1 Topics Covered

- Hall’s theorem
- Min-Max theorems

2 Hall’s Theorem

In this section, we define system of distinct representatives, state and prove Hall’s theorem.

**Definition 1. System of Distinct Representatives:** Given a collection of subsets $S_1, S_2, \ldots, S_m$ of $\mathcal{U}$. A set of points $x_1, x_2, \ldots, x_m$ s.t $x_i \in \mathcal{U}$ is called a system of distinct representatives if

- $x_i \in S_i$ for $i = 1, \ldots, m$
- $x_i \neq x_j$ for $i \neq j$

A basic requirement for such a system to exist is that $|\mathcal{U}| \geq m$. A set of necessary conditions are provided for such a system to exist. These together are called Hall’s condition.

**Definition 2. Hall’s Condition:** For any $I \subseteq [m]$

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|$$

It is easy to see that this is necessary for the existence of a system of distinct representatives. In the following theorem we show that this is also sufficient.

**Theorem 1. Hall’s Theorem** A system of Distinct Representatives exists for sets $S_1, S_2, \ldots, S_m$ iff $\forall I \subseteq [m]$,

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|$$
Proof. Proof is by induction on $m$.

Base case: Trivially true for $m=1$.

Induction hypothesis: Assume the statement holds for any $S_1, S_2, ... S_t$ for $t < m$. Now to prove for case of $m$ sets, from assumptions we have $\forall I \subseteq [m], |\bigcup_{i \in I} S_i| \geq |I|$. We now have 2 cases:

1. $\forall k : 1 \leq k < m$, and for any $I \subset [m]$ s.t. $|I| = k$,

$$\bigg|\bigcup_{i \in I} S_i\bigg| > k$$

Now for an arbitrary set $S_i$, we can pick an element $x_i \in S_i$ as a representative, and remove $x_i$ from all other sets. Remaining $m-1$ sets satisfy Hall’s condition, and hence by the induction hypothesis, there exists a system of direct representatives for them.

2. $\exists k : 1 \leq k < m$ and $\exists I \subset [m], |I| = k$ s.t.

$$\bigg|\bigcup_{i \in I} S_i\bigg| = k$$

For these sets $S_i$ for $i \in I$, applying the induction hypothesis we have a system of representatives $x_i$ for $i \in I$. Removing these from the remaining $m-k$ sets, we claim the sets $S_j, j \in [m] \setminus I$ satisfy Hall’s condition; for any $s \leq m-k$ any $s$ of these sets contain at least $s$ elements (for otherwise, the union of these $s$ sets and the $k$ sets $S_i, i \in I$ contain less than $s+k$ elements, which contradicts Hall’s condition). Hence by the induction hypothesis, the sets $S_j, j \in [m] \setminus I$ also have a system of distinct representatives.

\[ \square \]

**Definition 3. Matching** A matching in a bipartite graph is a collection of edges s.t. no two edges have a common vertex.

A perfect matching is a matching that covers all the nodes. Now we state Hall’s theorem for matching in bipartite graphs.

**Theorem 2.** A bipartite graph with vertex set $V_1$ and $V_2$ has a perfect matching (from $V_1$ to $V_2$) iff for any $X \subseteq V_1$ the union of neighbors of $v \in X$ has at least $|X|$ vertices.

**Proof.** For each $v \in V_1$ define the set $S_v \subseteq V_2$ as the set of neighbours of $v$, and apply Hall’s theorem. \[ \square \]

**3 Min-Max theorems**

A related question to the existence of a perfect matching in a bipartite graph is the size of a maximum sized matching in a graph; this result (König-Egerváry’s theorem) is similar to several such “min-max theorems” (such as Dilworth’s theorem, which we saw before). For example:
• **Dilworth’s theorem (Dilworth 1950)** The minimum number of chains (correspondingly antichains) which cover a partially ordered set is equal to the maximum size of an antichain (correspondingly chain) of the poset.

• **Menger’s theorem (Menger, 1927)** The minimum number of vertices separating two given vertices in a graph is equal to the maximum number of vertex-disjoint paths between them.

• **König-Egerváry’s theorem (König 1931, Egerváry 1931)** The size of a largest matching in a bipartite graph is equal to the size of the smallest vertex cover.