite graph on n vertices with properly connected 2-coloring c. If $\mathrm{pdiam}_2(G, c) = n - 1$, i.e. the proper diameter of G is given by the length of a properly colored Hamiltonian path $H = v_1 v_2 \ldots v_n$, then the chain C through H with links $\{v_{a_i} v_{b_i}\}_{i=0}^{i=z}$ is a second properly colored Hamiltonian path from v_1 to v_n.*

Proof. There are two things we must prove. We must show that C is both a path and that C is properly colored.

Let us begin by proving C is a path from v_1 to v_n. Recall that C is an alternating sequence of forward traversing links and backwards traversing subpaths of H. The only way C is not a path is if it is incident to the same vertex multiple times, but no edges are duplicated since C is defined as a trail. The only way for a vertex to be reached twice is if the compound inequality $a_i < b_{i-1} < a_{i+1} < b_i$ does not hold. By Lemma 2.4 and Definition 2.5, the first and last inequalities must hold, so the only one remaining is $b_{i-1} < a_{i+1}$. By way of contradiction, assume there exists a well defined chain C such that $b_{i-1} = a_{i+1}$ for at least one i where $1 \leq i \leq z - 1$. Then $c(v_{b_{i-1}-1} v_{b_{i-1}}) \neq c(v_{a_{i+1}} v_{a_{i+1}+1})$, by Theorem 2.2. Moreover, $c(v_{a_{i+1}} v_{a_{i+1}+1}) = c(v_{a_{i-1}} v_{b_{i-1}})$ and $c(v_{b_{i-1}-1} v_{b_{i-1}}) = c(v_{a_{i+1}} v_{b_{i+1}})$, meaning $c(v_{a_{i-1}} v_{b_{i-1}}) \neq c(v_{a_{i+1}} v_{b_{i+1}})$. Then define C^* to be the trail that uses C except all sections of C between and including $v_{a_{i-1}}$ and $v_{b_{i+1}}$ for all $1 \leq i \leq z - 1$ where $b_{i-1} = a_{i+1}$ are replaced by $v_{a_{i-1}} v_{b_{i-1}} v_{b_{i+1}}$ and then continuing along C. This gives that C^* from v_1 to v_n is a path since C^* omits the cycles in C, meaning no vertex is repeated. Theorem 2.2 gives that C^* is properly colored. Therefore, since C^* is a properly colored path that omits $v_{b_{i-1}-1}$ for the first i where $b_{i-1} = a_{i+1}$, this contradicts that G has proper diameter $n - 1$. Thus, the compound inequality $a_i < b_{i-1} < a_{i+1} < b_i$ must hold and C is a path.

It only remains to show that C is properly colored. By way of contradiction, suppose the chain C through H is not properly colored. Recall that by definition, C is an alternating sequence of forward traversing links and backwards traversing subpaths of H. Since H is properly colored, for C to not be a properly colored path, there must exist a link $v_{a_\ell} v_{b_\ell}$ in C for which at least one of the equalities $c(v_{a_\ell} v_{b_\ell}) = c(v_{a_\ell} v_{a_\ell+1})$ and $c(v_{a_\ell} v_{b_\ell}) = c(v_{b_\ell-1} v_{b_\ell})$ holds. Since G is bipartite, $c(v_{a_\ell} v_{a_\ell+1}) = c(v_{b_\ell-1} v_{b_\ell})$. Hence, for C to not be properly colored, there must exist a link $v_{a_\ell} v_{b_\ell}$ in C where $c(v_{a_\ell} v_{b_\ell}) = c(v_{a_\ell} v_{a_\ell+1}) = c(v_{b_\ell-1} v_{b_\ell})$. This contradicts the coloring for $v_{a_\ell} v_{b_\ell}$ required by Theorem 2.2.

□

Corollary 2.7. *Consider a 2-connected bipartite graph G on n vertices with a properly connected 2-coloring c and a Hamiltonian path $H = v_1 v_2 \ldots v_n$ that is a shortest properly colored path between v_1 and v_n. The chain C through H with links $\{v_{a_i} v_{b_i}\}_{i=0}^{i=z}$ has the following properties: $a_1 = a_0 + 1, b_z = b_{z-1} + 1$, and $a_\ell = b_{\ell-2} + 1$ when $2 \leq \ell \leq z$.*

Proof. By Theorem 2.6, C is Hamiltonian. If $a_1 \neq a_0 + 1$ or $b_z \neq b_{z-1} + 1$, then v_{a_0+1} or $v_{b_{(z-1)}+1}$, respectively, is not on C, thus contradicting that C is Hamiltonian. Similarly, for any ℓ where $2 \leq \ell \leq z$, if it is the case that $a_\ell \neq b_{\ell-2} + 1$, then the vertices between a_ℓ and $b_{\ell-2} + 1$ in H are not visited by C, again a contradiction. □

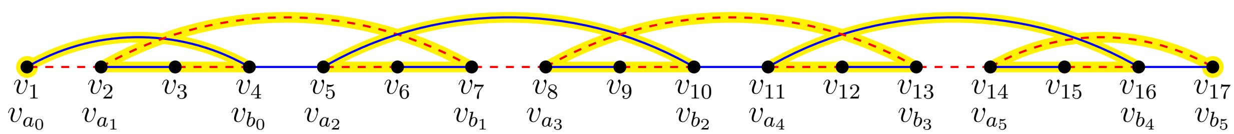


Figure 12: Hamiltonian C through H where $a_1 = a_0 + 1, b_z = b_{z-1} + 1$, and $a_\ell = b_{\ell-2} + 1$

Having considered many of the universal features of 2-colored bipartite graphs with proper diameter equal to one less than the number of vertices, we now introduce a new graph family to classify such graphs. We begin by giving several definitions to describe the structure of this new family, which we call *tau graphs* and denote by $\mathcal{T}_n$. We then prove Theorem 2.11, the main result of the section, in which we show a 2-connected bipartite graph G can be 2-colored to attain a proper diameter of $n-1$ if and only if G is a tau graph.

Definition 2.8. *Given an even cycle $u_1u_2\dots u_{2m}u_1$, a **band** on the cycle is defined as a newly inserted path of any length between vertices u_i and u_j where $i < j$ and $i + j = 2m + 2$. Label a band from u_i to u_j as B_i and its vertices $b_{i_1}, b_{i_2}, \dots, b_{i_\ell}$ where $b_{i_1} = u_i$ and $b_{i_\ell} = u_j$.*

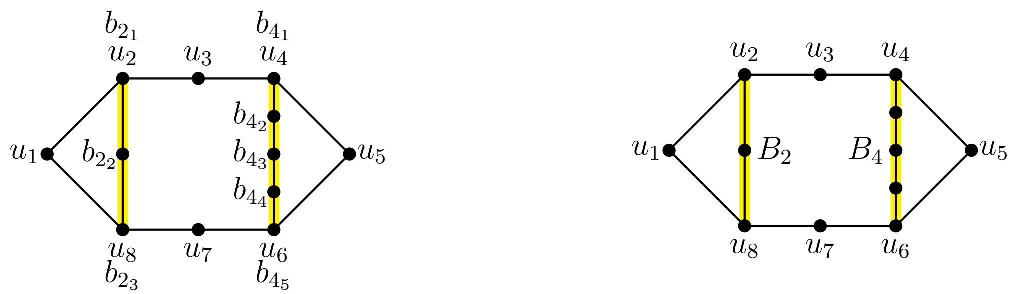


Figure 13: Cycle C_8 with bands B_2 and B_4

Definition 2.9. *Consider an even cycle $u_1u_2\dots u_{2m}u_1$. A **tau skeleton**, denoted T_n^*, is a bipartite graph on n vertices that results from adding a single band between each pair of vertices u_i, u_j where $i < j$ and $i + j = 2m + 2$, so that all bands have an odd number of vertices. We refer to u_1 and u_{m+1}, the two vertices not part of a band, as the **ears**. The collection of all T_n^* for a specific n will be denoted $\mathcal{T}_n^*$.*

Figures 14 and 15 show examples of tau skeletons, the most basic type of tau graphs.

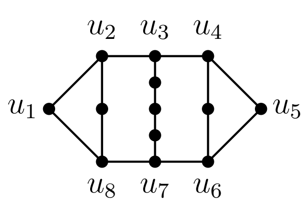


Figure 14: Tau skeleton on 13 vertices

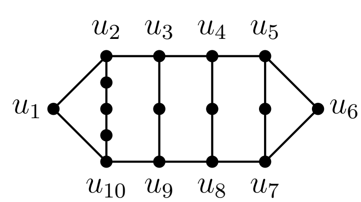


Figure 15: Tau skeleton on 16 vertices

Definition 2.10. *Let $T_n^* \in \mathcal{T}_n^*$ be a tau skeleton on n vertices with vertex labeling given by Definitions 2.8 and 2.9. A* ***tau graph*** *denoted by T_n is constructed by adding any number of extra edges of the form $b_{i_x}b_{i_y}$ where $1 \leq x < y \leq \ell$ to T_n^*. The collection of all possible tau graphs for a specific natural number n is given by $\mathcal{T}_n$. Note that $\mathcal{T}_n^* \subset \mathcal{T}_n$.*

Figure 16: Tau graphs on 13 vertices

Theorem 2.11. *Let G be a 2-connected bipartite graph on n vertices. Then* $\text{pdiam}_2(G) = n-1$ *if and only if G is a tau graph.*

Proof. ($\Leftarrow$) Let G be a tau graph on n vertices, labeled as in Definition 2.9. Since $\text{pdiam}_2(G) \leq n-1$, it is only necessary to construct one 2-coloring that achieves this proper diameter value. Define the coloring c by the following. Let $c(u_1u_2) = c(u_2u_3) = \cdots = c(u_mu_{m+1}) = 1$. Let $c(u_{m+1}u_{m+2}) = c(u_{m+2}u_{m+3}) = \cdots = c(u_{2m}u_1) = 2$. Alternate colors on the edges of each band B_i, starting with color 2 on $b_{i_1}b_{i_2}$ and ending with color 1 on $b_{i_{(\ell-1)}}b_{i_\ell}$. Color remaining edges of the form $b_{i_x}b_{i_y}$, $x < y$, color 1 if x is odd and color 2 if x is even. This coloring yields $\text{pdiam}_2(G, c) = n-1$. The proper diameter is given by $\text{pdist}_2(u_1, u_{m+1}, c)$ (see Figure 17).

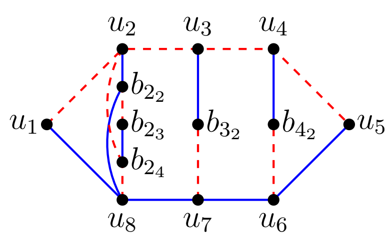


Figure 17: Tau graph on 13 vertices

($\Rightarrow$) Suppose there exists a coloring c such that $\text{pdiam}_2(G, c) = n-1$. Then there exists a properly colored Hamiltonian path $H = v_1v_2 \ldots v_n$ in G where H is a shortest such path from v_1 to v_n. By Theorem 2.6, the chain C through H with links $\{v_{a_i}v_{b_i}\}_{i=0}^{i=z}$ is a second properly colored Hamiltonian path from v_1 to v_n.

Let G' be the spanning subgraph $H \cup C$ of G. We now construct a tau skeleton that is isomorphic to G'. Let $T_n^* \in \mathcal{T}_n^*$ be the tau skeleton consisting of an even cycle $u_1u_2 \ldots u_{2m}u_1$ with $m-1$ bands where $m = z+1$. For $1 \leq j \leq m-1$, the number of interior vertices on band B_{j+1} corresponds to the number of interior vertices on the subpath $H_{a_j} = v_{a_j}v_{a_j+1} \ldots v_{b_{(j-1)}}$ of H. Since G is bipartite, G' must be as well, so all bands have an odd number of vertices, thus satisfying the definition of a tau skeleton. The isomorphism $\phi : V(G') \to V(T_n^*)$ is given as follows. Let $\phi(v_{a_0}) = u_1$ and $\phi(v_{b_0}) = u_{2m}$. If i is odd for $1 \leq i \leq z$, then $\phi(v_{a_i}) = u_{i+1}$ and $\phi(v_{b_i}) = u_{i+2}$. If i is even for $1 \leq i \leq z$, then $\phi(v_{a_i}) = u_{2m-i+1}$ and $\phi(v_{b_i}) = u_{2m-i}$ (see Figure 18).

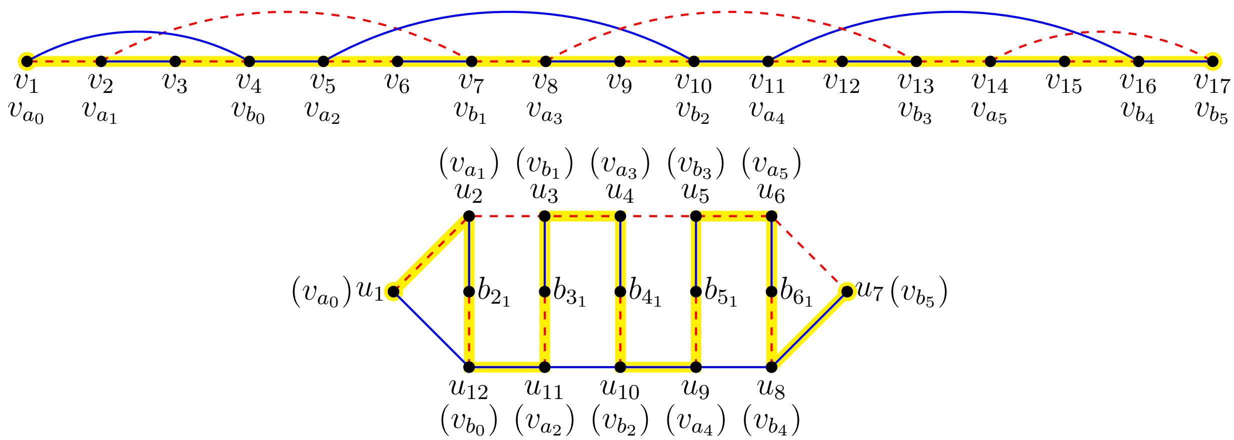


Figure 18: Illustration of the isomorphism with H and $\phi(H)$ highlighted

While the mapping ϕ shows that the tau skeleton T_n^* and $G' = H \cup C$ are isomorphic, it remains to show that the original graph G is also isomorphic to some tau graph, T_n. Since G' is a spanning subgraph of G, both G and G' have the same vertex set but G potentially has more edges. Any edge $v_iv_j \in E(G) - E(G')$ corresponds to an edge $\phi(v_i)\phi(v_j) \notin E(T_n^*)$. We must show that any such edge v_iv_j maps to an edge $\phi(v_i)\phi(v_j)$ that when added to T_n^* yields a tau graph. To show this, we partition all possible edges in $E(G) - E(G')$.

(a) Additional edges from v_1. These correspond to additional edges from u_1 in T_n.

(b) Additional edges from v_n. These correspond to additional edges from u_{m+1} in T_n.

(c) Edges between vertices of non-consecutive subpaths, that is, edges between H_{a_i} and $H_{a_{(i+j)}}$ where $j \geq 2$. These edges correspond to edges between vertices of non-consecutive bands in T_n.

(d) Edges between interior vertices of consecutive subpaths H_{a_i} and $H_{a_{(i+1)}}$, which correspond to edges between interior vertices of consecutive bands in T_n.

(e) Edges between endpoints of consecutive subpaths H_{a_i} and $H_{a_{i+1}}$, that is, between a_i and a_{i+1} or between b_i and b_{i+1}. (Note the edges a_ib_i and $a_{i+1}b_{i+1}$ are in the chain C and so are already in G' and thus are excluded from this case.)

(f) Edges between vertices in the same subpath H_{a_i}. These correspond to edges between vertices on the same band.

The edges described in case (f) are allowed by the definition of a tau graph and so when added to a tau graph yield another tau graph.

We now show that any edges described in (a)-(e) violate the hypothesis that $\text{pdiam}_2(G, c) = n - 1$. Let the edge under consideration in the following cases be denoted v_iv_j.

Case 1: Here we examine the edges v_iv_j considered in (a), so suppose $i = 1$. If $j = n$, then $\text{pdist}_2(v_1, v_n, c) = 1$. If $j \neq n$, then proceed with the following. Let $s \in \mathbb{N}$ such that $1 \leq s \leq z$ and s is the largest integer such that $a_s \leq j$. If $c(v_1v_j) = c(v_jv_{j+1})$, then the proper distance from v_1 to v_n is given by $v_1v_jv_{j-1}\dots v_{a_s+1}v_{a_s}$ and continuing from v_{a_s} along C. This path omits v_{b_0} so it has length less than $n-1$. If $c(v_1v_j) = c(v_{j-1}v_j)$, then the proper distance from v_1 to v_n is given by $v_1v_jv_{j+1}\dots v_n$. This paths omits v_2 so it has length less than $n-1$. Therefore, adding an edge described in (a) yields $\text{pdist}_2(v_1, v_n, c) < n-1$, which is a contradiction. Thus, these edges can't be added, $i \neq 1$, and v_1 has degree 2 (see Figure 19).

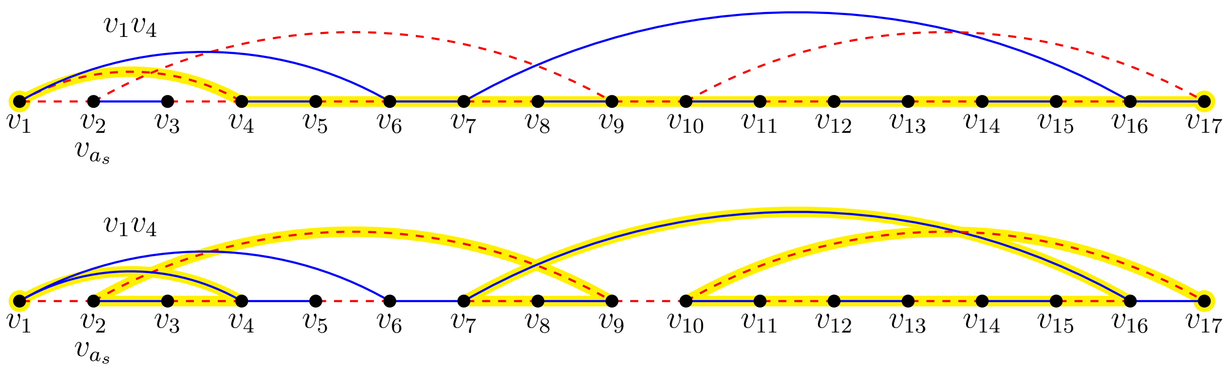


Figure 19: Illustration of contradiction for $i = 1$

Case 2: Here we examine the edges considered in (b). Suppose $j = n$. Since $i \neq 1$, proceed with the following. If $c(v_iv_n) = c(v_{i-1}v_i)$, then the proper distance from v_1 to v_n is given by traversing chain C until reaching v_i and then taking the edge v_iv_n. Since $i \neq a_z$, this path omits v_{a_z} so it has length less than $n-1$. If $c(v_iv_n) = c(v_iv_{i+1})$, then the proper distance from v_1 to v_n is given by $v_1v_2\dots v_{i-1}v_iv_n$. This omits v_{i+1} so it has length less than $n-1$. Therefore $\text{pdist}_2(v_1, v_n, c) < n-1$, which is a contradiction, so the edges in (b) can't be added, $j \neq n$, and v_n has degree 2 (see Figure 20).

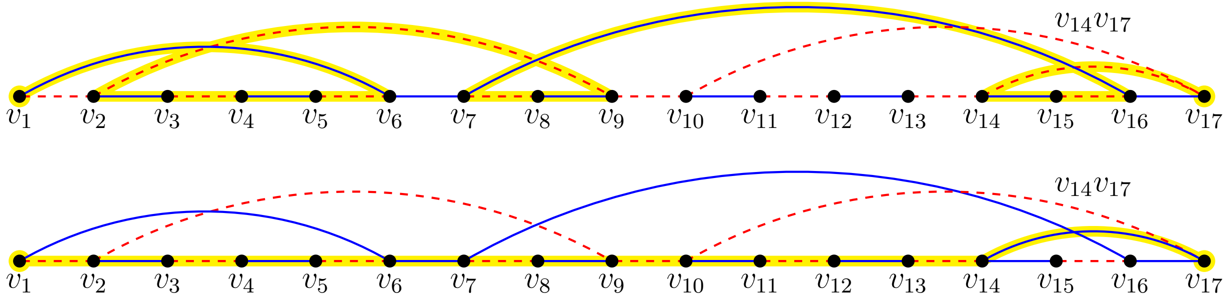


Figure 20: Illustration of contradiction for $j = n$

Case 3: Here we examine the edges considered in (c) and (d). We will consolidate these cases by supposing $i \neq 1$ and $j \neq n$. If $c(v_iv_j) = c(v_iv_{i+1}) = c(v_{j-1}v_j)$, then proceed along path H until reaching v_i, take v_iv_j, then continue along H until v_n. This path omits v_{i+1}. If $c(v_iv_j) = c(v_{i-1}v_i) = c(v_jv_{j+1})$, then proceed along C until reaching v_i, take v_iv_j, then continue along C until v_n. This path omits the vertex immediately after v_i when traversing the chain C (see Figure 21). Therefore, $\text{pdist}_2(v_1, v_n, c) < n - 1$, which is a contradiction.

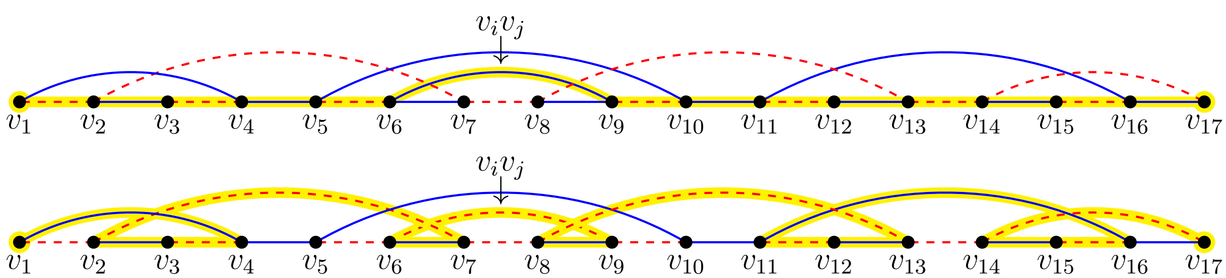


Figure 21: Illustration of contradiction for edges between different subpaths

Case 4: Here we examine the edges considered in (e). If $c(v_iv_j) = c(v_iv_{i+1})$, the shorter path is given by $v_1v_2 \ldots v_iv_jv_{j+1} \ldots v_n$ and omits v_{i+1}. If $c(v_iv_j) = c(v_{i-1}v_i)$ the shorter path is given by traversing C until reaching v_i, taking v_iv_j, then continuing along C until reaching v_n. This path omits v_{j+1}. As a result, adding this edge contradicts $\text{pdiam}_2(G, c) = n - 1$, so any edge in (e) cannot be added (see Figure 22).

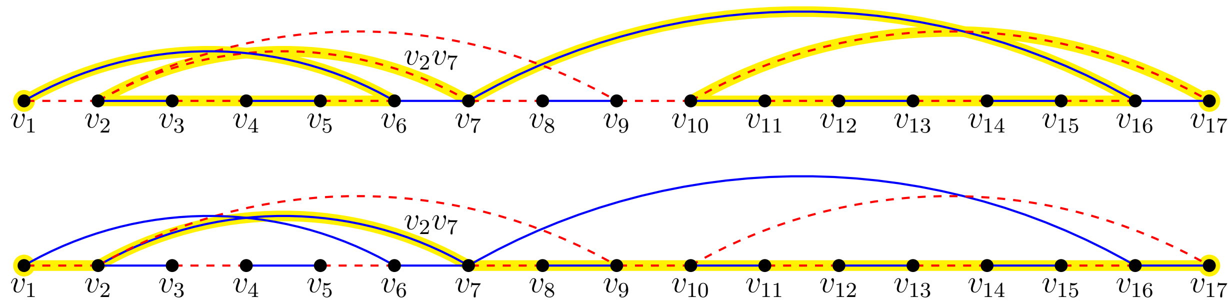


Figure 22: Illustration of forbidden edge between endpoints of consecutive subpaths

Since all v_iv_j in (a)-(e) lead to contradictions, all edges in $E(G) - E(G')$ must fit the form described in (f). Therefore, G is a tau graph. □

3 Bipartite Families

3.1 Ladder Graphs

Theorem 3.1. *Given the Ladder Graph on* $2n$ *vertices,* L_n*,* $n \leq \text{pdiam}_2(L_n, c) \leq 2n - 2$*. These bounds are tight and every intermediate proper diameter value is attainable.*

Proof. Consider the Ladder Graph on $2n$ vertices, L_n. A proper diameter less than n is unattainable since $\text{diam}(L_n) = n \leq \text{pdiam}_2(L_n, c)$. By Theorem 2.11, Tau Graphs are the only 2-connected bipartite graphs that may have a proper diameter given by a Hamiltonian path. While ladder graphs are 2-connected and bipartite, every vertex of degree 2 in a ladder graph is adjacent to another vertex of degree 2. This is not true of the ears of Tau Graphs, so no Ladder Graph is isomorphic to a Tau Graph. Thus, $\text{pdiam}_2(L_n, c) < 2n - 1$. We've now proven the bounds on $\text{pdiam}_2(L_n, c)$.

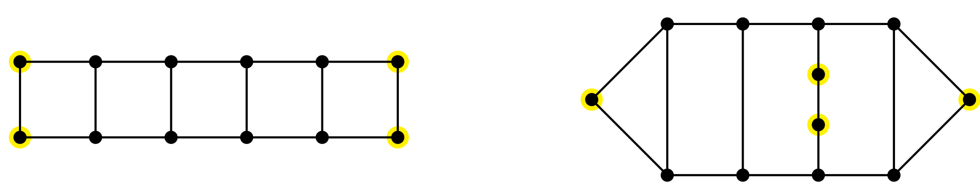


Figure 23: L_6 and a $\mathcal{T}_{12}$ with the degree 2 vertices highlighted to depict no isomorphism

We now work to show that all intermediate values can be attained. Let the vertices in the top path of L_n be denoted $t_1, t_2, \ldots, t_n$ and let the vertices in the bottom path be labeled $b_1, b_2, \ldots, b_n$. We will now define colorings $c_0, c_1, \ldots, c_{n-2}$ where this gives $\text{pdiam}_2(L_n, c_i) = 2n - 2 - i$ for $0 \leq i \leq n - 2$. We will continue by separating into cases.

Case 1: n is odd. Let c_0 be the following coloring on L_n. Let t_1b_1 be red and the remaining t_mb_m be blue for $2 \leq m \leq n$. Let t_mt_{m+1} be red for $1 \leq m \leq n - 1$. Let b_1b_2 be blue and the remaining $b_m b_{m+1}$ be red for $2 \leq m \leq n-1$. This gives $\text{pdiam}_2(L_n, c_0) = 2n-2$ because $\text{pdist}_2(t_1, t_n, c_0) = 2n-2$ is the longest proper distance within L_n (See Figure 24). Let c_1 be exactly the same as c_0 except $t_{n-1}t_n$ becomes blue. This gives $\text{pdiam}_2(L_n, c_1) = 2n-2-1 = 2n-3$ because $\text{pdist}_2(t_1, b_n, c_1) = 2n-3$ is the longest proper distance within L_n (See Figure 25).

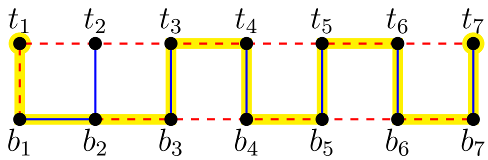


Figure 24: $\text{pdiam}_2(L_7, c_0) = 2n-2 = 12$

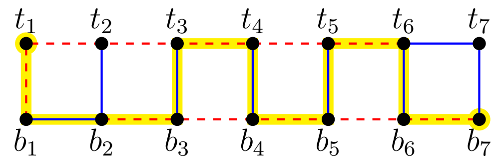


Figure 25: $\text{pdiam}_2(L_7, c_1) = 2n-3 = 11$

Let c_2 be exactly the same as c_1 except $b_{n-2}b_{n-1}$ becomes blue. This gives $\text{pdiam}_2(L_n, c_2)$ $= 2n-2-2 = 2n-4$ because $\text{pdist}_2(t_1, t_n, c_2) = 2n-4$ is the longest proper distance within L_n. In general, for $3 \leq i \leq n-2$, c_i will be the following. If i is odd, then c_i will be the same as c_{i-1} except $t_{n-i}t_{n-i+1}$ becomes blue. This gives $\text{pdiam}_2(L_n, c_i) = 2n-2-i$ because $\text{pdist}_2(t_1, b_n, c_i) = 2n-2-i$ is the longest proper distance within L_n (See Figure 27). If i is even, then c_i will be the same as c_{i-1} except $b_{n-i}b_{n-i+1}$ becomes blue. This gives $\text{pdiam}_2(L_n, c_i) = 2n-2-i$ because $\text{pdist}_2(t_1, t_n, c_i) = 2n-2-i$ is the longest proper distance within L_n (See Figure 26).

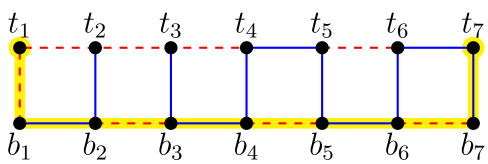


Figure 26: $\text{pdiam}_2(L_7, c_4) = 2n-6 = 8$

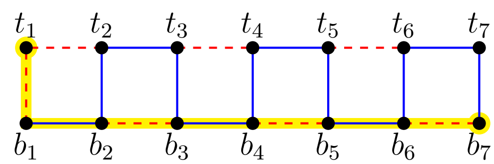


Figure 27: $\text{pdiam}_2(L_7, c_5) = 2n-7 = 7$

Case 2: n is even. Let c_0 be the following coloring on L_n. Let t_1b_1 be red and the remaining $t_m b_m$ be blue for $2 \leq m \leq n$. Let $t_m t_{m+1}$ be red for $1 \leq m \leq n-1$. Let b_1b_2 be blue and the remaining $b_m b_{m+1}$ be red for $2 \leq m \leq n-1$. This gives $\text{pdiam}_2(L_n, c_0) = 2n-2$ because $\text{pdist}_2(t_1, b_n, c_0) = 2n-2$ is the longest proper distance within L_n (See Figure 28). Let c_1 be exactly the same as c_0 except $b_{n-1}b_n$ becomes blue. This gives $\text{pdiam}_2(L_n, c_1) = 2n-2-1 = 2n-3$ because $\text{pdist}_2(t_1, t_n, c_1) = 2n-3$ is the longest proper distance within L_n (See Figure 29).

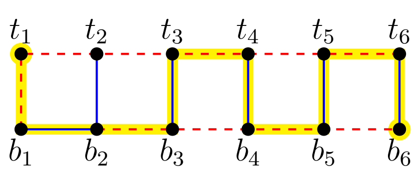


Figure 28: $\text{pdiam}_2(L_6, c_0) = 2n-2 = 10$

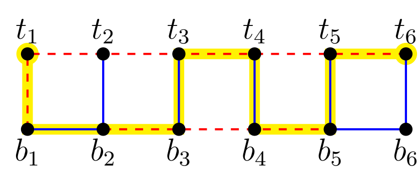


Figure 29: $\text{pdiam}_2(L_6, c_1) = 2n-3 = 9$

Let c_2 be exactly the same as c_1 except $t_{n-2}t_{n-1}$ becomes blue. This gives $\text{pdiam}_2(L_n, c_2) = 2n - 2 - 2 = 2n - 4$ because $\text{pdist}_2(t_1, b_n, c_2) = 2n - 4$ is the longest proper distance within L_n. For $3 \leq i \leq n - 2$, c_i will be the following. If i is odd, then c_i will be the same as c_{i-1} except $b_{n-i}b_{n-i+1}$ becomes blue. This gives $\text{pdiam}_2(L_n, c_i) = 2n - 2 - i$ because $\text{pdist}_2(t_1, t_n, c_i) = 2n - 2 - i$ is the longest proper distance within L_n (See Figure 30). If i is even, then c_i will be the same as c_{i-1} except $t_{n-i}t_{n-i+1}$ becomes blue. This gives $\text{pdiam}_2(L_n, c_i) = 2n - 2 - i$ because $\text{pdist}_2(t_1, b_n, c_i) = 2n - 2 - i$ is the longest proper distance within L_n (See Figure 31).

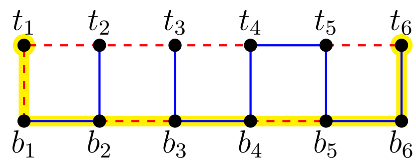


Figure 30: $\text{pdiam}_2(L_6, c_3) = 2n - 5 = 7$

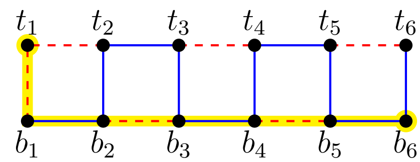


Figure 31: $\text{pdiam}_2(L_6, c_4) = 2n - 6 = 6$

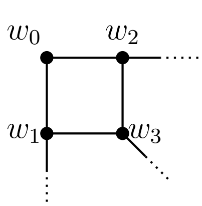
□

3.2 Grids

Lemma 3.2. *Let G be a bipartite graph, and let w_0 be a degree two vertex with two distinct paths of length 2 to vertex w_3. If w_3 has degree greater than 2, then w_0 is not isomorphic to the ear of some Tau Graph T_n.*

Proof. Let $Q_1 = w_0w_1w_3$ and $Q_2 = w_0w_2w_3$ be the two paths from w_0 to w_3 in G. Assume by way of contradiction that there exists a Tau Graph T_n such that w_0 is isomorphic to an ear in T with isomorphism $\phi : G \to T$. Use the vertex labeling on T that is outlined in Definition 2.9. Without loss of generality, let $\phi(w_0) = u_1$, $\phi(w_1) = u_2$, and $\phi(w_2) = u_{2m}$. Since w_1 and w_2 have two distinct paths of length two between them, the corresponding band B_2 in T must be length two. Note that w_3 is adjacent to both w_1 and w_2 in G, so $\phi(w_3) = b_{2_2}$. This is a contradiction since w_3 has degree greater than 2, but b_{2_2} must have degree two, since B_2 is of length 2. Therefore, w_0 is not isomorphic to an ear of T_n. □

Consequently, a corner vertex of a grid cannot be mapped to an ear of a Tau graph.



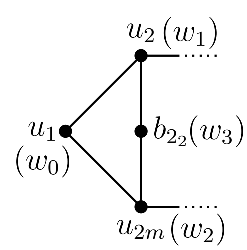


Figure 32: Illustration of the Isomorphism Contradiction

Theorem 3.3. *Let $G_{m,n}$ be a grid. A proper diameter of $nm-1$ is not attainable.*

Proof. By Theorem 2.11, since $G_{m,n}$ is 2-connected and bipartite, $\text{pdiam}_2(G_{m,n}) = nm-1$ is only attainable if $G_{m,n}$ is isomorphic to a Tau Graph T_{nm}. We now show that $G_{m,n}$ is not isomorphic to T_{nm}.

Assume by way of contradiction that $G_{m,n}$ is isomorphic to T_{nm} with isomorphism $\phi : G_{m,n} \to T_{nm}$. Use the vertex labeling on T_{nm} that is outlined in Definition 2.9. Note that there are four vertices in $G_{m,n}$ with degree two. By Lemma 3.2, none of these vertices can be mapped by ϕ to an ear. Therefore, no vertex in $G_{m,n}$ can be mapped to an ear in T_{nm}, so $G_{m,n}$ is not isomorphic to T_{nm} and a proper diameter of $nm-1$ is not attainable. □

Theorem 3.4. *For some grid $G_{m,n}$, $\text{pdiam}_2(G_{m,n}) = nm-2$ if and only if at least one of m and n is even.*

Proof. (⇒) We seek to prove the contrapositive, which says if m and n are both odd, then a proper diameter of $nm-2$ is unattainable. Consider $G_{m,n}$ where m and n are both odd. Suppose there exists a coloring c such that $\text{pdiam}(G_{m,n}, c) = nm-2$. Therefore, there exist $s, t \in V(G_{m,n})$ such that $\text{pdist}_2(s,t,c) = nm-2$. Let P be a properly colored path between s and t of length $nm-2$. So there exists vertex z not on P. Consider graph $G' = G_{m,n} - \{z\}$. Note that G' has $nm-1$ vertices and $\text{pdiam}_2(G') = nm-2$, so P is Hamiltonian in G'. By Theorem 2.11, if G' is 2-connected, then it is isomorphic to some tau graph, T_{nm-1}. First, suppose G' is 2-connected. We have two cases.

Case 1: Suppose $m = n = 3$. If z is the degree 4 vertex in $G_{m,n}$, then G' is a cycle, which is not isomorphic to a tau graph. Additionally, z cannot be a degree 3 vertex because G' would not be 2-connected. If z is a degree 2 vertex, then there do not exist two degree 2 vertices where neither is adjacent to another degree 2 vertex. Therefore, G' is not isomorphic to a tau graph, so at least one of n and m is not equal to 3.

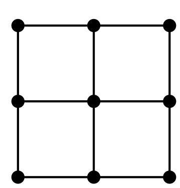


Figure 33: $G_{3,3}$

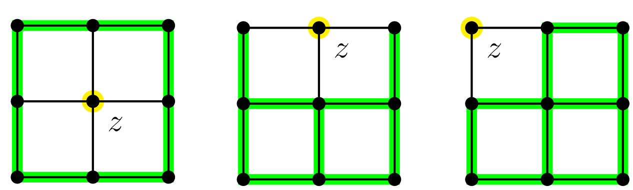


Figure 34: Varying $\deg(z)$ and illustrating G'

Case 2: Suppose at most one of m and n is equal to 3. Without loss of generality, let $m \leq n$. Place the grid on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1,1)$, and the grid expands into the first quadrant. Label the vertices $w_{x,y}$ according to their x and y coordinates. Note that by Lemma 3.2, no corner vertex of a grid can be mapped to an ear in a tau graph.

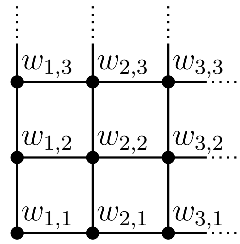


Figure 35: Illustration of $G_{m,n}$ in the Cartesian Plane

Case 2.1: Suppose z is a corner vertex. Without loss of generality, assume $z = w_{1,1}$. Then $w_{1,2}$ and $w_{2,1}$ are the only degree 2 vertices in G' that are not in a corner position of $G_{m,n}$. By Lemma 3.2, $w_{1,2}$ and $w_{2,1}$ are the only vertices in $G_{m,n}$ that can map to the ears of T_{nm-1}. This is a contradiction, because there is exactly one path of length 2 between $w_{1,2}$ and $w_{2,1}$, while the ears of a tau graph must have either 0 or 2 paths of length 2 between them. Therefore z cannot be a corner vertex.

Case 2.2: Suppose z is a vertex of degree three. If z is adjacent to a corner vertex, then G' is not two connected and this contradicts the prior supposition that G' is two connected. Thus z cannot be adjacent to a corner vertex. Without loss of generality, say $z = w_{q,1}$ with $3 \leq q \leq n-2$. The proof from Lemma 3.2 still applies to all corners of G', since each corner still has two edge-disjoint paths to a vertex with degree greater than two. Thus, if G' is isomorphic to some tau graph T_{nm-1} then the isomorphism maps both of $w_{q-1,1}$ and $w_{q+1,1}$ to ears since they are the only two remaining degree two vertices. Since $3 \leq q \leq n-2$, this means $w_{q-2,2}$ is defined and has degree at least three (equal to three if $q = 3$ and four otherwise). Therefore $w_{q-1,1}w_{q-2,1}w_{q-2,2}$ and $w_{q-1,1}w_{q-1,2}w_{q-2,2}$ are two edge-disjoint paths of length two in G' from $w_{q-1,1}$ to $w_{q-2,2}$. Since $w_{q-2,2}$ has degree greater than two, Lemma 3.2 shows $w_{q-1,1}$ can not be mapped to an ear in T_{nm-1}, further contradicting that G' is isomorphic to some tau graph T_{nm-1}. Therefore z is not a vertex of degree three.

Case 2.3: Suppose z is a vertex of degree 4 with distance 2 from a corner vertex. Without loss of generality, say $z = w_{2,2}$. As noted before, the corner vertices $w_{n,1}$ and $w_{n,m}$ cannot be mapped to ears in T_{nm-1}. If $m = 3$ then $w_{1,1}, w_{2,1}, w_{1,2}, w_{1,3}$, and $w_{2,3}$ are all adjacent to a degree 2 vertex and thus cannot be mapped to ears. If $m \neq 3$ then $w_{1,1}, w_{2,1}$, and $w_{1,2}$ are all adjacent to a degree 2 vertex and thus cannot be mapped to ears. The corner vertex $w_{1,m}$ cannot be an ear by Lemma 3.2. Therefore z cannot be a vertex of degree 4 with distance 2 from a corner vertex.

Case 2.4: Suppose z is an interior vertex of degree 4, with distance greater than 2 from any corner vertex. If $m > 3$ then the removal of z cannot lower the degree of more than one vertex to 2. Then by Lemma 3.2, there cannot be enough vertices in G' that correspond to ears in T_{nm-1}. If $m = 3$, then without loss of generality let $z = w_{q,2}$, $3 \leq q \leq n-2$. Then $w_{q,1}$ and $w_{q,3}$ have degree 2 and therefore must be mapped to the ears of T_{nm-1}. This is a contradiction because $w_{q,1}$ and $w_{q,3}$ form a cut set, which can never be true of the ears of T_{nm-1}. Therefore z cannot be an interior vertex of degree 4, with distance more than 2 from any corner vertex.

Thus, z must be adjacent to a corner vertex in $G_{m,n}$, so G' is not 2-connected. Note that

$G_{m,n}$ is bipartite and all vertices adjacent to a corner vertex are in the smaller partite set. So the smaller partite set in G' has two vertices fewer than the larger partite set. So there cannot exist a Hamiltonian path in G', which is a contradiction. Therefore, $\text{pdiam}_2(G_{n,m}) < nm-2$ when n and m are both odd.

($\Leftarrow$) Place $G_{m,n}$ on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1, 1)$, and the grid expands into the first quadrant. Label the vertices $w_{x,y}$ according to their x and y coordinates.

Either m or n is even. Without loss of generality, let m be even. Let $G' = G - \{w_{2,m}\}$. We will construct coloring c' such that $\text{pdiam}_2(G', c') = mn - 2$ in the following way. Consider the Hamiltonian path

$$P = w_{1,m}w_{1,m-1} \dots w_{1,1}w_{2,1} \dots w_{n,1}w_{n,2}w_{n-1,2} \dots \\ w_{2,2}w_{2,3}w_{3,3} \dots w_{n,3} \dots w_{n,m-1}w_{n,m}w_{n-1,m} \dots w_{4,m}w_{3,m}.$$

The penultimate elipsis in this construction indicates that the path traverses all the vertices in the i-th row followed by moving up to the $(i+1)$-th row until reaching the m-th row. Let $c'(w_{1,m}w_{1,m-1}) = 1$ and alternate between colors 1 and 2 for the remaining edges of P. By Theorem 2.2, the colors of $E(G') - E(P)$ are fixed (See Figure 36).

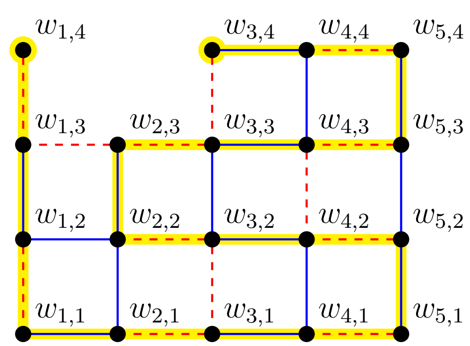


Figure 36: Properly colored path in G' with coloring c'

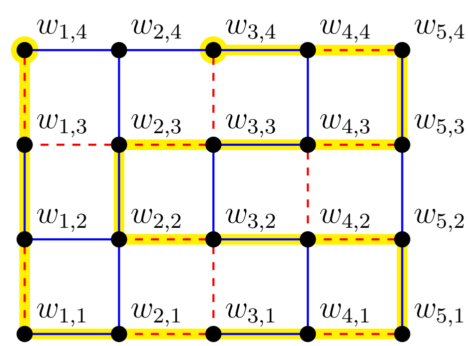


Figure 37: Properly colored path in G with coloring c

Let c on G be equal to c' with the addition of $c(w_{2,m}w_{1,m}) = c(w_{2,m}w_{3,m}) = c(w_{2,m}w_{2,m-1}) = 2$. This gives $\text{pdiam}_2(G_{m,n}, c) = mn-2$ because $\text{pdist}_2(w_{1,m}w_{3,m}, c) = mn-2$ is the longest proper distance within $G_{m,n}$ (See Figure 37).

□

Theorem 3.5. *Let $G_{m,n}$ be a grid. If n and m are odd, then $n+m-2 \le \text{pdiam}_2(G_{m,n}, c) \le nm - 3$. These bounds are tight and every intermediate value is attainable.*

Proof. By Theorem 3.3 and Theorem 3.4, $\text{pdiam}_2(G_{m,n}) \le nm - 3$. Again, place $G_{m,n}$ on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1, 1)$, and the grid expands into the first quadrant. Without loss of generality, let $m \le n$ and let the side of length m extend parallel to the y-axis. Label the vertices $w_{x,y}$ according to their x and y coordinates.

To see that the upper bound is tight, define graph $G' = G - \{w_{2,m}, w_{2,m-1}\}$ and coloring c' on G' in the following way. Consider the Hamiltonian path

$$P = w_{1,m}w_{1,m-1}\dots w_{1,1}w_{2,1}\dots w_{n,1}w_{n,2}w_{n-1,2}\dots w_{2,2}w_{2,3}w_{3,3}\dots w_{n,3}w_{n,4}\dots w_{2,m-3}w_{2,m-2}\dots$$
$$w_{n,m-2}w_{n,m-1}w_{n,m}w_{n-1,m}w_{n-1,m-1}w_{n-2,m-1}w_{n-2,m}w_{n-3,m}w_{n-3,m-1}w_{n-4,m-1}\dots w_{3,m-1}w_{3,m}.$$

Alternate the colors of P with 1 and 2, beginning with $c'(w_{1,m}w_{1,m-1}) = 1$ and continuing for the remaining edges. With this alternating path, G' is properly connected. Per Theorem 2.2, for P to be a shortest properly colored Hamiltonian path between its endpoints, there is only one possible coloring of the edges in $E(G')-E(P)$. Color these edges as such. (See Figure 38.)

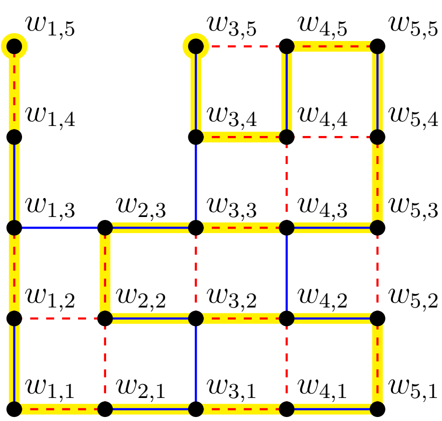


Figure 38: Properly colored path in G' with coloring c'

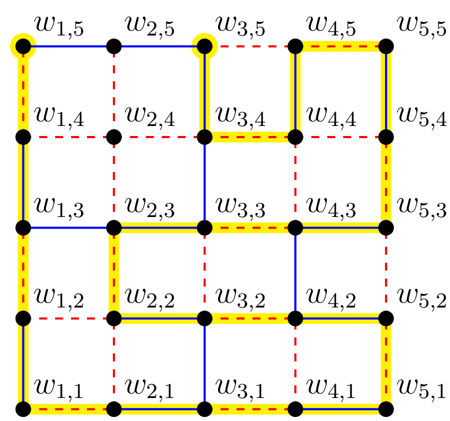


Figure 39: Properly colored path in G with coloring c

Let c on G be equal to c' with the addition of $c(w_{2,m-1}w_{2,m}) = c(w_{2,m-1}w_{3,m-1}) = c(w_{2,m-1}w_{2,m-2}) = c(w_{2,m-1}w_{1,m-1}) = 1$ and $c(w_{2,m}w_{1,m}) = c(w_{2,m}w_{3,m}) = 2$. This gives $\text{pdiam}_2(G_{m,n}, c) = nm - 3$ because $\text{pdist}_2(w_{1,m}w_{3,m}, c) = nm - 3$ is the longest proper distance within $G_{m,n}$ (See Figure 39).

Any intermediate proper diameter value, as well as the lower bound, can be attained with the following algorithm. We will define c_a such that $\text{pdiam}_2(G_{m,n}, c_a) = a$ for $n + m - 2 \leq a < nm - 3$. Let $b = a - (n + m - 2)$. For each a, $\text{pdiam}_2(G_{m,n}, c_a)$ will be given by $\text{pdist}_2(w_{1,m}, w_{x_a,y_a})$ where

$$y_a = \left\lceil \frac{b}{n-1} \right\rceil + 1,\ \ x_a = \begin{cases} n - [(b-1) \bmod (n-1)] & y_a \textit{ even} \\ 2 + [(b-1) \bmod (n-1)] & y_a \textit{ odd} \end{cases}$$

if $a \leq mn - 2n + 1$, and

$$y_a = \begin{cases} m & \text{if } (nm-3-a) \mod (4) = 0, 3 \\ m-1 & \text{if } (nm-3-a) \mod (4) = 1, 2 \end{cases},\ \ x_a = 3 + \left\lfloor \frac{nm-3-a}{2} \right\rfloor$$

if $a > mn - 2n + 1$.

Note that in the previous formulas, b counts the length of the $w_{n,1} - w_{x_a,y_a}$ sub-path within the properly colored $w_{1,m} - w_{x_a,y_a}$ path. In the first case, the restriction $a \leq mn - 2n + 1 = (m-2)n + 1$ implies that $y_a \leq m - 2$. In this case, $n - 1$ counts the number of sub-paths of the form $w_{n,j-1}w_{n,j}w_{n-1,j} \dots w_{3,j}w_{2,j}$ or $w_{2,j-1}w_{2,j}w_{3,j}w_{4,j} \dots w_{n,j}$. So $\lceil \frac{b}{n-1} \rceil$ is equivalent to the number of edges in the path of the form $w_{i,j-1}w_{i,j}$, $2 \leq j \leq y_a$, and $y_a = \lceil \frac{b}{n-1} \rceil + 1$. To see how to obtain x_a, first note that $b - 1$ counts the length of the path $w_{n,2}w_{n-1,2} \dots w_{x_a,y_a}$, so $n - 1$ gives the number of sub-paths of the form $w_{n,j}w_{n-1,j} \dots w_{3,j}w_{2,j}w_{2,j+1}$ or $w_{2,j}w_{3,j} \dots w_{n,j}w_{n,j+1}$. Thus, when y_a is even, $(b-1) \bmod (n-1)$ counts the length of the path $w_{n,y_a}w_{n-1,y_a} \dots w_{x_a,y_a}$, so $x_a = n - [(b-1) \bmod (n-1)]$. When y_a is odd, $(b-1) \bmod (n-1)$ counts the length of the path $w_{2,y_a}w_{3,y_a} \dots w_{x_a,y_a}$. Hence, $x_a = 2 + [(b-1) \bmod (n-1)]$.

In the second case, $a > mn - 2n + 1$ implies $y_a \geq m - 1$. Here, the first step in obtaining x_a is to count $w_{2,m}$, $w_{2,m-1}$, and the number of vertices in the $w_{1,m} - w_{x_a,y_a}$ path of proper diameter a, and subtract that quantity from the total number of vertices in the grid. This gives $nm - 3 - a$, the number of vertices in the path $w_{x_{a+1},y_{a+1}}w_{x_{a+2},y_{a+2}} \dots w_{x_{nm-3},y_{nm-3}}$. Then $\lfloor \frac{nm-3-a}{2} \rfloor = x_a - x_{nm-3}$. Since $x_{nm-3} = 3$, we have $x_a = 3 + \lfloor \frac{nm-3-a}{2} \rfloor$. Additionally, since $nm - 3 - a$ counts the length of path $w_{x_{nm-3},y_{nm-3}}w_{x_{nm-2},y_{nm-2}} \dots w_{x_{a+2},y_{a+2}}w_{x_{a+1},y_{a+1}}w_{x_a,y_a}$ and $y_{nm-3} = m$, $y_a = m - 1$ when $(nm - 3 - a) \mod (4) \in \{1, 2\}$ and $y_a = m$ otherwise.

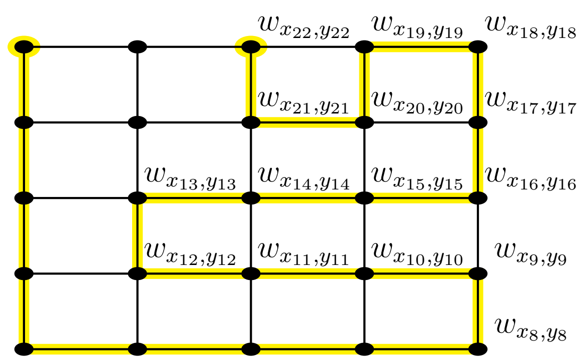


Figure 40: Illustration of w_{x_a,y_a}

Use the following process to form coloring c_a from c for each $n + m - 2 \leq a \leq nm - 4$.

1. Let $c_a(w_{2,m}w_{3,m}) = 1$.

2. If $a = nm - 4$, $c_a(w_{2,m-1}w_{2,m}) = 1$. If $a < nm - 4$, $c_a(w_{2,m-1}w_{2,m}) = 2$ and $w_{x_a,y_a}w_{x_{a+1},y_{a+1}}$ changes color.

3. If $a < \lfloor \frac{nm}{2} \rfloor - 1$, change coloring such that $w_{1,m}w_{2,m} \dots w_{n,m}$ is a path of alternating color, beginning with $c_a(w_{1,m}w_{2,m}) = 2$ and ending with $c_a(w_{n-1,m}w_{n,m}) = 1$. Additionally, change colors as needed such that $c_a(w_{i,j}w_{i,j+1}) = 1$ if j is odd and $c_a(w_{i,j}w_{i,j+1}) = 2$ if j is even, for all $2 \leq i \leq$ n and $y_a < j < m$.

4. If $a = n + m - 2$, change coloring such that $c_a(w_{n,2}w_{n,3}) = 2$.

Let C be a cycle in $G_{m,n}$ containing path P and path $w_{1,m}w_{2,m}w_{3,m}$. Let C' denote the path $C \setminus \{w_{x_a,y_a}w_{x_{a+1},y_{a+1}}\}$. In this way, C' is a properly colored path in $G_{m,n}$ containing $V(G_{m,n})\setminus\{w_{2,m-2}\}$. The above adaptations from coloring c to form c_a ensures that $(G_{m,n}, c_a)$ is properly connected. Furthermore, similar to the coloring c, the proper distance between $w_{1,m}$ and w_{x_a,y_a} is a. It remains to show that no longer proper distance exists in $G_{m,n}$ with coloring c_a. Clearly no two vertices of the segment of C' between $w_{1,m}$ and w_{x_a,y_a} have proper distance exceeding a. Additionally, step 3 above ensures that no two vertices outside the segment of C' between $w_{1,m}$ and w_{x_a,y_a} have proper distance exceeding a, noting that if $a \geq \lfloor \frac{nm}{2} \rfloor - 1$, then C' gives a properly colored path of length at most a (even if one of the two vertices considered is $w_{2,m-1}$). It only remains to consider a vertex of the segment of C' between $w_{1,m}$ and w_{x_a,y_a} and another vertex off this path. The properly colored path between two such vertices can be constructed by traversing horizontal segments opposite the direction implied by path P and vertical segments of $G_{m,n}$. A simple check of the coloring construction shows that paths of this form exist.

Therefore, each c_a gives $\text{pdiam}_2(G_{m,n}, c_a) = a$ for all $a \in [n+m-2, nm-4]$, showing that, in addition to the upper bound, the lower bound and all intermediate proper diameter values are attainable. □

Theorem 3.6. *Let $G_{m,n}$ be a grid. If either n or m is even, then $n+m-2 \leq \text{pdiam}_2(G_{m,n}, c) \leq nm-2$. These bounds are tight and every intermediate value is attainable.*

Proof. Without loss of generality, let m be even. Let a be the desired proper diameter, with $n+m-2 \leq a \leq nm-2$. Place $G_{m,n}$ on a Cartesian plane such that each edge is of length 1, one corner vertex of the grid is at $(1,1)$, and the grid expands into the first quadrant. Let the side of length m extend parallel to the y-axis. Label the vertices $w_{x,y}$ according to their x and y coordinates.

We will define c_a such that $\text{pdiam}_2(G_{m,n}, c_a) = a$ for $n+m-2 \leq a \leq nm-2$. Let $b = a-(n+m-2)$. For each a, $\text{pdiam}_2(G_{m,n}, c_a)$ will be given by $\text{pdist}_2(w_{1,m}, w_{x_a,y_a})$, where

$$y_a = \left\lceil \frac{b}{n-1} \right\rceil + 1,\ \ x_a = \begin{cases} n-(b-1) \bmod (n-1) & y_a \textit{ even} \\ 2+(b-1) \bmod (n-1) & y_a \textit{ odd} \end{cases}$$

Note that these formulas are the same as those in the first case of Theorem 3.5 and in this case they apply to the entire grid.

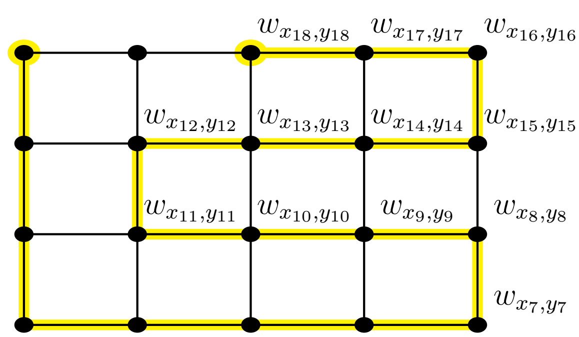


Figure 41: Illustration of w_{x_a,y_a}

Let c_{nm-2} be identical to the coloring of the grid in the backward direction of the proof in Theorem 3.4. Define c_a to be identical to c_{nm-2} except for the following.

If $y_a = m$, switch the colors of the edges $w_{i,y_a-1}w_{i,y_a}$ for $3 \leq i \leq x_a - 1$. Otherwise, follow these steps.

1. If y_a is even, switch the color of edges $w_{i,y_a-1}w_{i,y_a}$ for $2 \leq i \leq x_a - 1$.

   If y_a is odd, switch the color of edges $w_{i,y_a-1}w_{i,y_a}$ for $x_a + 1 \leq i \leq n$.

2. For $y > y_a$, preserve the coloring of $w_{1,y}w_{2,y}$ and alternate the colors along the path $w_{1,y}w_{2,y} \dots w_{n,y}$.

3. For $x > 1$, color $w_{x,m}w_{x,m-1}$ such that $c_a(w_{x,m}w_{x,m-1}) \neq c_a(w_{x-1,m}w_{x,m})$ and then alternate colors along the vertical path $w_{x,m}w_{x,m-1} \dots w_{x,y_a+1}$.

4. For $x > 1$, color each edge $w_{x,y_a+1}w_{x,y_a}$ the same color as $w_{x-1,y_a+1}w_{x,y_a+1}$. However, if $y_a \leq 2$, then color $w_{n,y_a+1}w_{n,y_a}$ the opposite color as $w_{n-1,y_a+1}w_{n,y_a+1}$.

Similar to the description provided after the construction for c_a in Theorem 3.5, this yields $\text{pdiam}_2(G_{m,n}, c_a) = a$ for all $a \in [n + m - 2, nm - 2]$, so the bounds are tight and all intermediate proper diameter values are attainable. □

4 Conclusion

The work presented here is strictly related to bipartite graphs. One clear question asks about extending the results to general 2-connected graphs. The following definitions extend the notion of Tau graphs in this paper and motivate a conjecture about general 2-connected graphs on n vertices with proper diameter $n - 1$.

Definition 4.1. *Consider a tau skeleton S on a total of n vertices consisting of an even cycle $u_1u_2 \dots u_{2m}u_1$ and bands all having an odd number of vertices (called an odd skeleton) or an even number of vertices (called an even skeleton) labeled as in Definition 2.8. A* ***tau graph*** *$\mathcal{T}_n$ is a family of graphs on n vertices that results from adding any or none of the following edges to S.*

- *Edges of the form $b_{i_x}b_{i_y}$ where $1 \leq x < y \leq \ell$ may be added on any band B_i.*
- *In an even skeleton, edges of the form $u_iu_{2m-(i+1)}$ for $1 \leq i \leq m - 2$ and $u_iu_{2m-(i-1)}$ for $2 \leq i \leq m - 1$ may be added to S, and we call such an edge a* ***lace****.*

We call a tau graph with an odd skeleton an ***odd tau graph*** *or an* ***odd*** *$\mathcal{T}_n$. Similarly, a tau graph with an even skeleton is an* ***even tau graph*** *or an* ***even*** *$\mathcal{T}_n$.*

Figure 42: Even $\mathcal{T}_{12}$ graphs. Laces are highlighted.

Conjecture 4.2. *A 2-connected graph on n vertices has proper diameter $n-1$ iff G is a tau graph as defined in Definition 4.1.*

Between the work of this paper and the above conjecture, the authors consider all 2-connected graphs which attain the upper bound given by the result of Coll et. al. [4]. We ask the same question of graphs on n vertices which are k-connected for $3 \leq k \leq n-2$. Given a graph k-connected graph G which attains the maximum proper diameter given by the result of Coll et al. [4], is the subgraph of G induced by the path that realizes the proper diameter isomorphic to some tau graph presented earlier?

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References

[1] A. Abouelaoualim, K. Ch. Das, W. Fernandez de la Vega, M. Karpinski, Y. Manoussakis, C. A. Martinhon, and R. Saad. Cycles and paths in edge-colored graphs with given degrees. J. Graph Theory, 64(1):63–86, 2010.

[2] E. Andrews, E. Laforge, C. Lumduanhom, and P. Zhang. On proper-path colorings in graphs. J. Combin. Math. Combin. Comput., 97:189–207, 2016.

[3] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, and Z. Tuza. Proper connection of graphs. Discrete Math., 312(17):2550–2560, 2012.

[4] V. Coll, J. Hook, C. Magnant, K. McCready, and K. Ryan. The proper diameter of a graph. Discuss. Math. Graph Theory, 40:107–125, 2020.

[5] S. Fujita and C. Magnant. Properly colored paths and cycles. Discrete Appl. Math., 159:1391–1397, 2011.

[6] X. Li and C Magnant. Properly colored notions of connectivity - a dynamic survey. Theory and Application of Graphs, 0(1), 2015.

[7] X. Li, C. Magnant, and Z. Qin. Properly Colored Connectivity of Graphs. SpringerBriefs in Mathematics. Springer International Publishing, 2018.

[8] V. G. Vizing. On an estimate of the chromatic class of a p-graph. Diskret. Analiz No., 3:25–30, 1964.

[9] G. Wang and H. Li. Color degree and alternating cycles in edge-colored graphs. Discrete Math., 309:4349–4354, 2009.