Parallel-prefix computation

Properties of functions

- f: DxD $\rightarrow$ D
  - Function takes two arguments from set D
  - Returns element of D
  - Example: +,*: $\Re$x$\Re$ $\rightarrow$ $\Re$
  - Not an example: division (why?)
- Commutative function: f(x,y) = f(y,x)
  - Examples: +, * (even in floating-point arithmetic)
  - Not commutative: matrix multiplication
- Associative function: f(x,f(y,z)) = f(f(x,y),z))
  - Intuitively, we can parenthesize the operands any way
  - Examples: +, * on real numbers, matrix multiplication
  - Not associative: floating-point +, *
- Reduction operation: both commutative and associative
  - Example: +, * on real numbers

Map and Reduce

- map f $<x_1,x_2,\ldots,x_n> = <f(x_1),f(x_2),\ldots,f(x_n)>$
  - Types
    - f: $D \rightarrow R$
    - $<x_1,\ldots x_n>$: array<D>
    - $<f(x_1),\ldots,f(x_n)>$: array<R>
- reduce f $<x_1,x_2,\ldots,x_n> = f(x_1,f(x_2,\ldots f(x_{n-1},x_n)))$
  - Types
    - f: $DxD \rightarrow D$
    - $<x_1,\ldots x_n>$: array<D>
    - Output: D
  - Usually, f is commutative and associative

Outline

- Prefix-sum problem
  - Scan computation: generalization in which addition is replaced by an *associative* operation like *, min, max, and, or etc.
- Parallel prefix computation
  - Divide and conquer algorithms that expose parallelism that is not obvious from get-go
- Applications of parallel prefix computation
  - Many seemingly sequential problems can be parallelized

The prefix-sum problem

val prefix_sum : int array -> int array

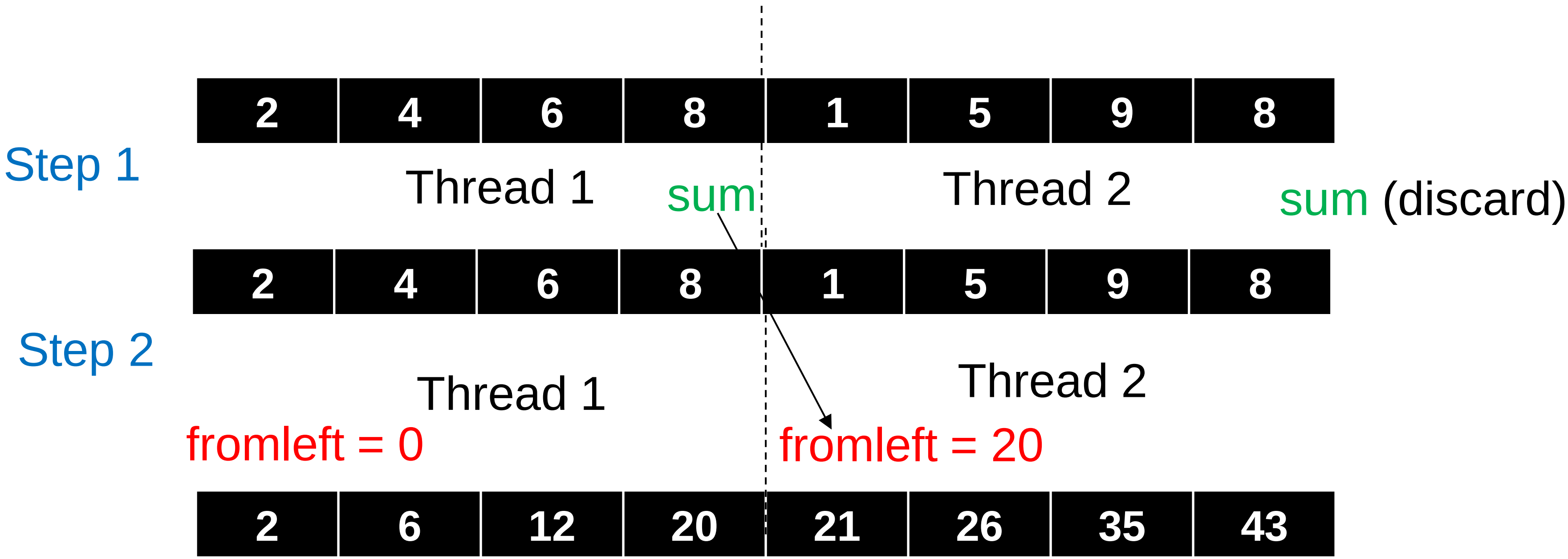
input	6	4	16	10	16	14	2	8
output	6	10	26	36	52	66	68	76

fromleft

The simple sequential algorithm: accumulate the sum from left to right

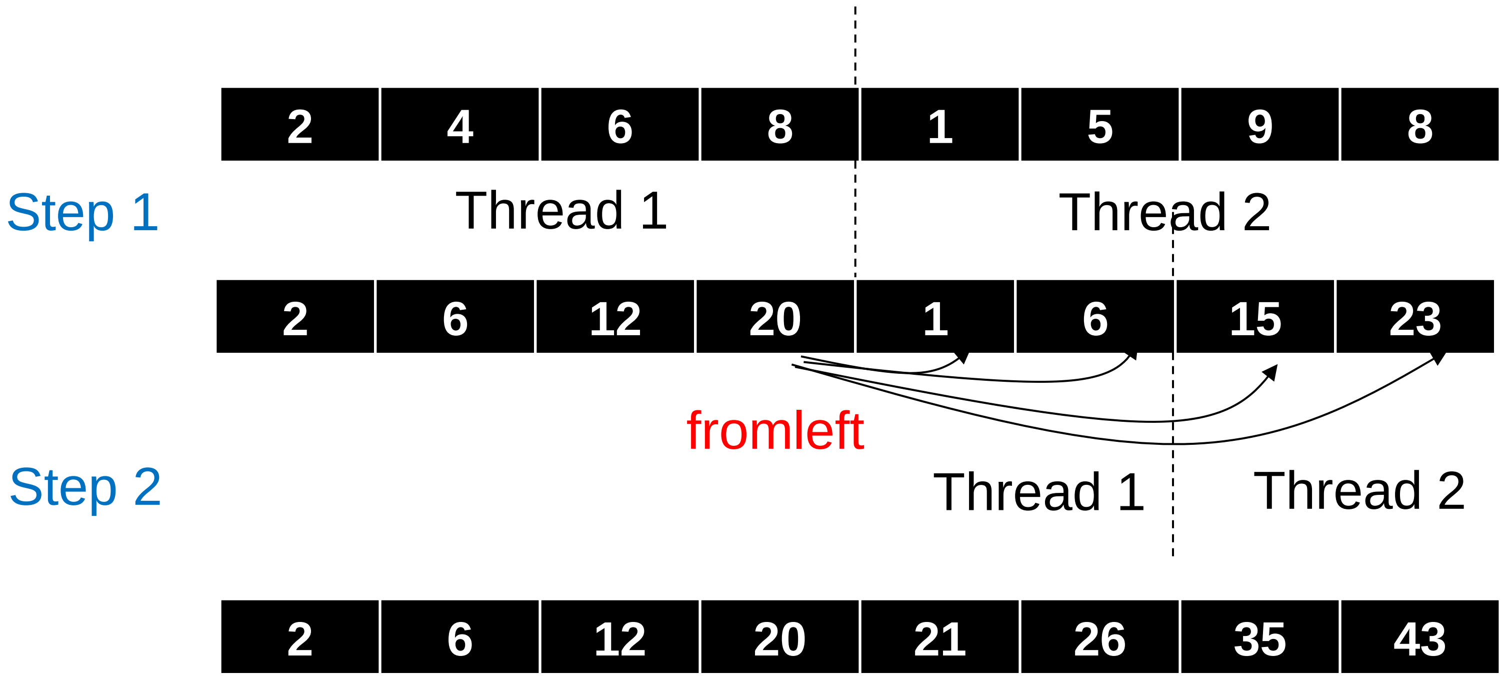
- Sequential algorithm: Work: $O(n)$, Span: $O(n)$
- Goal: a parallel algorithm with Work: $O(n)$, Span: O(log n)

Parallelization: two threads



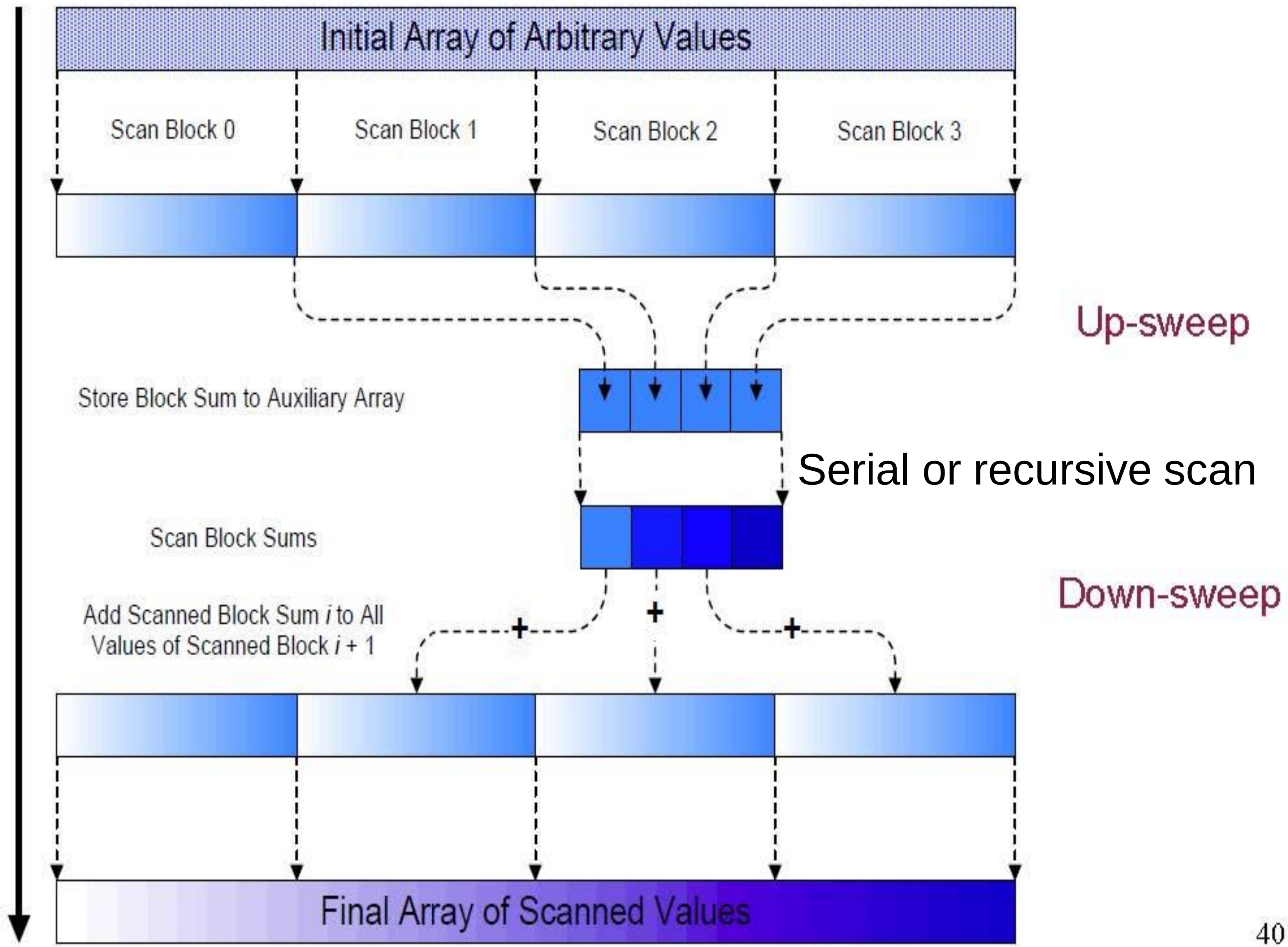
- Step 1: each thread computes sum of left/right half of array in parallel without updating array
- Step 2:
  - fromleft values
    - fromleft = 0 for Thread 1
    - fromleft = sum from Thread 1 for Thread 2
  - compute prefix-sum for left and right sub-arrays, using fromleft values to initialize the prefix-sum computations

Alternative strategy



- Step 1: threads compute prefix-sum for left and right halves of array in parallel using some algorithm (say sequential algorithm)
- Step 2: add final element from first half to elements of second half
  - Divide work between threads
  - Block partitioning so no ping-ponging of cache lines
- We will go with the first strategy since it is easier to generalize to more threads

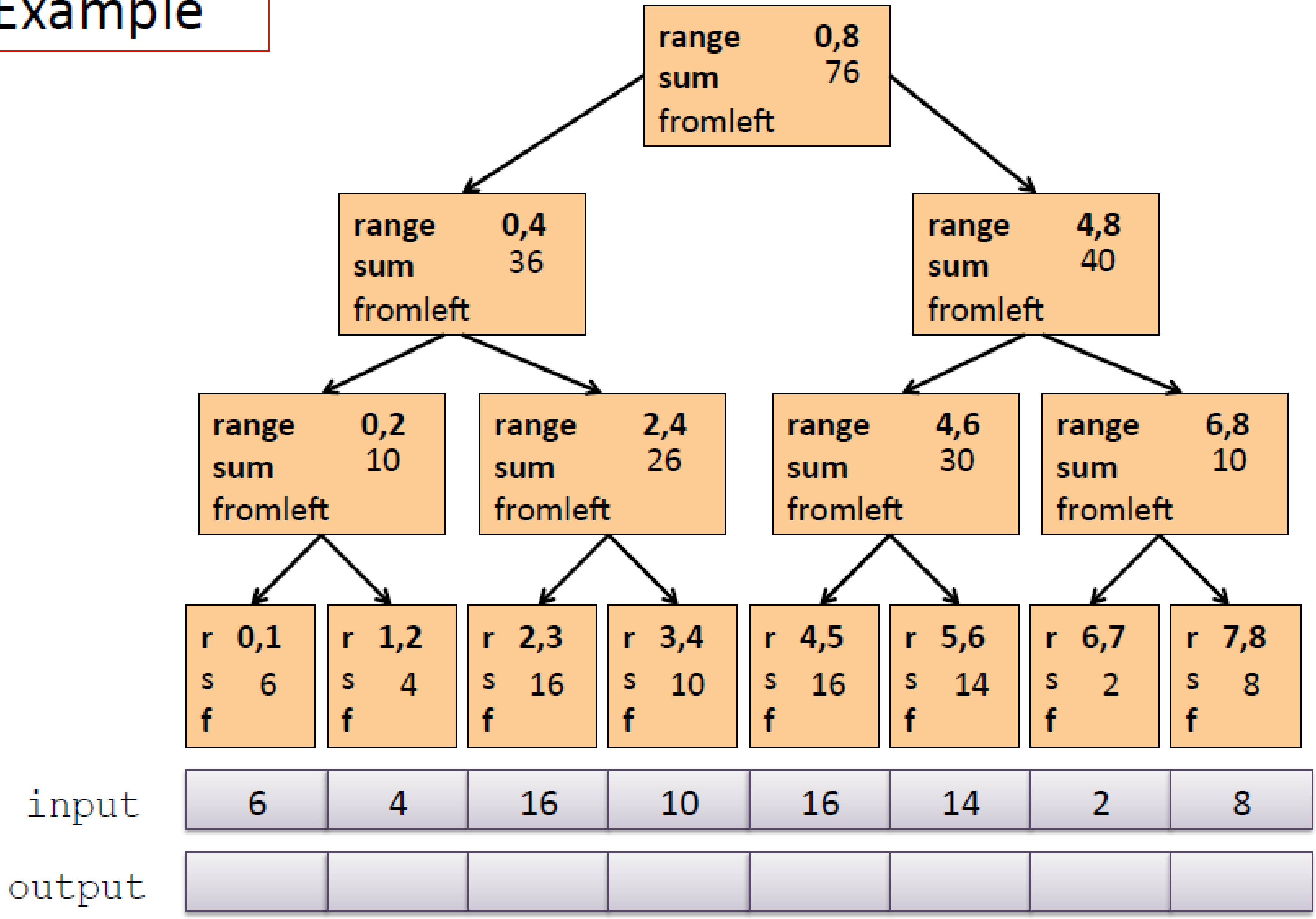
Generalize to t (=4) threads



In the limit

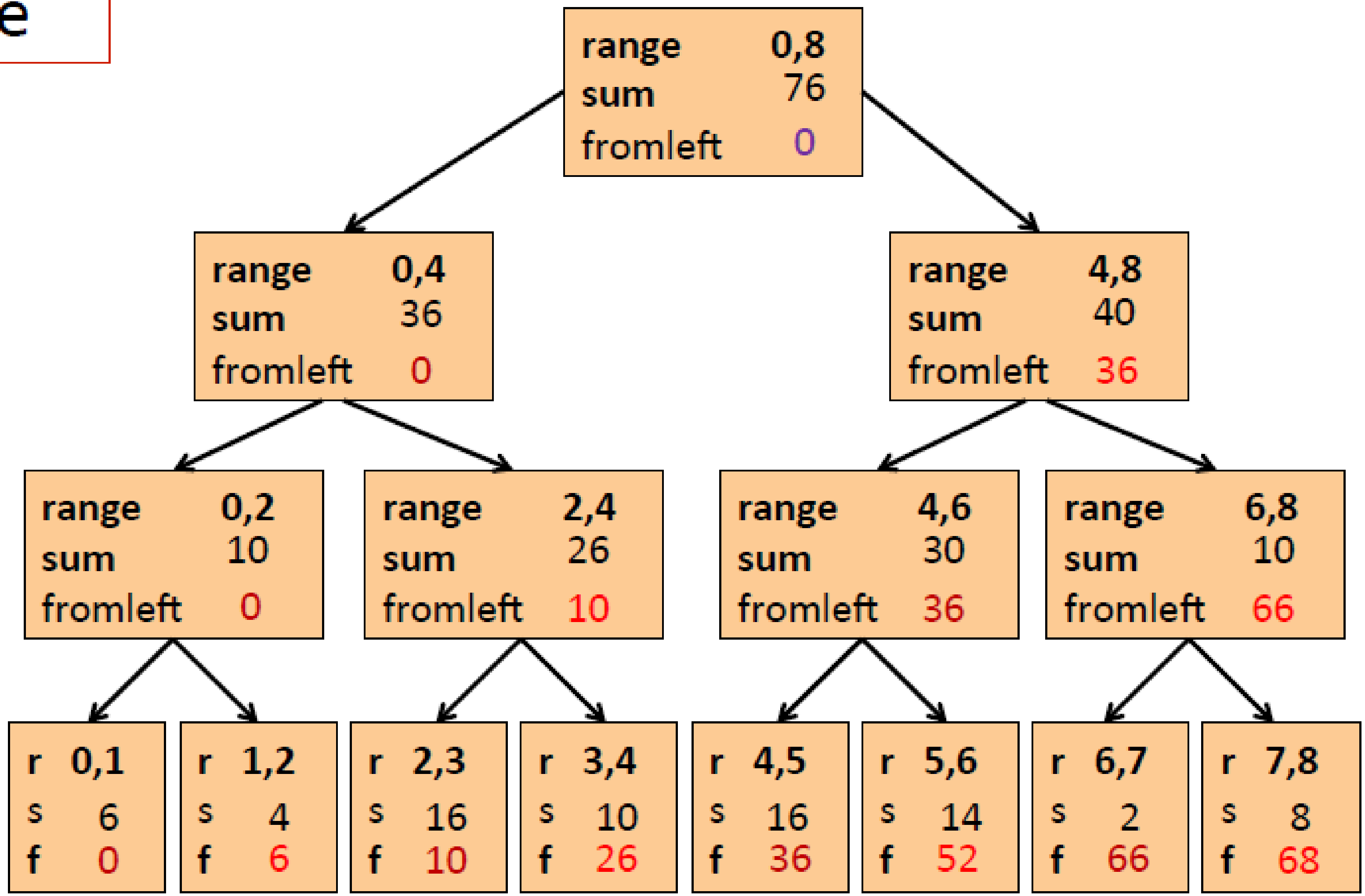
- Assume large array, unbounded # of processors
- Up-sweep:
  - Divide input array into segments of length 2
  - Collect from-left values from each segment into another array like in previous slide
  - This array will be large too so perform previous two steps recursively on this array as well
  - Recursion stops when from-left array is size 1
- Down-sweep:
  - Update from-left arrays successively

Example



range 0,8
sum 76
fromleft
range 0,4
sum 36
fromleft
range 4,8
sum 40
fromleft
range 0,2
sum 10
fromleft
range 2,4
sum 26
fromleft
range 4,6
sum 30
fromleft
range 6,8
sum 10
fromleft
r 0,1
s 6
f
r 1,2
s 4
f
r 2,3
s 16
f
r 3,4
s 10
f
r 4,5
s 16
f
r 5,6
s 14
f
r 6,7
s 2
f
r 7,8
s 8
f
input
6
4
16
10
16
14
2
8
output

Example



input	6	4	16	10	16	14	2	8
output	6	10	26	36	52	66	68	76

The algorithm, pass 1

1. Up: Build a binary tree where
   - Root has sum of the range [**`x,y`**)
   - If a node has sum of [**`lo,hi`**) and **`hi>lo`**,
     - Left child has sum of [**`lo,middle`**)
     - Right child has sum of [**`middle,hi`**)
     - A leaf has sum of [**`i,i+1`**), i.e., **`input[i]`**

This is an easy parallel divide-and-conquer algorithm: "combine" results by actually building a binary tree with all the range-sums

- Tree built bottom-up in parallel

Analysis: $O(n)$ work, $O(\mathbf{log}\ n)$ span

The algorithm, pass 2

2. Down: Pass down a value **`fromLeft`**
   - Root given a **`fromLeft`** of 0
   - Node takes its **`fromLeft`** value and
     - Passes its left child the same **`fromLeft`**
     - Passes its right child its **`fromLeft`** plus its left child's **`sum`**
       - as stored in part 1
   - At the leaf for array position **`i`**,
     - **`output[i]=fromLeft+input[i]`**

This is an easy parallel divide-and-conquer algorithm: traverse the tree built in step 1 and produce no result

- Leaves assign to **`output`**
- Invariant: **`fromLeft`** is sum of elements left of the node's range

Analysis: $O(n)$ work, $O(\mathbf{log}\ n)$ span

Sequential cut-off

For performance, we need a sequential cut-off:

- Up:

  just a sum, have leaf node hold the sum of a range

- Down:

```
output.(lo) = fromLeft + input.(lo);
for i=lo+1 up to hi-1 do
  output.(i) = output.(i-1) + input.(i)
```

Parallel prefix, generalized

Just as map and reduce are the simplest examples of a common pattern, prefix-sum illustrates a pattern that arises in many, many problems

- Minimum, maximum of all elements *to the left of* `i`
- Is there an element *to the left of* `i` satisfying some property?
- Count of elements *to the left of* `i` satisfying some property
  - This last one is perfect for an efficient parallel filter …
  - Perfect for building on top of the "parallel prefix trick"

Filter

Given an array **`input`**, produce an array **`output`** containing only elements such that (**`f elt)`** is **`true`**

Example: let f x = x > 10

```
    filter f <17, 4, 6, 8, 11, 5, 13, 19, 0, 24>
 == <17, 11, 13, 19, 24>
```

Parallelizable?

- Finding elements for the output is easy
- *But getting them in the right place seems hard*

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements

```
input  <17, 4, 6, 8, 11, 5, 13, 19, 0, 24>
bits   <1,  0, 0, 0,  1, 0,  1,  1, 0,  1>
```

2. Parallel-prefix sum on the bit-vector

```
bitsum <1,  1, 1, 1,  2, 2,  3,  4, 4,  5>
```

3. Parallel map to produce the output

```
output <17, 11, 13, 19, 24>
```

Quicksort review

Recall quicksort was sequential, in-place, expected time $O(n \texttt{ log } n)$

	Best / expected case *work*
1. Pick a pivot element	O(1)
2. Partition all the data into:	O(n)
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	2T(n/2)

How should we parallelize this?

Quicksort

	Best / expected case *work*
1. Pick a pivot element	$O(1)$
2. Partition all the data into:	$O(n)$
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	$2T(n/2)$

Easy: Do the two recursive calls in parallel

- Work: unchanged. Total: $O(n \log n)$
- Span: now $T(n) = O(n) + 1T(n/2) = O(n)$

Doing better

We get a $O(\texttt{log}\ n)$ speed-up with an *infinite* number of processors. That is a bit underwhelming

- Sort 10^9 elements 30 times faster

(Some) Google searches suggest quicksort cannot do better because the partition cannot be parallelized

- The Internet has been known to be wrong ☺
- But we need auxiliary storage (no longer in place)
- In practice, constant factors may make it not worth it

Already have everything we need to parallelize the partition…

Parallel partition (not in place)

Partition all the data into:

A. The elements less than the pivot
B. The pivot
C. The elements greater than the pivot

This is just two filters!

- We know a parallel filter is $O(n)$ work, $O(\log n)$ span
- Parallel filter elements less than pivot into left side of **`aux`** array
- Parallel filter elements greater than pivot into right size of **`aux`** array
- Put pivot between them and recursively sort
- With a little more cleverness, can do both filters at once but no effect on asymptotic complexity

With $O(\log n)$ span for partition, the total best-case and expected-case span for quicksort is

$$T(n) = O(\log n) + 1T(n/2) = O(\log^2 n)$$

Example

Step 1: pick pivot as median of three

8	1	4	9	**0**	3	5	2	7	**6**

Steps 2a and 2c (combinable): filter less than, then filter greater than into a second array

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	**7**

Step 3: Two recursive sorts in parallel

– Can copy back into original array (like in mergesort)

More Algorithms

- To add multi precision numbers.
- To evaluate polynomials
- To solve recurrences.
- To implement radix sort
- To delete marked elements from an array
- To dynamically allocate processors
- To perform lexical analysis. For example, to parse a program into tokens.
- To search for regular expressions. For example, to implement the UNIX grep program.
- To implement some tree operations. For example, to find the depth of every vertex in a tree
- To label components in two dimensional images.

See Guy Blelloch "Prefix Sums and Their Applications"