The PRAM Model and Algorithms

Advanced Topics Spring 2008
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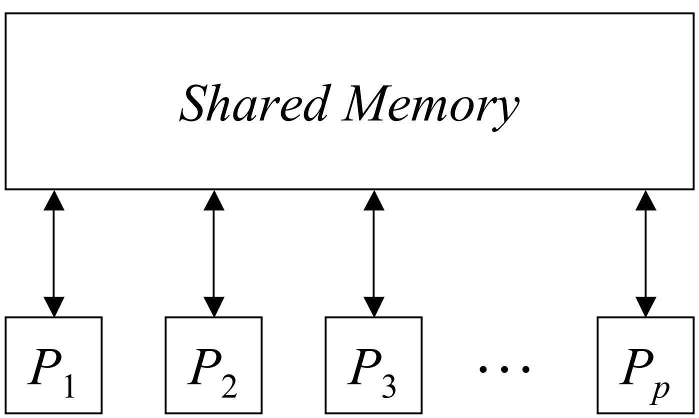
Overview

- The PRAM model of parallel computation
- Simulations between PRAM models
- Work-time presentation framework of parallel algorithms
- Example algorithms

The PRAM Model of Parallel Computation

- Parallel Random Access Machine (PRAM)
- Natural extension of RAM: each processor is a RAM
- Processors operate synchronously
- Earliest and best-known model of parallel computation



Shared memory with m locations

p processors, each with private memory

All processors operate synchronously, by executing load, store, and operations on data

Synchronous PRAM

- Synchronous PRAM is a SIMD-style model
  - All processors execute the same program
  - All processors execute the same PRAM step instruction stream in "lock-step"
  - Effect of operation depends on local data
  - Instructions can be selectively disabled (if-then-else flow)
- Asynchronous PRAM
  - Several competing models
  - No lock-step

Classification of PRAM Model

- A PRAM step ("clock cycle") consists of three phases
  1. *Read*: each processor may read a value from shared memory
  2. *Compute*: each processor may perform operations on local data
  3. *Write*: each processor may write a value to shared memory
- Model is refined for concurrent read/write capability
  - Exclusive Read Exclusive Write (EREW)
  - Concurrent Read Exclusive Write (CREW)
  - Concurrent Read Concurrent Write (CRCW)
- CRCW PRAM
  - Common CRCW: all processors must write the same value
  - Arbitrary CRCW: one of the processors succeeds in writing
  - Priority CRCW: processor with highest priority succeeds in writing

Comparison of PRAM Models

- A model *A* is less powerful compared to model *B* if either
  - The time complexity is asymptotically less in model *B* for solving a problem compared to *A*
  - Or the time complexity is the same and the work complexity is asymptotically less in model *B* compared to *A*
- From weakest to strongest:
  - EREW
  - CREW
  - Common CRCW
  - Arbitrary CRCW
  - Priority CRCW

Simulations Between PRAM Models

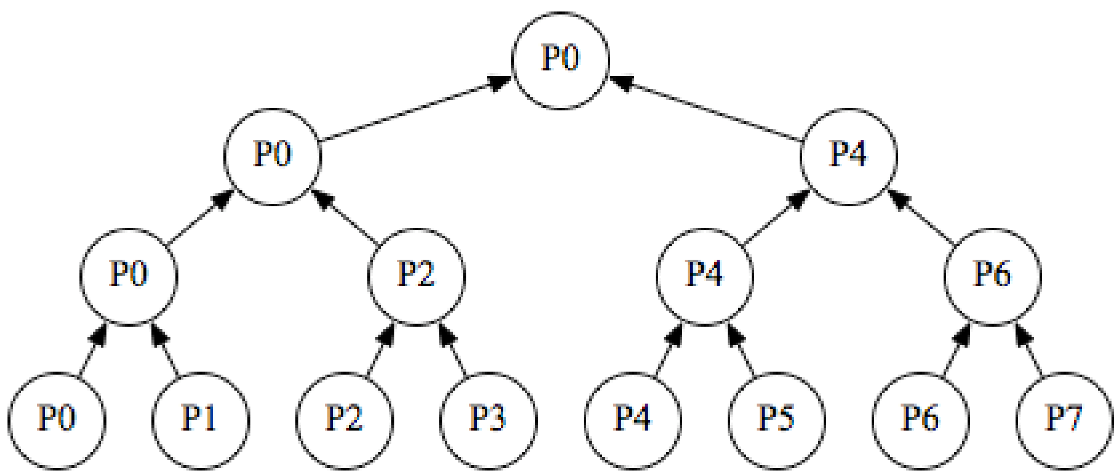
- An algorithm designed for a weaker model can be executed within the same time complexity and work complexity on a stronger model
- An algorithm designed for a stronger model can be *simulated* on a weaker model, either with
  - Asymptotically more processors (more work)
  - Or asymptotically more time

Simulating a Priority CRCW on an EREW PRAM

- Theorem: An algorithm that runs in T time on the p-processor priority CRCW PRAM can be simulated by EREW PRAM to run in $O(T \log p)$ time
  - A concurrent read or write of an p-processor CRCW PRAM can be implemented on a p-processor EREW PRAM to execute in $O(\log p)$ time
  - $Q_1,\ldots,Q_p$ CRCW processors, such that Q_i has to read (write) $M[j_i]$
  - $P_1,\ldots,P_p$ EREW processors
  - $M_1,\ldots,M_p$ denote shared memory locations for special use
  - P_i stores $<j_i,i>$ in M_i
  - Sort pairs in lexicographically non-decreasing order in $O(\log p)$ time using EREW merge sort algorithm
  - Pick representative from each block of pairs that have same first component in $O(1)$ time
  - Representative P_i reads (writes) from $M[k]$ with $<k,\_>$ in M_i and copies data to each M in the block in O(log p) time using EREW segmented parallel prefix algorithm
  - P_i reads data from M_i

Reduction on the EREW PRAM

- Reduce p values on the p-processor EREW PRAM in O(log p) time
- Reduction algorithm uses exclusive reads and writes
- Algorithm is the basis of other EREW algorithms



Sum on the EREW PRAM

Sum of n values using n processors (i)
Input: $A[1,\ldots,n]$, $n = 2^k$
Output: S
begin
  $B[i] := A[i]$
  for $h = 1$ **to** $\log n$ **do**
    if $i \leq n/2^h$ **then**
      $B[i] := B[2i\text{-}1] + B[2i]$
  if $i = 1$ **then**
    $S := B[i]$
end

Matrix Multiplication

- Consider $n \times n$ matrix multiplication with n^3 processors
- Each $c_{ij} = \sum_{k=1..n} a_{ik}\, b_{kj}$ can be computed on the CREW PRAM in parallel using n processors in O(log n) time

- On the EREW PRAM exclusive reads of a_{ij} and b_{ij} values can be satisfied by making n copies of a and b, which takes O(log n) time with n processors (broadcast tree)
- Total time is still O(log n)
- Memory requirement is huge

Matrix Multiplication on the CREW PRAM

Matrix multiply with n^3 processors (i,j,l)
Input: $n \times n$ matrices A and B, $n = 2^k$
Output: $C = AB$
begin
  $C'[i,j,l] := A[i,l]B[l,j]$
  for $h = 1$ **to** $\log n$ **do**
    if $i \leq n/2^h$ **then**
      $C'[i,j,l] := C'[i,j,2l\text{-}1] + C'[i,j,2l]$
  if $l = 1$ **then**
    $C[i,j] := C'[i,j,1]$
end

The WT Scheduling Principle

- The work-time (WT) scheduling principle schedules p processors to execute an algorithm
  - Algorithm has $T(n)$ time steps
  - A time step can be parallel, i.e. **pardo**
- Let $W_i(n)$ be the number of operations (work) performed in time unit i, $1 \leq i \leq T(n)$
- Simulate each set of $W_i(n)$ operations in $\lceil W_i(n)/p \rceil$ parallel steps, for each $1 \leq i \leq T(n)$
- The p-processor PRAM takes

$$\sum_i \lceil W_i(n)/p \rceil \leq \sum_i (\lfloor W_i(n)/p \rfloor + 1) \leq \lfloor W(n)/p \rfloor + T(n)$$

steps, where $W(n)$ is the total number of operations

Work-Time Presentation

- The WT presentation can be used to determine computation and communication requirements of an algorithm
- The upper-level WT presentation framework describes the algorithm in terms of a sequence of time units
- The lower-level follows the WT scheduling principle

Matrix Multiplication on the CREW PRAM WT-Presentation

Input: $n \times n$ matrices A and B, $n = 2^k$
Output: $C = AB$
begin
  for $1 \leq i, j, l \leq n$ **pardo**
    $C'[i,j,l] := A[i,l]B[l,j]$
  for $h = 1$ **to** $\log n$ **do**
    for $1 \leq i, j \leq n,\ 1 \leq l \leq n/2^h$ **pardo**
      $C'[i,j,l] := C'[i,j,2l-1] + C'[i,j,2l]$
  for $1 \leq i, j \leq n$ **pardo**
    $C[i,j] := C'[i,j,1]$
end

WT scheduling principle: $O(n^3/p + \log n)$ time

PRAM Recursive Prefix Sum Algorithm

Input: Array of $(x_1, x_2, \ldots, x_n)$ elements, $n = 2^k$
Output: Prefix sums s_i, $1 \leq i \leq n$
begin
  if n = 1 **then** $s_1 = x_1$; **exit**
  for $1 \leq i \leq n/2$ **pardo**
    $y_i := x_{2i-1} + x_{2i}$
  Recursively compute prefix sums of y and store in z
  for $1 \leq i \leq n$ **pardo**
   if i is even **then** $s_i := z_{i/2}$
   else if $i = 1$ **then** $s_1 := x_1$
   else $s_i := z_{(i-1)/2} + x_i$
end

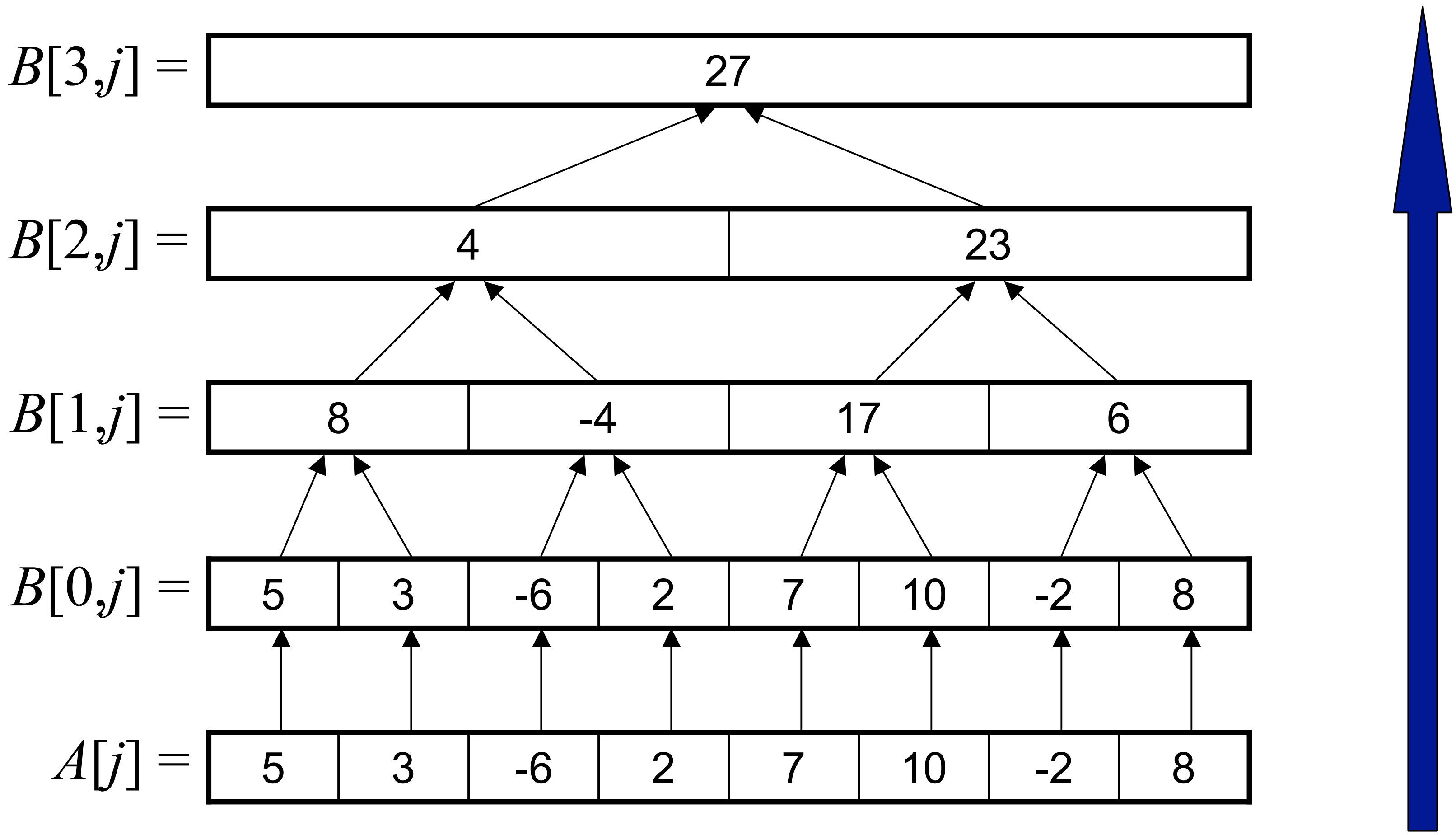
Proof of Work Optimality

- Theorem: The PRAM prefix sum algorithm correctly computes the prefix sum and takes $T(n) = O(\log n)$ time using a total of $W(n) = O(n)$ operations
- Proof by induction on k, where input size $n = 2^k$
  - Base case $k = 0$: $s_1 = x_1$
  - Assume correct for $n = 2^k$
  - For $n = 2^{k+1}$
    - For all $1 \leq j \leq n/2$ we have
      $z_j = y_1 + y_2 + \ldots + y_j = (x_1 + x_2) + (x_3 + x_4) \ldots + (x_{2j-1} + x_{2j})$
    - Hence, for $i = 2j \leq n$ we have $s_i = s_{2j} = z_j = z_{i/2}$
    - And $i = 2j+1 \leq n$ we have $s_i = s_{2j+1} = s_{2j} + x_{2j+1} = z_j + x_{2j+1} = z_{(i-1)/2} + x_i$
  - $T(n) = T(n/2) + a \quad \Rightarrow T(n) = O(\log n)$
  - $W(n) = W(n/2) + bn \quad \Rightarrow W(n) = O(n)$

PRAM Nonrecursive Prefix Sum

Input: Array A of size $n = 2^k$
Output: Prefix sums in $C[0,j]$, $1 \leq j \leq n$
begin
  for $1 \leq j \leq n$ **pardo**
    $B[0,j] := A[j]$
  for $h = 1$ **to** $\log n$ **do**
    for $1 \leq j \leq n/2^h$ **pardo**
      $B[h,j] := B[h\text{-}1,2j\text{-}1] + B[h\text{-}1,2j]$
  for $h = \log n$ **to** 0 **do**
    for $1 \leq j \leq n/2^h$ **pardo**
      if j is even **then** $C[h,j] := C[h+1,j/2]$
      else if $i = 1$ **then** $C[h,1] := B[h,1]$
      else $C[h,j] := C[h+1,(j\text{-}1)/2] + B[h,j]$
end

First Pass: Bottom-Up

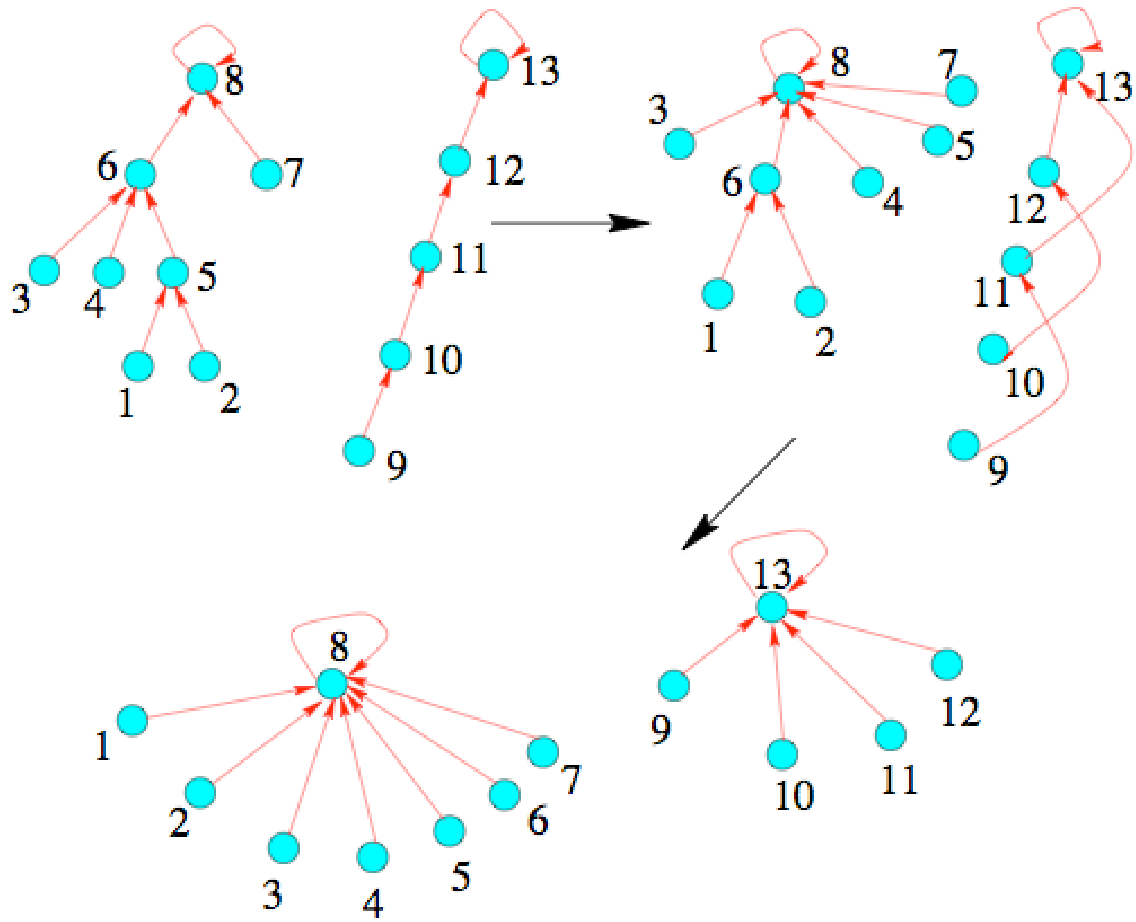


Second Pass: Top-Down

<table>
<tr><td>$B[3,j] =$</td><td colspan="8">27</td></tr>
<tr><td>$C[3,j] =$</td><td colspan="8">27</td></tr>
<tr><td>$B[2,j] =$</td><td colspan="4">4</td><td colspan="4">23</td></tr>
<tr><td>$C[2,j] =$</td><td colspan="4">4</td><td colspan="4">27</td></tr>
<tr><td>$B[1,j] =$</td><td colspan="2">8</td><td colspan="2">-4</td><td colspan="2">17</td><td colspan="2">6</td></tr>
<tr><td>$C[1,j] =$</td><td colspan="2">8</td><td colspan="2">4</td><td colspan="2">21</td><td colspan="2">27</td></tr>
<tr><td>$B[0,j] =$</td><td>5</td><td>3</td><td>-6</td><td>2</td><td>7</td><td>10</td><td>-2</td><td>8</td></tr>
<tr><td>$C[0,j] =$</td><td>5</td><td>8</td><td>2</td><td>4</td><td>11</td><td>21</td><td>19</td><td>27</td></tr>
<tr><td>$A[j] =$</td><td>5</td><td>3</td><td>-6</td><td>2</td><td>7</td><td>10</td><td>-2</td><td>8</td></tr>
</table>

Pointer Jumping

- Finding the roots of a forest using pointer-jumping



Pointer Jumping on the CREW PRAM

Input: A forest of trees, each with a self-loop at its root, consisting of arcs $(i,P(i))$ and nodes i, where $1 \leq i \leq n$
Output: For each node i, the root $S[i]$
begin
  for $1 \leq i \leq n$ **pardo**
    $S[i] := P[i]$
    while $S[i] \neq S[S[i]]$ **do**
      $S[i] := S[S[i]]$
end

$T(n) = O(\log h)$ with h the maximum height of trees
$W(n) = O(n \log h)$

PRAM Model Summary

- PRAM removes algorithmic details concerning synchronization and communication, allowing the algorithm designer to focus on problem properties
- A PRAM algorithm includes an explicit understanding of the operations performed at each time unit and an explicit allocation of processors to jobs at each time unit
- PRAM design paradigms have turned out to be robust and have been mapped efficiently onto many other parallel models and even network models
  - A SIMD network model considers *communication diameter*, *bisection width*, and *scalability* properties of the network topology of a parallel machine such as a mesh or hypercube

Further Reading

- An Introduction to Parallel Algorithms, by J. JaJa, 1992