Egalitarian Resource Sharing Over Multiple Rounds*

Fu Li C. Gregory Plaxton Vaibhav B. Sinha

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Abstract

It is often beneficial for agents to pool their resources in order to better accommodate fluctuations in individual demand. Many multi-round resource allocation mechanisms operate in an online manner: in each round, the agents specify their demands for that round, and the mechanism determines a corresponding allocation. In this paper, we focus instead on the offline setting in which the agents specify their demand for each round at the outset. We formulate a specific resource allocation problem in this setting, and design and analyze an associated mechanism based on the solution concept of lexicographic maximin fairness. We present an efficient implementation of our mechanism, and prove that it is envy-free, non-wasteful, resource monotonic, population monotonic, and group strategyproof. We also prove that our mechanism guarantees each agent at least half of the utility that they can obtain by not sharing their resources. We complement these positive results by proving that no maximin fair mechanism can improve on the aforementioned factor of one-half.

1 Introduction

Problems related to computational resource allocation lie at the intersection of economics and computer science, and have received a lot of attention in the literature. In particular, the theory of fair division, including such concepts as the egalitarian equivalent rule, provides a suitable game-theoretic framework for tackling modern technological challenges arising in cloud computing environments. This connection has inspired several mechanisms with strong game-theoretic properties, including mechanisms for coping with fluctuating demands [5, 8, 27], and for allocating multiple resource types such as CPU, disk, and bandwidth [10, 11] when agents do not know their resource demands [13] or when there is a stream of resources [1].

In this paper, we consider a group of agents sharing a single type of resource (e.g., a collection of identically-equipped servers in the cloud) over a set of rounds. (Later we allow for the possibility that the total supply of the shared resources may vary from one round to the next.) Each agent owns a specific fraction of the shared resources, and reports a demand for each round. For a given round, an agent accrues utility equal to the (possibly

*Department of Computer Science, University of Texas at Austin. Email: {fuli, plaxton, vaibhavsinha}@utexas.edu.

fractional) number of units allocated to them, as long as the allocation does not exceed their demand; any allocation beyond this threshold does not provide additional utility. We assume that no monetary exchange occurs between the agents, as is often the case in resource sharing applications (e.g., within a single organization). Even in this simple setting, there are many interesting questions that can be investigated. Indeed, a number of works (e.g., [8, 12, 13, 27]) have proposed and analyzed allocation mechanisms for this setting. These works have emphasized the natural online variant in which the agent demands for any given round are not revealed until the start of that round. In some applications, it may be possible to accurately estimate future demands, e.g., due to periodicity. Can we design mechanisms that effectively exploit such (partial) knowledge of future demands? From a theoretical perspective, a natural starting point for addressing this question is to consider the offline variant in which all of the future demands are known at the outset; this is the approach taken in the present paper.

Strategic (coalitions of) agents might misreport their demands to try to achieve higher utility, often at the expense of other agents. For example, an agent might under-report their demand in a given round, hoping for any loss of utility in that round to be more than offset by the net gain realized in the remaining rounds. Accordingly, we seek to design strategyproof (SP) or group strategyproof (GSP) mechanisms that incentivize truthful reporting.

An allocation satisfies the sharing incentives (SI) property if it ensures that each agent achieves utility at least as high as they can achieve by not sharing their resources. The mechanism that allocates resources to the agents in proportion to their relative endowments (ownership shares) is GSP and SI. Such a mechanism can be wasteful in the sense that it can allocate resources to an agent in excess of their demand while leaving the demand of another agent unmet. Thus, we seek to design mechanisms that only produce non-wasteful (NW) allocations.

Given an allocation, we say that agent a envies agent a' if a prefers the allocation of a' (scaled to account for the relative endowments of a and a') to their own. An allocation is envy-free (EF) if no agent envies another. An allocation is frugal if it does not allocate more resources to an agent than they demand. We consider two notions of fair allocations: lexicographically maximin fair (LMMF), and a weaker notion, maximin fair (MMF).

We seek mechanisms that are resource monotonic (RM), that is, if the supply of one or more resources is increased, no agent experiences a decrease in utility. We seek mechanisms that are also population monotonic (PM), that is, if the endowment of one or more agents is decreased, no other agent experiences a decrease in utility.

For the offline setting, we present an egalitarian mechanism for allocating resources to agents over multiple rounds and provide an efficient algorithm to compute the allocation. Our mechanism is frugal, LMMF, GSP, $\frac{1}{2}$-SI (a relaxation of SI), NW, EF, RM, and PM. We also show that there is no MMF z-SI mechanism for any $z > \frac{1}{2}$. The significance of our work is discussed in greater detail below, but first we review relevant prior work.

Related work. A lexicographic maximin solution maximizes the minimum utility, and subject to this, maximizes the second-lowest utility, and subject to this, maximizes the third-lowest utility, and so on. Lexicographic maximin solutions have been studied in many area of research, including computing the nucleolus of cooperative games [23], combinatorial optimization [2], network flows [18, 19], and as one of the standard fairness concepts in telecommunications and network applications [21, 22]. For more details, we refer the reader to the recent work of Ogryczak et al. [20, Section 2.1]. Below we briefly discuss the works that are most relevant to the present paper.

Some widely used online schedulers (e.g., the fair scheduler implemented in Hadoop and Spark) enforce LMMF. In the online setting, there are two senses in which we can seek to achieve the LMMF property: static and dynamic. In the static sense, we produce an LMMF allocation for each round independently. In the dynamic sense, we produce an allocation for any given round that enforces LMMF over the entire history up to that round (subject to the constraint that the allocations determined for previous rounds cannot be changed).

Our work is inspired by Freeman et al. [8], who studied the game-theoretic aspects of online resource sharing, with a primary focus on the SP, SI, and NW properties. They prove that the static version discussed above satisfies these desiderata, while the dynamic version fails to satisfy SI and SP. They then consider a more general utility function, where agents derive a fixed "high" utility per unit of resource up to their demands and a fixed "low" utility beyond that threshold. With this utility function, Freeman et al. show that the three aforementioned properties are incompatible in a dynamic setting and thus appropriate trade-offs need to be considered. They propose two mechanisms that partly satisfy the desiderata. Hossain [12] has subsequently presented another mechanism for this setting.

Kandasamy et al. [13] study mechanism design for online resource sharing when agents do not know their resource requirements. Like Freeman et al., they focus on satisfying the SP, SI, and NW properties. Tang et al. [27] propose a dynamic allocation policy for the online setting that is similar to the dynamic version of LMMF.

Bogomolnaia and Moulin [3] study random assignment problems with dichotomous preferences from a game-theoretical perspective. Dichotomous preferences can be viewed as a special case of fractional demands. Bogomolnaia and Moulin consider several mechanisms, including the LMMF mechanism. They prove that the latter mechanism is GSP, EF, RM, PM, and fair-share (the special case of SI where all demands are zero or infinite). Compared with the model of Bogomolnaia and Moulin, our setting allows for fractional demands, unequal agent endowments, and an unequal supply of resources between rounds.

The work of Katta and Sethuraman [14] addresses the random assignment problem with general agent preferences (i.e., where indifference is allowed in the agent preferences). They use parametric network flow to achieve LMMF for the special case of dichotomous preferences. For general preferences, they extend the parametric flow algorithm to compute an EF and ordinally efficient assignment, and they prove that no mechanism is SP, EF, and ordinally efficient.

Ghodsi et al. [11] consider the LMMF mechanism in the context of a random assignment problem where the agent endowments need not be the same. Our work strengthens their SP result to GSP while allowing for fractional demands.

Offline resource sharing has been studied in the context of multiperiod resource allocation with equal agent endowments [16, 17]. This line of research is focused on the design of efficient algorithms for computing a lexicographic maximin solution (via linear programming), as opposed to analyzing the associated game-theoretic properties.

Offline resource sharing can be viewed as the problem of allocating different kinds of substitutable resources to different populations of agents. This allocation problem has been studied in various specific settings, e.g., distribution of coal among power companies [4], multiperiod manufacturing of high-tech products [15], and allocation of vaccines to different populations [25]. To the best of our knowledge, the prior work in this area studies this allocation problem from a computational perspective, rather than a game-theoretic perspective. Sethuraman's survey paper on house allocation problems [24] discusses the connection between allocation with substitutable resources and random assignment with dichotomous preferences.

Significance of our work. Freeman et al. [8] study the game-theoretic properties of several online resource allocation mechanisms: the previously known static and dynamic LMMF mechanisms, and the newly-proposed Flexible Lending and T-period mechanisms [8]. In settings where future demands are known, or can be accurately estimated, we can hope to significantly improve upon the fairness guarantees of such online mechanisms. As a simple example, consider an instance with n agents and n rounds, where each agent contributes a single unit per round. Suppose agents 2 through n each demand two units in every round, and agent 1 demands n units in round 1 and no units thereafter. Clearly, an egalitarian allocation gives a utility of n to every agent. On the other hand, all of the aforementioned online mechanisms give agent 1 a utility of 1, which is only a $1/n$ fraction of the egalitarian share. Our first main contribution is to provide an efficient implementation of a suitable egalitarian mechanism for the offline setting (i.e., where future demands are known); see the first part of Section 3.

Our second main contribution is to establish various fundamental game-theoretic properties of our egalitarian mechanism. To do so, we leverage a connection between the random assignment problem of Bogomolnaia and Moulin [3] and our resource sharing problem. Specifically, they assume that the agent preferences are dichotomous, which corresponds to the special case of our setting in which the fractional demands of the agents are all 0 or 1. While the work of Bogomolnaia and Moulin provides us with an invaluable roadmap, we need to overcome some technical challenges in order to handle arbitrary fractional demands. (We also handle unequal agent endowments and unequal supplies over the rounds, but generalizing our results in these directions is quite straightforward.) In Section 3.1, we establish a number of useful structural properties of lexicographic maximin allocations. In Section 3.2, we use these structural properties to establish various game-theoretic properties of frugal LMMF allocations. In Section 3.3, we establish that our egalitarian mechanism (and in fact any frugal LMMF mechanism) is GSP.

Our third main contribution is to establish possibility and impossibility results related to the SI property. The SI property is of particular importance in the setting of resource sharing, where we need to ensure that agents are not discouraged from pooling their resources. In Section 3.2, we show that any frugal LMMF allocation is $\frac{1}{2}$-SI. In Section 4, we show that, for any $z > \frac{1}{2}$, no mechanism is MMF and z-SI. Since no mechanism is MMF and SI, we consider a natural relaxation: mechanisms that are MMF subject to being SI. (In other words, we require the mechanism to be SI, and we only enforce the MMF property with respect to the set of SI allocations.) In Section 4, we show that no such mechanism is SP. All of our proofs appear in the appendix.

2 Preliminaries

As discussed in the introduction, we wish to model a setting in which a number of agents share a set of resources over multiple rounds, and where the demand of each agent in each round is known in advance. For the purposes of our formal presentation, we find it convenient to refer to the pool of available resources in a given round as an object. (We will allow the size of this pool to vary from one round to the next; see the notion of "supply" defined below.) Thus, we use k objects to model a k-round instance. Since we are restricting attention to the offline setting, the ordering of these objects is immaterial.

For any set of agents A, we define endowments(A) as the set of all endowment functions $\alpha : A \to \mathbb{R}_{>0}$, and for any subset A' of A we define $\alpha(A')$ as $\sum_{a \in A'} \alpha(a)$.[1] For any set of objects B, we define supplies(B) as the set of all supply functions $\beta : B \to \mathbb{R}_{\geq 0}$.

For any set of agents A and any set of objects B, we define demands(A, B) as the set of all demand functions $d : A \times B \to \mathbb{R}_{\geq 0}$. For any subset A' of A and any d in demands(A, B), $d_{A'}$ denotes the demand function in demands(A', B) such that $d_{A'}(a, b) = d(a, b)$ for all agents a in A' and all objects b in B.

For any set of agents A, any set of objects B, any α in endowments(A), any β in supplies(B), and any d in demands(A, B), the tuple (A, B, α, β, d) denotes an instance of object allocation with fractional demands (OAFD). We think of each agent a in A as owning a $\alpha(a)/\alpha(A)$ fraction of each object in B.

For any OAFD instance $I = (A, B, \alpha, \beta, d)$, we define allocs($I$) as the set of all allocation functions $\mu : A \times B \to \mathbb{R}_{\geq 0}$ such that $\mu(A, b) \leq \beta(b)$ for all objects b in B.

An OAFD mechanism M takes as input an OAFD instance I and outputs a subset $M(I)$ of allocs(I).[2]

For any OAFD instance $I = (A, B, \alpha, \beta, d)$, and any μ in allocs(I), we define the utility of agent a from object b as $u(\mu, d, a, b) = \min(\mu(a, b), d(a, b))$. We assume that the utility of any agent a, denoted $u(\mu, d, a)$, is equal to $\sum_{b \in B} u(\mu, d, a, b)$.

For any OAFD instance $I = (A, B, \alpha, \beta, d)$, any μ in allocs(I), and any b in B, the definition of utility implies that $\sum_{a \in A} u(\mu, d, a, b) \leq \min(\beta(b), d(A, b))$. We let β_I in supplies(I) denote the supply such that $\beta_I(b) = \min(\beta(b), d(A, b))$ for all objects b in B. For any OAFD instance $I = (A, B, \alpha, \beta, d)$, any μ in allocs(I), and any subset A' of A, the definition of utility also implies that $\sum_{a \in A'} u(\mu, d, a) \leq \sum_{b \in B} \min(\beta_I(b), d(A', b))$. We let cap($I, A'$) denote $\sum_{b \in B} \min(\beta_I(b), d(A', b))$.

Game-theoretic desiderata for allocations. For any OAFD instance $I = (A, B, \alpha, \beta, d)$, an allocation μ in allocs(I) is (proportionally) envy-free (EF) if

$$u(\mu, d, a) \geq \sum_{b \in B} \min\left(\frac{\alpha(a)}{\alpha(a')}\mu(a', b), d(a, b)\right)$$

[1] In the remainder of the paper, we implicitly define similar overloads for a number of other functions associated with supply, demand, allocation, flow, and capacity.

[2] In the present paper, it is convenient to assume that the output of an OAFD mechanism is a set of allocations, as opposed to a single allocation, because the OAFD mechanism $\mathcal{M}$ that we present in Section 3 has this characteristic. For any OAFD instance $I = (A, B, \alpha, \beta, d)$, all of the agents in A are indifferent between the allocations in $\mathcal{M}(I)$. For any OAFD instance I, our efficient implementation of $\mathcal{M}$ computes a single allocation in $\mathcal{M}(I)$.

for all agents a and a' in A. Intuitively, no agent prefers the appropriately scaled (i.e., taking into account relative endowments) version of another agent's allocation to their own allocation.

The sharing incentives (SI) property requires that any agent a who provides a truthful report achieves utility at least as high as they would achieve with an $\alpha(a)/\alpha(A)$ fraction of every object. Formally, for any OAFD instance $I = (A, B, \alpha, \beta, d)$ and any z in $[0, 1]$, an allocation μ in allocs(I) is said to be z-SI if $u(\mu, d, a) \geq z \sum_{b \in B} \min\left(\frac{\alpha(a)}{\alpha(A)}\beta(b), d(a, b)\right)$ for all agents a in A. We say that an allocation is SI if it is 1-SI.

In our model, the maximum utility that an agent a can achieve from an object b is $d(a, b)$; accordingly, in our setting, there is no reason to allocate more than $d(a, b)$ units of object b to agent a. For any OAFD instance $I = (A, B, \alpha, \beta, d)$, we say that an allocation μ in allocs(I) is frugal if $\mu(a, b) \leq d(a, b)$ for all (a, b) in $A \times B$. We let frugal(I) denote the set of all frugal allocations in allocs(I). For any OAFD instance $I = (A, B, \alpha, \beta, d)$ and any μ in frugal(I), we say that μ is non-wasteful (NW) if for any object b in B, either $\mu(A, b) = \beta(b)$ or $\mu(a, b) = d(a, b)$ for all agents a in A.

An allocation μ in allocs(I) that maximizes $\min_{a \in A} u(\mu', d, a)/\alpha(a)$ over all μ' in allocs(I) is said to be maximin fair (MMF). We let MMF(I) denote the set of all MMF allocations in allocs(I). We let $\mathbf{u}(I, \mu)$ denote the length-$|A|$ vector whose jth component denotes the jth smallest $u(\mu, d, a)/\alpha(a)$ for all agents a in A. An allocation μ in allocs(I) is lexicographically maximin fair (LMMF) if $\mathbf{u}(I, \mu)$ is lexicographically at least $\mathbf{u}(I, \mu')$ for all μ' in allocs(I). We let LMMF(I) denote the set of all LMMF allocations in allocs(I). Note that LMMF is a stricter notion of fairness than MMF.

For any OAFD instance $I = (A, B, \alpha, \beta, d)$, any subset A' of A, and any μ in LMMF(I), we let sub(I, A', μ) denote the OAFD instance $(A \setminus A', B, \alpha', \beta', d_{A \setminus A'})$ where $\alpha'(a) = \alpha(a)$ for all agents a in $A \setminus A'$ and $\beta'(b) = \beta(b) - \mu(A', b)$ for all objects b in B. Lemma 1 below establishes an optimal substructure property of LMMF allocations.

Lemma 1. *Let $I = (A, B, \alpha, \beta, d)$ be an* OAFD *instance, let A' be a subset of A, and let μ belong to* LMMF(I). *Let μ' be the restriction of μ to $A \setminus A'$, that is, $\mu' : (A \setminus A') \times B \to \mathbb{R}_{\geq 0}$ is such that $\mu'(a, b) = \mu(a, b)$ for all (a, b) in $(A \setminus A') \times B$. Then μ' belongs to* LMMF(sub(I, A', μ)).

Game-theoretic desiderata for mechanisms. In order to define the strategyproof (SP) and group strategyproof (GSP) properties, it is convenient to first define the k-SP property for any given positive integer k. An OAFD mechanism is k-SP if no coalition of k agents can misrepresent their demands in such a way that some member of the coalition gains and no member of the coalition loses. Formally, an OAFD mechanism M is said to be k-SP if for any OAFD instance $I = (A, B, \alpha, \beta, d)$, any μ in $M(I)$, any subset A' of A such that $|A'| = k$, any d^* in demands(A', B), any OAFD instance $I' = (A, B, \alpha, \beta, d')$ where $d' = (d_{A \setminus A'}, d^*)$, and any μ' in $M(I')$, either there is no agent a in A' such that $u(\mu, d, a) < u(\mu', d, a)$, or there is an agent a in A' such that $u(\mu, d, a) > u(\mu', d, a)$. A mechanism is SP if it is 1-SP. A mechanism is GSP if it is k-SP for all k.

An OAFD mechanism is said to be resource monotonic (RM) if increasing the supply of one or more objects does not decrease the utility of any agent. Formally, an OAFD mechanism M is said to be RM if for any instances $I = (A, B, \alpha, \beta, d)$ and $I' = (A, B, \alpha, \beta', d)$

such that $\beta(b) \leq \beta'(b)$ for all objects b in B, any agent a in A, any allocation μ in $M(I)$, and any allocation μ' in $M(I')$, we have $u(\mu, d, a) \leq u(\mu', d, a)$.

An OAFD mechanism is said to be population monotonic (PM) if decreasing the endowments of one or more agents does not decrease the utility of any other agent. Formally, given any instance $I = (A, B, \alpha, \beta, d)$, we define shrink$(I)$ as the set of all OAFD instances $I' = (A', B, \alpha', \beta, d_{A'})$ such that A' is a subset of A and $\alpha'(a) \leq \alpha(a)$ for all agents a in A'. An OAFD mechanism M is said to be PM if for any OAFD instances $I = (A, B, \alpha, \beta, d)$ and $I' = (A', B, \alpha', \beta, d_{A'})$ in shrink(I), any allocations μ in $M(I)$ and μ' in $M(I')$, and any agent a in A' such that $\alpha'(a) = \alpha(a)$, we have $u(\mu, d, a) \leq u(\mu', d, a)$.

An OAFD mechanism M is EF (resp., NW, z-SI) if for any OAFD instance I, every allocation in $M(I)$ is EF (resp., NW, z-SI). An OAFD mechanism M is frugal (resp., MMF, LMMF) if for any OAFD instance I, the set of allocations $M(I)$ is contained in frugal(I) (resp., MMF(I), LMMF(I)).

Lexicographic flow. We now briefly review the lexicographic flow problem, which we utilize to obtain an efficient implementation of our mechanism. We refer readers unfamiliar with the parametric maximum flow to Appendix A for more details.

Given a flow network $G = (V, E)$ with source s and sink t, and a subset S of $V - t$ such that s is in S, we write $(S, \overline{S})$ to denote the associated cut of G. There is a minimum cut $(S, \overline{S})$ such that S contains S' for all minimum cuts $(S', \overline{S'})$. We refer to this minimum cut $(S, \overline{S})$ as the source-heavy minimum cut.

In this paper, we consider parametric flow networks where each edge leaving s has a capacity proportional to a parameter λ and all other edge capacities are independent of λ. For any parametric flow network G, we let $G(\lambda)$ denote the flow network associated with a particular value of λ. We use the standard terminology of breakpoints as defined in Gallo et al. [9]. As the value of λ increases from 0, the vertices in $V \setminus \{s, t\}$ move from the sink side to the source side of the source-heavy minimum cut. For any parametric flow network G, the breakpoint function $\Lambda(v)$ maps any given vertex v in $V \setminus \{s, t\}$ to the breakpoint value of λ at which v moves from the sink side to the source side of the source-heavy minimum cut [26].

We now define the notion of a lexicographic flow [18, 19]. Assume that the edges leaving s reach the vertices $\{v_1, \ldots, v_k\}$, and that t does not belong to this set. Let the capacity of the edge (s, v_i) be $w_i\lambda$. For a flow f in $G(\infty)$, let $\theta(G, f)$ denote the length-k vector whose jth component is the jth smallest $f(s, v_i)/w_i$, for i in $[k]$. A lexicographic flow f of G is a maximum flow f in $G(\infty)$ that is lexicographically at least $\theta(G, f')$ for all maximum flows f' in $G(\infty)$. Gallo et al. describe an algorithm that computes the breakpoint function and a lexicographic flow in $O(|V||E| \log(|V|^2/|E|))$ time [9].

3 Frugal Lexicographic Maximin Fair Mechanism

Let $\mathcal{M}$ denote the OAFD mechanism such that $\mathcal{M}(I) = \text{LMMF}(I) \cap \text{frugal}(I)$ for all OAFD instances I. In Section 3.1 we establish that all agents are indifferent between allocations in $\mathcal{M}(I)$ (see Lemma 4). In this section, we describe an efficient non-deterministic algorithm $\mathcal{A}$ that implements $\mathcal{M}$ in the following sense: on any input OAFD instance I, the set of possible allocations produced by $\mathcal{A}$ is $\mathcal{M}(I)$. The algorithm $\mathcal{A}$ is based on a reduction to the lexicographic flow problem on a parametric flow network. Given an OAFD instance $I = (A, B, \alpha, \beta, d)$ as input, algorithm $\mathcal{A}$ first creates a parametric flow network $G_I = (A \cup B \cup \{s, t\}, E)$ with the edge capacities defined by the functions α, d, and β_I described below. The network G_I has an agent (resp., object) vertex for each agent (resp., object) in the input. We denote the set of agent (resp., object) vertices by A (resp., B). For any agent vertex a and any object vertex b, there is a directed edge of capacity $\alpha(a)\lambda$ from s to a, there is a directed edge of capacity $d(a, b)$ from a to b, and there is a directed edge of capacity $\beta_I(b)$ from b to t. It is easy to check that Observation 1 below holds.

Observation 1. *For any* OAFD *instance $I = (A, B, \alpha, \beta, d)$, there is a one-to-one correspondence between flows f in $G_I(\infty)$ and allocations μ in* frugal(I) *such that $f(a, b) = \mu(a, b)$ for all (a, b) in $A \times B$.*

Given as input an OAFD instance I, algorithm $\mathcal{A}$ non-deterministically selects a lexicographic flow in $G_I(\infty)$ and outputs the corresponding allocation in allocs(I). The algorithm of Gallo et al. can be used to compute a lexicographic flow in $O((|A| + |B|)|A||B| \log((|A| + |B|)^2/|A||B|))$ time. Using Observation 1, it is straightforward to prove that there is a one-to-one correspondence between frugal LMMF allocations in $\mathcal{M}(I)$ and lexicographic flows in G_I. Thus we obtain Lemma 2 below.

Lemma 2. *For any* OAFD *instance I, the set of possible allocations produced by algorithm $\mathcal{A}$ on input I is equal to $\mathcal{M}(I)$.*

We introduce some notations that are helpful in analysis of mechanism $\mathcal{M}$. The algorithm of Gallo et al. to find the lexicographic flow computes the breakpoint function Λ_I such that $\Lambda_I(a)$ denotes the breakpoint at which agent vertex a moves from the sink side to the source side of the source-heavy minimum cut, for all agent vertices a in A. Let $\text{num}(I)$ denote $|\{\Lambda_I(a) \mid a \in A\}|$. For any i in $[\text{num}(I)]$, let $\text{brkpts}(I, i)$ denote the ith smallest value in $\{\Lambda_I(a) \mid a \in A\}$, and let $\text{agents}(I, i)$ denote the set $\{a \in A \mid \Lambda_I(a) \leq \text{brkpts}(I, i)\}$. We define $\text{agents}(I, 0)$ as $\emptyset$. For any i in $[\text{num}(I)]$, and object b in B, let $\text{cap}(I, i, b)$ denote $\beta_I(b) - d(\text{agents}(I, i-1), b)$. With $\text{objects}(I, 0)$ defined as $\emptyset$, for any i in $[\text{num}(I)]$, let $\text{objects}(I, i)$ be recursively defined as the union of $\text{objects}(I, i-1)$ and

$$\{b \in B \setminus \text{objects}(I, i-1) \mid d(\text{agents}(I, i) \setminus \text{agents}(I, i-1), b) > \text{cap}(I, i, b)\}.$$

3.1 Technical Properties of Mechanism $\mathcal{M}$

In this section, we establish some basic technical results concerning mechanism $\mathcal{M}$. These results are used in Section 3.2 (resp., Section 3.3) to derive certain game-theoretic properties of frugal LMMF allocations (resp., mechanisms). Throughout this section, let $I = (A, B, \alpha, \beta, d)$ denote an OAFD instance and let G denote G_I. We let Λ and k denote the breakpoint function Λ_I and the value $\text{num}(I)$, respectively. For any i in $[k]$, we let λ_i, A_i, and B_i denote $\text{brkpts}(I, i)$, $\text{agents}(I, i)$, and $\text{objects}(I, i)$, respectively. In addition, let A_0 denote $\text{agents}(I, 0)$ and let B_0 denote $\text{objects}(I, 0)$. For any i in $[k]$ and any object b in B, we let $c_i(b)$ denote $\text{cap}(I, i, b)$. For any i in $[k]$, and any non-empty subset A' of $A \setminus A_{i-1}$, let $C_i(A')$ denote $\sum_{b \in B \setminus B_{i-1}} \min(c_i(b), d(A', b))$. Notice that for any non-empty subset A' of A, we have $C_1(A') = \text{cap}(I, A')$.

Lemma 3 characterizes the values of the breakpoints of G, and the breakpoint associated with each agent vertex. It also establishes a connection between a lexicographic flow in G and the sets $A_1, \ldots, A_k, B_1, \ldots, B_k$. The result of Lemma 3 is similar in spirit to Theorem 4.6 of Megiddo [18] for general parametric flow networks. Since we work with parametric flow networks with a special structure, we are able to obtain a more specific result and we can characterize a lexicographic flow in greater detail. Our proof of Lemma 3 is not based on Megiddo's proof; instead, we provide a simpler proof for our special case. Our formulation of Lemma 3 generalizes Megiddo's result in one aspect, since it allows for agents with different endowments; this generalization is straightforward.

Consider a sequence of parametric flow networks $G_1, \ldots, G_k$, where G_i is the subgraph of G induced by $(A \setminus A_{i-1}) \cup (B \setminus B_{i-1}) \cup \{s, t\}$, except that for any object vertex b in G_i, the capacity of edge (b, t) is defined to be $c_i(b)$. Remark: It follows easily from Lemma 3 below that $c_i(b) \geq 0$.

For any i in $[k]$, we define the following predicates: $\Gamma_1(i)$ denotes "the minimum breakpoint of any agent vertex in G_i is λ_i"; $\Gamma_2(i)$ denotes "λ_i is equal to $\min_{\emptyset \neq \tilde{A} \subseteq A \setminus A_{i-1}} C_i(\tilde{A})/\alpha(\tilde{A})$"; $\Gamma_3(i)$ denotes "A_i is equal to $A_{i-1} \cup \bigcup\{\tilde{A} \subseteq A \setminus A_{i-1} \mid C_i(\tilde{A}) = \alpha(\tilde{A})\lambda_i\}$"; $\Gamma_4(i)$ denotes "for any lexicographic flow f in G, we have $f(a, b) = d(a, b)$ for all (a, b) in $(A_i \setminus A_{i-1}) \times (B \setminus B_i)$"; $\Gamma_5(i)$ denotes "for any lexicographic flow f in G, we have $f(A_i, b) = f(b, t) = \beta_I(b)$ and $f(a, b) = 0$ for all (a, b) in $(A \setminus A_i) \times (B_i \setminus B_{i-1})$."

Lemma 3. *Predicate $\Gamma_j(i)$ holds for all i in $[k]$ and all j in $\{1, \ldots, 5\}$.*

We now prove some results about frugal LMMF allocations. Throughout the remainder of the section, let μ denote an allocation in $\mathcal{M}(I)$. The definition of mechanism $\mathcal{M}$ implies that μ is frugal and LMMF. Recall that algorithm $\mathcal{A}$ first computes a lexicographic flow f in G such that $\mu(a, b) = f(a, b)$ for all (a, b) in $A \times B$.

Corollaries 1 and 2 below describe structural properties of the allocation μ that follow immediately from predicates Γ_4 and Γ_5, respectively.

Corollary 1. *For any i in $[k]$, any agent a in $A_i \setminus A_{i-1}$, and any object b in $B \setminus B_i$, we have $\mu(a, b) = d(a, b)$.*

Corollary 2. *For any i in $[k]$, any agent a in $A \setminus A_i$, and any object b in B_i, we have $\mu(a, b) = 0$.*

Lemma 4 below establishes some basic results that are useful for many of our subsequent proofs. For example, we use Lemma 4 along with Corollaries 1 and 2 to prove that any frugal LMMF allocation is EF (Theorem 2).

Lemma 4. *Let a be an agent in A and let b be an object in B. Then, $\mu(a, b)$ belongs to $[0, d(a, b)]$, and $u(\mu, d, a) = \mu(a, B) = \alpha(a)\Lambda(a)$.*

We use Lemma 5 below, along with Lemma 4 and Corollary 1, to prove that any frugal LMMF allocation is $\frac{1}{2}$-SI (Theorem 3).

Lemma 5. *Let i be in $[k]$. Then $\sum_{j \in [i]} \alpha(A_j \setminus A_{j-1})\lambda_j = \beta(B_i) + d(A_i, B \setminus B_i)$.*

We use Lemma 6 below, along with Lemma 4 and the result that any frugal LMMF allocation is NW (Theorem 1), to prove that any frugal LMMF mechanism is RM (Theorem 4). We use Lemmas 1, 3, and 4 and the result that any frugal LMMF mechanism is RM (Theorem 4) to prove that any frugal LMMF mechanism is PM (Theorem 5).

Lemma 6. *Let i be in $[k]$, let a (resp., a') be an agent in A_i (resp., $A \setminus A_i$), and let b be an object in B such that $\mu(a', b) > 0$. Then $\mu(a, b) = d(a, b)$.*

We use Lemma 7 below, along with Lemmas 1, 3, and 4 and the result that any frugal LMMF mechanism is RM (Theorem 4), to prove that any frugal LMMF mechanism is GSP (Theorem 6).

Lemma 7. *Let i belong to $[k]$ and let μ' be an allocation in $\text{allocs}(I)$ such that $u(\mu', d, a) \geq \alpha(a)\Lambda(a)$ for all agents a in A_i. Then $\mu'(A_i, b) \geq \mu(A_i, b)$ for all objects b in B.*

3.2 Game-Theoretic Properties of Frugal LMMF Allocations

In this section we establish some game-theoretic properties of frugal LMMF allocations. Throughout this section, let $I = (A, B, \alpha, \beta, d)$ be an OAFD instance and let μ belong to $\mathcal{M}(I)$. The definition of $\mathcal{M}$ implies that μ is an arbitrary frugal LMMF allocation.

Theorem 1. *Allocation μ is NW.*

Theorem 2 below shows that any frugal LMMF allocation is EF. Bogomolnaia and Moulin [3] show that any frugal LMMF allocation is EF when all demands are 0 or 1, all agent endowments are equal, and all object supplies are equal. To generalize this result to our setting, the main issue is to handle fractional demands; Corollaries 1 and 2 are useful in this regard.

Theorem 2. *Allocation μ is EF.*

Theorem 3 below shows that any frugal LMMF allocation is $\frac{1}{2}$-SI. Lemma 8 in Section 4 implies that no frugal LMMF mechanism is z-SI for any $z > \frac{1}{2}$.

Theorem 3. *Allocation μ is $\frac{1}{2}$-SI.*

3.3 Game-Theoretic Properties of Frugal LMMF Mechanisms

In this section, we establish that any frugal LMMF mechanism is RM (Theorem 4), PM (Theorem 5), and GSP (Theorem 6). Bogomolnaia and Moulin [3] prove these properties for the special case where the demands and supplies are all 0-1. To handle arbitrary fractional demands and supplies, we employ a similar high-level proof framework, with some additional low-level details. We now sketch our RM proof, which proceeds by contradiction. (See Appendix B.11 for the full proof.) Let $I = (A, B, \alpha, \beta, d)$ and $I' = (A, B, \alpha, \beta', d)$ denote OAFD instances such that $\beta(b) \leq \beta'(b)$ for all objects b in B, let μ belong to $\mathcal{M}(I)$, and let μ' belong to $\mathcal{M}(I')$. We first use Lemmas 4 and 6 to prove that for any agents a and a', if there is an object b such that $\mu'(a, b) < \mu(a, b)$ and $\mu'(a', b) > \mu(a', b)$, then $\Lambda_{I'}(a') \leq \Lambda_{I'}(a)$ and $\Lambda_I(a) \leq \Lambda_I(a')$. Next, to derive a contradiction, we consider the set A' of agents who suffer a loss from switching I to I'. A straightforward counting argument shows that there is an agent a' in $A \setminus A'$ and an object b' such that $\mu'(a, b') < \mu(a, b')$ and $\mu'(a', b') > \mu(a', b')$, implying that $\Lambda_{I'}(a') \leq \Lambda_{I'}(a)$ and $\Lambda_I(a) \leq \Lambda_I(a')$. Since agent a suffers a loss from switching I to I', we conclude that a' also suffers a loss from switching I to I'. This is a contradiction since a' belongs to $A \setminus A'$ and A' contains all agents who suffer a loss from switching I to I'.

With regard to establishing the PM and GSP properties, a key difference between our setting and that of Bogomolnaia and Moulin may be illustrated by considering the set of agents receiving the minimum utility, i.e., agents$(I, 1)$. When all of the demands are either 0 or 1, the objects demanded by agents in agents$(I, 1)$ are not available to the remaining agents, and hence there is a clean partitioning of the objects into the subset demanded by agents in agents$(I, 1)$ and the remaining objects. However, in our setting, the objects fractionally demanded by agents in agents$(I, 1)$ may still be partly available for the remaining agents, which allows for a more complicated interplay between the two subproblems. We use some new ideas to cope with this added complexity. For example, in the main case (Case 2) of our proof of Theorem 5 and in the main case (Case 4) of our proof of Theorem 6, we find it convenient to leverage the RM property established in Theorem 4. We remark that Bogomolnaia and Moulin do not use RM to establish PM or GSP.

4 Impossibility Results

In this section, we show that fairness and SI are incompatible properties. Lemma 8 below establishes that for any $z > \frac{1}{2}$, no OAFD mechanism is z-SI and MMF. As mentioned in Section 2, MMF is a weaker notion of fairness than LMMF. Thus for any $z > \frac{1}{2}$, no OAFD mechanism is z-SI and LMMF.

Lemma 8. *For any $z > \frac{1}{2}$, no* OAFD *mechanism is z-SI and MMF.*

Since no mechanism can be MMF and SI, we consider the following natural relaxation: mechanisms that are MMF subject to being SI. Formally, for any OAFD instance $I = (A, B, \alpha, \beta, d)$, we say that an allocation μ in allocs(I) is MMF-SI if μ maximizes $\min_{a \in A} u(\mu', d, a)/\alpha(a)$ over all μ' in allocs(I) such that μ' is SI. An OAFD mechanism M is MMF-SI if for any OAFD instance I, every allocation in $M(I)$ is MMF-SI. Lemma 9 below shows that the SP and MMF-SI properties are incompatible.

Lemma 9. *No* OAFD *mechanism is SP and MMF-SI.*

5 Concluding Remarks

In this paper, we introduced the OAFD problem and we presented a lexicographically maximin fair OAFD mechanism that enjoys a number of desirable game-theoretic properties: GSP, 1/2-SI, NW, EF, PE, RM, and PM. We also showed that no maximin fair mechanism can be z-SI for any $z > 1/2$. Further, we showed that no MMF-SI mechanism is SP.

We briefly mention some possible directions for future research. First, we have shown that our mechanism is z-SI for $z = 1/2$, but on most real world instances it might achieve z-SI for a significantly higher value of z. It would be interesting to benchmark our mechanism on real data. Second, our work assumes perfect knowledge of future demands. It would be interesting to develop mechanisms whose performance degrades gracefully as the knowledge of future demands becomes more unreliable. Finally, we have studied lexicographic maximin fairness in this paper. It would also be interesting to study other notions of fairness.

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A Lexicographic Flow

The problem of computing a maximum flow in a flow network has been extensively studied. We describe this problem and discuss a parameterized version of the problem. We utilize parametric maximum flow to propose an efficient implementation of our algorithm.

A flow network is a directed graph $G = (V, E)$ with the vertex set V, the edge set E, having a source vertex s, a sink vertex t, and a non-negative capacity $c(e)$ for each edge e in E. A function $f : E \to R_{\geq 0}$ is said to be a flow if $f(e) \leq c(e)$ (the capacity constraint for edge e) holds for each edge e in E and $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,u)\in E} f(v, u)$ (the flow conservation constraint for vertex v) holds for each vertex v in $V \setminus \{s, t\}$. The value of flow f may be defined as the net flow out of the source s. The goal of the maximum flow problem is to determine a flow of maximum value [7].

A cut of a flow network $G = (V, E)$ is a partition $(S, \overline{S})$ of V such that s belongs to S and t belongs to $\overline{S}$. The capacity of a cut $(S, \overline{S})$ is defined as the total capacity of all edges going from some vertex in S to some vertex in $\overline{S}$. A minimum cut is a cut of minimum capacity. The famous max-flow min-cut theorem states that in any flow network, the value of a maximum flow is equal to the capacity of a minimum cut. A standard result in network flow theory states that there is a minimum cut $(S, \overline{S})$ such that S contains S' for all minimum capacity cuts $(S', \overline{S'})$. We refer to this minimum cut $(S, \overline{S})$ as the source-heavy minimum cut.

In a parametric flow network, each edge capacity is a function of a parameter λ. In this paper, we restrict our attention to parametric networks where each edge leaving s has a capacity proportional to λ and all other edge capacities are independent of λ. Parametric networks have been widely studied; we refer readers to [9] for more general settings and other results. For any parametric flow network G, we let $G(\lambda)$ denote the flow network associated with a particular value of λ.

In a parametric flow network, the capacity of the minimum cut changes as the value of λ changes. We let the minimum cut capacity function $\kappa(\lambda)$ denote the capacity of the minimum cut as a function of the parameter λ. It is well known that $\kappa(\lambda)$ is a non-decreasing, concave, and piecewise-linear function with at most $|V| - 2$ breakpoints, where a breakpoint is a value of λ at which the slope of $\kappa(\lambda)$ changes [6, 26]. Each of the $|V| - 1$ or fewer line segments that form the graph of $\kappa(\lambda)$ corresponds to a cut. Notice that $\kappa(0) = 0$. As the value of λ increases, the vertices in $V \setminus \{s, t\}$ move from the sink side to the source side of the source-heavy minimum cut. For any parametric flow network G, the breakpoint function $\Lambda(v)$ maps any given vertex v in $V \setminus \{s, t\}$ to the breakpoint value of λ at which v moves from the sink side to the source side of the source-heavy minimum cut. The breakpoint function describes the sequence of cuts associated with $\kappa(\lambda)$ [26].

We now define the notion of a lexicographic flow. Assume that the edges leaving s reach the vertices $\{v_1, \ldots, v_k\}$, and that t does not belong this set. Let the capacity of the edge (s, v_i) be $w_i\lambda$. For a flow f in $G(\infty)$, let $\theta(G, f)$ denote the length-k vector whose jth component is the jth smallest $f(s, v_i)/w_i$, for i in $[k]$. A lexicographic flow f of G is a maximum flow f in $G(\infty)$ that is lexicographically at least $\theta(G, f')$ for all maximum flows f' in $G(\infty)$.

The lexicographic flow problem has been studied by Megiddo [18, 19]. An efficient algorithm for this problem was proposed by Gallo et al. [9]. We describe this algorithm here.

First, find all the breakpoints of $\kappa(\lambda)$ for a given parametric flow network G. Also, determine the breakpoint $\lambda(v_i)$ for each vertex v_i in $\{v_1, \ldots, v_k\}$ at which v_i moves from the sink side to the source side of the source-heavy minimum cut. Let G' be the flow network obtained by setting the capacity of edge (s, v_i) to $w_i\lambda(v_i)$ for each vertex v_i in $\{v_1, \ldots, v_k\}$ in G. Any maximum flow f of G' is a lexicographic flow of G. Gallo et al. describe an algorithm that computes the breakpoint function and a lexicographic flow in $O(|V||E|\log(|V|^2/|E|))$ time.

B Proofs

B.1 Proof of Lemma 1

Proof. For any OAFD instance $\hat{I} = (\hat{A}, \hat{B}, \hat{\alpha}, \hat{\beta}, \hat{d})$, any $\hat{\mu}$ in $\mathrm{LMMF}(\hat{I})$, and any subset $\hat{A}'$ of $\hat{A}$, let $\mathbf{u}(\hat{I}, \hat{\mu}, \hat{A}')$ denote the length-$|\hat{A}'|$ vector whose jth component denotes the jth smallest value of $u(\hat{\mu}, \hat{d}, a)/\alpha(a)$ over all agents a in $\hat{A}'$. Then

$$\mathbf{u}(\hat{I}, \hat{\mu}) = \mathrm{sort}(\mathbf{u}(\hat{I}, \hat{\mu}, \hat{A}') + \mathbf{u}(\hat{I}, \hat{\mu}, \hat{A} \setminus \hat{A}')),$$

where + denotes concatenation and sort is a function that sorts the input vector.

Let $\tilde{I}$ denote $\mathrm{sub}(I, A', \mu)$. Notice that μ' belongs to $\mathrm{allocs}(\tilde{I})$. Assume for the sake of contradiction that μ' does not belong to $\mathrm{LMMF}(\tilde{I})$. Let $\tilde{\mu}$ belong to $\mathrm{LMMF}(\tilde{I})$. Then $\mathbf{u}(\tilde{I}, \mu') = \mathbf{u}(I, \mu, A \setminus A') \neq \mathbf{u}(\tilde{I}, \tilde{\mu})$. Since $\tilde{\mu}$ belongs to $\mathrm{LMMF}(\tilde{I})$ and μ' does not belong to $\mathrm{LMMF}(\tilde{I})$, we deduce that $\mathbf{u}(\tilde{I}, \tilde{\mu})$ is lexicographically greater than $\mathbf{u}(I, \mu, A \setminus A')$.

Let $\mu^* : A \times B \to \mathbb{R}_{\geq 0}$ be defined by $\mu^*(a, b) = \tilde{\mu}(a, b)$ for all (a, b) in $(A \setminus A') \times B$ and $\mu^*(a, b) = \mu(a, b)$ for all (a, b) in $A' \times B$. Thus μ^* belongs to $\mathrm{allocs}(I)$ and

$$\mathbf{u}(I, \mu^*) = \mathrm{sort}(\mathbf{u}(I, \mu^*, A') + \mathbf{u}(I, \mu^*, A \setminus A')) = \mathrm{sort}(\mathbf{u}(I, \mu, A') + \mathbf{u}(\tilde{I}, \tilde{\mu}))$$

is lexicographically greater than $\mathrm{sort}(\mathbf{u}(I, \mu, A') + \mathbf{u}(I, \mu, A \setminus A')) = \mathbf{u}(I, \mu)$, a contradiction since μ belongs to $\mathrm{LMMF}(I)$. □

B.2 Proof of Lemma 2

Proof. It is straightforward to verify that the following observations hold.

Observation 2. *For any* OAFD *instance I, all allocations in* $\mathrm{LMMF}(I)$ *are NW.*

Observation 3. *For any* OAFD *instance I, the capacity of a minimum cut of $G_I(\infty)$ is $\beta_I(B)$.*

We begin by proving the following useful claim.

Claim 1: Let $I = (A, B, \alpha, \beta, d)$ be an OAFD instance. Let μ be an allocation in $\mathrm{frugal}(I)$ and let f be a flow in $G_I(\infty)$ such that $f(a, b) = \mu(a, b)$ for all (a, b) in $A \times B$. Then f is a maximum flow in $G_I(\infty)$ if and only if μ is NW.

Proof: We first prove the only if direction. Using the max-flow min-cut theorem and Observation 3, we deduce that the value of flow f is $\beta_I(B)$. Since μ is frugal and $\mu(a, b) = f(a, b)$ for all (a, b) in $A \times B$, we have $\mu(A, B) = \beta_I(B)$. Since μ is frugal and $\mu(A, B) = \beta_I(B)$, we deduce that $\mu(A, b) = \beta_I(b)$ for all objects b in B, which further implies that μ is NW.

Now, we prove the if direction. Since μ belongs to frugal(I) and μ is NW, we deduce that $\mu(A, B) = \beta_I(B)$. Thus, the value of flow f is $\beta_I(B)$. Hence the max-flow min-cut theorem and Observation 3 imply that flow f is a maximum flow in $G_I(\infty)$. This concludes the proof of Claim 1.

Let $I = (A, B, \alpha, \beta, d)$ be an OAFD instance. Let $\mathcal{A}(I)$ denote the set of possible allocations produced by algorithm $\mathcal{A}$ on input I. Since $\mathcal{M}(I) = \text{frugal}(I) \cap \text{LMMF}(I)$, it suffices to prove that $\mathcal{A}(I) = \text{frugal}(I) \cap \text{LMMF}(I)$. Thus Claims 2 and 3 below imply that the lemma holds.

Claim 2: $\text{frugal}(I) \cap \text{LMMF}(I) \subseteq \mathcal{A}(I)$.

Proof: Let μ be an allocation in $\text{frugal}(I) \cap \text{LMMF}(I)$. Observation 1 implies that there is a flow in $G_I(\infty)$, call it f, such that $\mu(a, b) = f(a, b)$ for all (a, b) in $A \times B$. Observation 2 implies that μ is NW, and hence Claim 1 implies that flow f is a maximum flow in $G_I(\infty)$. To prove that μ belongs to $\mathcal{A}(I)$, it suffices to prove that f is a lexicographic flow in G_I. Assume for the sake of contradiction that f is not a lexicographic flow in G_I. Hence there is a maximum flow f' in $G_I(\infty)$ such that $\theta(G_I, f')$ is lexicographically greater than $\theta(G_I, f)$. Observation 1 implies that there is a frugal allocation, call it μ', such that $\mu'(a, b) = f(a, b)$ for all (a, b) in $A \times B$. Since $\mu(a, b) = f(a, b)$ and $\mu'(a, b) = f'(a, b)$ for all (a, b) in $A \times B$, we deduce that $\mathbf{u}(I, \mu) = \theta(G_I, f)$ and $\mathbf{u}(I, \mu') = \theta(G_I, f')$, respectively. Since $\mathbf{u}(I, \mu) = \theta(G_I, f)$, $\mathbf{u}(I, \mu') = \theta(G_I, f')$, and $\theta(G_I, f')$ is lexicographically greater than $\theta(G_I, f)$, we deduce that $\mathbf{u}(I, \mu')$ is lexicographically greater than $\mathbf{u}(I, \mu)$, a contradiction since μ belongs to LMMF(I). This concludes the proof of Claim 2.

Claim 3: $\mathcal{A}(I) \subseteq \text{frugal}(I) \cap \text{LMMF}(I)$.

Proof: Let μ be an allocation function in $\mathcal{A}(I)$. Let f denote the lexicographic flow in G_I selected by algorithm $\mathcal{A}$; thus f corresponds to μ. Since any lexicographic flow in G_I is a maximum flow in $G_I(\infty)$, we deduce from Observation 1 that μ belongs to frugal(I). It remains to prove that μ belongs to LMMF(I). Assume for the sake of contradiction that μ does not belong to LMMF(I). Hence there is an allocation μ' in LMMF(I) such that $\mathbf{u}(I, \mu')$ is lexicographically greater than $\mathbf{u}(I, \mu)$. Let μ'' be an allocation in allocs(I) such that $\mu''(a, b) = \min(\mu'(a, b), d(a, b))$ for all (a, b) in $A \times B$. The definition of μ'' implies that μ'' belongs to frugal(I). Since the maximum utility an agent a can achieve from an object b is $d(a, b)$, we have $\mathbf{u}(I, \mu'') = \mathbf{u}(I, \mu')$. Since $\mathbf{u}(I, \mu'') = \mathbf{u}(I, \mu')$ and μ' belongs to LMMF(I), we deduce that μ'' belongs to LMMF(I). Since μ'' belongs to LMMF(I), Observation 2 implies that μ'' is NW. Since $\mathbf{u}(I, \mu')$ is lexicographically greater than $\mathbf{u}(I, \mu)$ and $\mathbf{u}(I, \mu'') = \mathbf{u}(I, \mu')$, we conclude that $\mathbf{u}(I, \mu'')$ is lexicographically greater than $\mathbf{u}(I, \mu)$. Since μ'' belongs to frugal(I) and μ'' is NW, Observation 1 and Claim 1 imply that there is a maximum flow in $G_I(\infty)$, call it f'', such that $f''(a, b) = \mu''(a, b)$ for all (a, b) in $A \times B$. Since $\mu(a, b) = f(a, b)$ and $\mu''(a, b) = f''(a, b)$ for all (a, b) in $A \times B$, we deduce that $\mathbf{u}(I, \mu) = \theta(G_I, f)$ and $\mathbf{u}(I, \mu'') = \theta(G_I, f'')$, respectively. Since $\mathbf{u}(I, \mu) = \theta(G_I, f)$, $\mathbf{u}(I, \mu'') = \theta(G_I, f'')$, and $\mathbf{u}(I, \mu'')$ is lexicographically greater than $\mathbf{u}(I, \mu)$, we deduce that $\theta(G_I, f'')$ is lexicographically greater than $\theta(G_I, f)$, a contradiction since f is a lexicographic flow in G_I. This concludes the proof of Claim 3. □

B.3 Proof of Lemma 3

Before proving Lemma 3, we show Lemma 10 below which establishes important properties of the minimum breakpoint of any agent vertex in each parametric flow network G_i. Throughout this section, we make the following definitions for all i in $[k]$: Λ_i^* denotes the breakpoint function of G_i; λ_i^* denotes the minimum breakpoint of an agent vertex in G_i; λ_i^{**} denotes $\min_{\emptyset \neq A' \subseteq A \setminus A_{i-1}} C_i(A')/\alpha(A')$; A_i^* denotes $\bigcup\{A' \subseteq A \setminus A_{i-1} \mid C_i(A') = \alpha(A')\lambda_i^*\}$; B_i^* denotes $\{b \in B \setminus B_{i-1} \mid d(A_i^*, b) > c_i(b)\}$; $\Psi_1(i)$ denotes the predicate "$\lambda_i^* = \lambda_i^{**}$"; $\Psi_2(i)$ denotes the predicate "$\Lambda_i^*(a) = \lambda_i^*$ for all agents a in A_i^*"; $\Psi_3(i)$ denotes the predicate "for any flow in $G_i(\infty)$ such that $f(s,a) = \alpha(a)\lambda_i^*$ for all agent vertices a in $A \setminus A_{i-1}$, we have $f(a,b) = d(a,b)$ for all (a,b) in $A_i^* \times ((B \setminus B_{i-1}) \setminus B_i^*)$"; $\Psi_4(i)$ denotes the predicate "for any flow in $G_i(\infty)$ such that $f(s,a) = \alpha(a)\lambda_i^*$ for all agent vertices a in $A \setminus A_{i-1}$, we have $f(A_i^*, b) = f(b,t) = c_i(b)$ for all object vertices b in B_i^*."

Lemma 10. *Let i be in $[k]$. Then predicate $\Psi_j(i)$ holds for all j in $\{1, \ldots, 4\}$.*

Proof. Recall that the set of agent (resp., object) vertices in G_i is $A \setminus A_{i-1}$ (resp., $B \setminus B_{i-1}$). We first establish the following useful claim, which implies that $\lambda_i^* \geq \lambda_i^{**}$.

Claim 1: A maximum flow in $G_i(\lambda_i^{**})$ has value $\alpha(A \setminus A_{i-1})\lambda_i^{**}$.

Proof: To prove that a maximum flow in $G_i(\lambda_i^{**})$ has value $\alpha(A \setminus A_{i-1})\lambda_i^{**}$, it is sufficient to argue that a minimum cut in $G_i(\lambda_i^{**})$ has capacity $\alpha(A \setminus A_{i-1})\lambda_i^{**}$. Let the source-heavy minimum cut in $G_i(\lambda_i^{**})$ be $(S, \overline{S})$ and let A' denote $S \cap (A \setminus A_{i-1})$. We begin by showing that $S \cap (B \setminus B_{i-1}) = \{b \in B \setminus B_{i-1} \mid d(A', b) \geq c_i(b)\}$. Assume for the sake of contradiction that this equation does not hold. We consider three cases.

Case 1: There is an object vertex b in $\overline{S} \cap (B \setminus B_{i-1})$ such that $d(A', b) > c_i(b)$. Hence the capacity of the cut $(S + b, \overline{S} - b)$ is $d(A', b) - c_i(b) > 0$ less than the capacity of the cut $(S, \overline{S})$, a contradiction since $(S, \overline{S})$ is a minimum capacity cut.

Case 2: There is an object vertex b in $S \cap (B \setminus B_{i-1})$ such that $c_i(b) > d(A', b)$. Hence the capacity of the cut $(S - b, \overline{S} + b)$ is $c_i(b) - d(A', b) > 0$ less than the capacity of the cut $(S, \overline{S})$, a contradiction since $(S, \overline{S})$ is a minimum capacity cut.

Case 3: There is an object b in $\overline{S} \cap (B \setminus B_{i-1})$ such that $c_i(b) = d(A', b)$. Hence the cuts $(S + b, \overline{S} - b)$ and $(S, \overline{S})$ have the same capacity, but $(S + b, \overline{S} - b)$ has a larger source side, a contradiction.

From the above case analysis, $S \cap (B \setminus B_{i-1}) = \{b \in B \setminus B_{i-1} \mid d(A', b) \geq c_i(b)\}$. Thus the capacity of the cut $(S, \overline{S})$ is

$$\alpha(A \setminus (A_{i-1} \cup A'))\lambda_i^{**} + \sum_{b \in B \setminus B_{i-1}} \min(c_i(b), d(A', b)) = \alpha(A \setminus (A_{i-1} \cup A'))\lambda_i^{**} + C_i(A').$$

The definition of λ_i^{**} implies that $\alpha(A')\lambda_i^{**} \leq C_i(A')$. Thus the capacity of the minimum cut is at least $\alpha(A \setminus A_{i-1})\lambda_i^{**}$. Moreover, the capacity of cut $(s, V \setminus s)$ is $\alpha(A \setminus A_{i-1})\lambda_i^{**}$. Thus the capacity of a minimum cut of $G_i(\lambda_i^{**})$ is $\alpha(A \setminus A_{i-1})\lambda_i^{**}$. This concludes the proof of Claim 1.

Let λ' be a value greater than λ_i^{**}. We show that there is no flow in $G_i(\lambda')$ such that every agent vertex a has incoming flow $\alpha(a)\lambda'$. Assume for the sake of contradiction that there is a flow such that every agent vertex a has incoming flow $\alpha(a)\lambda'$. The total capacity of the edges leaving $A_i^* \cup B_i^*$ is

$$\sum_{b \in B \setminus B_{i-1}} \min(c_i(b), d(A_i^*, b)) = C_i(A_i^*).$$

Since $C_i(A_i^*) = \alpha(A_i^*)\lambda_i^{**} < \alpha(A_i^*)\lambda'$, the total capacity of the edges leaving $A_i^* \cup B_i^*$ is less than the total flow into the set $A_i^* \cup B_i^*$, a contradiction. This result, together with Claim 1, establishes that $\Psi_1(i)$ and $\Psi_2(i)$ hold.

Let f be a flow in $G_i(\infty)$ such that $f(s,a) = \alpha(a)\lambda_i^*$ for all agent vertices a in $A \setminus A_{i-1}$ Since the total capacity of the edges leaving $A_i^* \cup B_i^*$ is $C_i(A_i^*)$, which is equal to the total flow $\alpha(A_i^*)\lambda_i^*$ into $A_i^* \cup B_i^*$ in f, we deduce that $f(e) = c(e)$ for all edges e leaving $A_i^* \cup B_i^*$. Thus $f(b,t) = c_i(b)$ for all object vertices b in B_i^*, and $f(a,b) = d(a,b)$ for all (a,b) in $A_i^* \times ((B \setminus B_{i-1}) \setminus B_i^*)$. Moreover, since the total flow into $A_i^* \cup B_i^*$ is $C_i(A_i^*) = c_i(B_i^*) + d(A_i^*, (B \setminus B_{i-1}) \setminus B_i^*)$, we have $f(A_i^*, B_i^*) = c_i(B_i^*)$. It follows that $f(A_i^*, b) = f(b,t)$ for all object vertices b in B_i^*. We conclude that $\Psi_3(i)$ and $\Psi_4(i)$ hold. $\square$

We now present a proof of Lemma 3.

Proof. Let f denote a lexicographic flow in G and for any i in $[k]$, let $P(i)$ denote the predicate "$\Gamma_j(i)$ holds for all j in $\{1,\ldots,5\}$." We prove by induction that $P(i)$ holds for all i in $[k]$.

Base case: Since $A_0 = \emptyset$, we have $c_1(b) = \beta_I(b)$ for all object vertices b in B. Thus $G_1 = G$, and hence $\Gamma_1(1)$ holds. Lemma 10 implies that $\Psi_1(1)$ and $\Psi_2(1)$ hold; hence $\Gamma_2(1)$ and $\Gamma_3(1)$ hold. Since $\Psi_3(1)$ and $\Psi_4(1)$ hold by Lemma 10, $\lambda_1^* = \lambda_1$, $A_1^* = A_1 \setminus A_0$, and $B_1^* = B_1 \setminus B_0$, we deduce that $\Gamma_4(1)$ and $\Gamma_5(1)$ hold.

Induction step: Let i belong to $\{2,\ldots,k\}$ and assume that $P(i')$ holds for all i' in $[i-1]$. We need to prove that $P(i)$ holds. Let b be an object vertex in $B \setminus B_{i-1}$. Since the induction hypothesis implies that $\Gamma_4(i')$ holds for all i' in $[i-1]$, we deduce that $f(a,b) = d(a,b)$ for all a in A_{i-1}. Thus $f(A \setminus A_{i-1}, b) \le \beta_I(b) - d(A_{i-1}, b) = c_i(b)$. Moreover, since the induction hypothesis implies that $\Gamma_5(i')$ holds for all i' in $[i-1]$, we deduce that $f(a,b') = 0$ for all (a,b') in $(A \setminus A_{i-1}) \times B_{i-1}$. From the aforementioned results, it is straightforward to verify that $\Gamma_1(i)$ holds. Lemma 10 implies that $\Psi_1(i)$ and $\Psi_2(i)$ hold; hence $\Gamma_2(i)$ and $\Gamma_3(i)$ hold. Since $\Psi_3(i)$ holds by Lemma 10, $\lambda_i^* = \lambda_i$, $A_i^* = A_i \setminus A_{i-1}$, and $B_i^* = B_i \setminus B_{i-1}$, we deduce that $\Gamma_4(i)$ holds. Let b' be an object vertex in $B_i \setminus B_{i-1}$. Predicate $\Psi_4(i)$ implies that $f(A_i \setminus A_{i-1}, b') = c_i(b') = \beta_I(b') - d(A_{i-1}, b')$. Since the induction hypothesis implies $\Gamma_4(i')$ holds for all i' in $[i-1]$, we deduce that $f(a,b') = d(a,b')$ for all a in A_{i-1}. Hence $f(A_{i-1}, b') = d(A_{i-1}, b')$. Since $f(A_{i-1}, b') = d(A_{i-1}, b')$ and $f(A_i \setminus A_{i-1}, b') = \beta_I(b') - d(A_{i-1}, b')$, we deduce that $f(A_i, b') = \beta_I(b') = f(b', t)$, where the last equality holds because the capacity of edge (b', t) is $\beta_I(b')$. Since $f(A_i, b') = \beta_I(b')$ and $\beta_I(b')$ is the capacity of edge (b', t), we deduce that $f(a,b) = 0$ for all a in $A \setminus A_i$, which establishes $\Gamma_5(i)$. We conclude that $P(i)$ holds, as required. $\square$

B.4 Proof of Lemma 4

Proof. The capacity of edge (a,b) in G is $d(a,b)$. Hence $\mu(a,b) = f(a,b)$ belongs to $[0, d(a,b)]$. Flow f satisfies $\alpha(a)\Lambda(a) = f(s,a) = f(a,B) = \mu(a,B) = u(\mu,d,a)$. $\square$

B.5 Proof of Lemma 5

Proof. We begin by proving the following useful claim.

Claim 1: Let i belong to $[k]$. Let b be an object in $B_i \setminus B_{i-1}$. Then $\beta_I(b) = \beta(b)$.

To prove Claim 1, observe that the definition of $B_i \setminus B_{i-1}$ implies that $d(A_i \setminus A_{i-1}, b) > c_i(b) = \beta_I(b) - d(A_{i-1}, b)$. Thus $d(A, b) \geq d(A_i, b) > \beta_I(b)$. Since $d(A, b) > \beta_I(b)$ and $\beta_I(b) = \min(\beta(b), d(A, b))$, we have $\beta_I(b) = \beta(b)$. This completes the proof of Claim 1.

Notice that $\sum_{j\in[i]} \alpha(A_j \setminus A_{j-1})\lambda_j$ is the total flow into A_i in f. The total flow out of A_i in f is $f(A_i, B)$. For any agent vertex a in A_i and any object vertex b in $B \setminus B_i$, Lemma 3 implies that $f(a, b) = d(a, b)$. For any object b in B_i, Lemma 3 implies that $f(A_i, b) = \beta_I(b)$. Thus $f(A_i, B) = \beta_I(B_i) + d(A_i, B \setminus B_i)$. Since the net flow into A_i is 0, we obtain $\sum_{j\in[i]} \alpha(A_j \setminus A_{j-1})\lambda_j = \beta_I(B_i) + d(A_i, B \setminus B_i) = \beta(B_i) + d(A_i, B \setminus B_i)$, where the last equality follows from Claim 1. □

B.6 Proof of Lemma 6

Proof. Corollary 2 and $\mu(a', b) > 0$ imply that b belongs to $B \setminus B_i$. Hence Corollary 1 implies that $\mu(a, b) = d(a, b)$. □

B.7 Proof of Lemma 7

Proof. For any object b in B, we let $u(b)$ (resp., $u'(b)$) denote $\sum_{a\in A_i} u(\mu, d, a, b)$ (resp., $\sum_{a\in A_i} u(\mu', d, a, b)$). We begin by establishing a useful claim.

Claim 1: We have $u'(b) \leq u(b)$ for all b in B.

Proof: Let b be an object in B. We consider two cases.

Case 1: $b \in B_i$. The definition of $\beta_I(b)$ implies that $u'(b) \leq \beta_I(b)$. Using Lemma 3 and the definition of μ, we deduce that $\mu(A_i, b) = f(A_i, b) = \beta_I(b)$. Lemma 4 implies that $\mu(a, b)$ belongs to $[0, d(a, b)]$ for all a in A_i. Using Lemma 4, we conclude that $u(b) = \mu(A_i, b) = \beta_I(b)$. Thus $u'(b) \leq u(b)$.

Case 2: $b \in B \setminus B_i$. We have $u'(b) \leq d(A_i, b)$. Using Lemma 3 and the definition of μ, we deduce that for any agent a in A_i, $\mu(a, b) = f(a, b) = d(a, b)$. Using Lemma 4, we conclude that $u(b) = d(A_i, b)$. Thus, $u'(b) \leq u(b)$.

We have

$$\sum_{b\in B} u'(b) = \sum_{a\in A_i} u(\mu', d, a) \geq \sum_{a\in A_i} \Lambda(a)\alpha(a) = \sum_{a\in A_i} u(\mu, d, a) = \sum_{b\in B} u(b),$$

where the second equality follows from Lemma 4. Together with Claim 1, we deduce that $u'(b) = u(b)$ for all b in B. Thus $\mu'(A_i, b) \geq u'(b) = u(b) = \mu(A_i, b)$ for all b in B, where the last equality follows from Lemma 4. □

B.8 Proof of Theorem 1

Proof. The definition of μ implies that μ belongs to frugal(I). Assume for the sake of contradiction that μ is not NW. Hence there is an agent a in A and an object b in B such that $\mu(a, b) < d(a, b)$ and $\mu(A, b) < \beta(b)$. Let μ' be the allocation in allocs(I) such that

$\mu'(a,b) = \min(d(a,b), \beta(b) - \mu(A-a,b)) > \mu(a,b)$, and $\mu'(a',b') = \mu(a',b')$ for all (a',b') in $A \times B - (a,b)$. Thus $u(\mu',d,a) > u(\mu,d,a)$ and $u(\mu',d,a') = u(\mu,d,a')$ for all agents a' in $A - a$, a contradiction since μ is LMMF. $\square$

B.9 Proof of Theorem 2

Proof. Assume for the sake of contradiction that there are agents a and a' such that agent a envies the allocation of agent a', that is,

$$u(\mu,d,a) < \sum_{b \in B} \min\left(\frac{\alpha(a)}{\alpha(a')}\mu(a',b), d(a,b)\right). \tag{1}$$

As in Section 3.1, let k denote the value num(I). For any i in $[k]$, let λ_i, A_i, and B_i denote brkpts(I,i), agents(I,i), and objects(I,i), respectively. Let i and i' in $[k]$ be such that agent a (resp. a') belongs to $A_i \setminus A_{i-1}$ (resp., $A_{i'} \setminus A_{i'-1}$). We consider two cases.

Case 1: In this case we have $i' > i$. We deduce that

$$\begin{aligned}
u(\mu,d,a) &= \sum_{b \in B} \min(\mu(a,b), d(a,b)) \\
&= \sum_{b \in B \setminus B_i} \min(\mu(a,b), d(a,b)) + \sum_{b \in B_i} \min(\mu(a,b), d(a,b)) \\
&= d(a, B \setminus B_i) + \sum_{b \in B_i} \min(\mu(a,b), d(a,b)) \\
&\geq d(a, B \setminus B_i),
\end{aligned}$$

where the last equality follows from Corollary 1. Corollary 2 implies that $\mu(a',b) = 0$ for all objects b in B_i. Therefore,

$$\begin{aligned}
\sum_{b \in B} \min\left(\frac{\alpha(a)}{\alpha(a')}\mu(a',b), d(a,b)\right) &= \sum_{b \in B \setminus B_i} \min\left(\frac{\alpha(a)}{\alpha(a')}\mu(a',b), d(a,b)\right) \\
&\leq d(a, B \setminus B_i).
\end{aligned}$$

The inequalities derived above imply that

$$\sum_{b \in B} \min\left(\frac{\alpha(a)}{\alpha(a')}\mu(a',b), d(a,b)\right) \leq u(\mu,d,a),$$

contradicting inequality (1).

Case 2: $i' \leq i$. Since $\lambda_1, \ldots, \lambda_k$ is an increasing sequence, $\lambda_{i'} \leq \lambda_i$. Lemma 4 implies that $\mu(a',B) = \alpha(a')\lambda_{i'}$. Thus

$$\begin{aligned}
\sum_{b \in B} \min\left(\frac{\alpha(a)}{\alpha(a')}\mu(a',b), d(a,b)\right) &\leq \frac{\alpha(a)}{\alpha(a')}\mu(a',B) \\
&= \alpha(a)\lambda_{i'} \\
&\leq \alpha(a)\lambda_i \\
&= u(\mu,d,a),
\end{aligned}$$

where the first and second equalities follow from Lemma 4. This inequality contradicts inequality (1). $\square$

B.10 Proof of Theorem 3

Proof. Let a be an agent in A and let $\mathrm{SI}(a)$ denote $\sum_{b\in B}\min\left(\frac{\alpha(a)}{\alpha(A)}\beta(b), d(a,b)\right)$. We need to show that $u(\mu,d,a)\geq \mathrm{SI}(a)/2$.

As in Section 3.1, let Λ and k denote the breakpoint function Λ_I and the value $\mathrm{num}(I)$, respectively. For any i in $[k]$, let λ_i, A_i, and B_i denote $\mathrm{brkpts}(I,i)$, $\mathrm{agents}(I,i)$, and $\mathrm{objects}(I,i)$, respectively. Let i in $[k]$ be such that agent a belongs to $A_i\setminus A_{i-1}$. Thus $\Lambda(a)=\lambda_i$.

Lemma 4 implies that $u(\mu,d,a)=\alpha(a)\lambda_i$. We have

$$\mathrm{SI}(a)=\sum_{b\in B}\min\left(\frac{\alpha(a)}{\alpha(A)}\beta(b), d(a,b)\right)\leq \frac{\alpha(a)}{\alpha(A)}\beta(B_i)+d(a,B\setminus B_i).$$

Thus, to prove that $u(\mu,d,a)\geq \mathrm{SI}(a)/2$, it suffices to prove that $u(\mu,d,a)\geq d(a,B\setminus B_i)$ and $u(\mu,d,a)\geq \frac{\alpha(a)}{\alpha(A)}\beta(B_i)$. Observe that $u(\mu,d,a)=\mu(a,B)\geq\mu(a,B\setminus B_i)=d(a,B\setminus B_i)$, where the first equality follows from Lemma 4 and the last equality follows from Corollary 1. Thus $u(\mu,d,a)\geq d(a,B\setminus B_i)$.

It remains to prove that $u(\mu,d,a)\geq\frac{\alpha(a)}{\alpha(A)}\beta(B_i)$. Since $u(\mu,d,a)=\alpha(a)\lambda_i$, it suffices to prove that $\lambda_i\geq\beta(B_i)/\alpha(A)$. Note that $\alpha(A_i)\lambda_i\geq\sum_{j\in[i]}\alpha(A_j\setminus A_{j-1})\lambda_j=\beta(B_i)+d(A_i,B\setminus B_i)\geq\beta(B_i)$, where the first inequality follows since $\lambda_1,\ldots,\lambda_k$ is an increasing sequence and the equality follows from Lemma 5. Since $\alpha(A_i)\lambda_i\geq\beta(B_i)$, we find that $\lambda_i\geq\beta(B_i)/\alpha(A_i)\geq\beta(B_i)/\alpha(A)$, where the last inequality holds because A_i is a subset of A and hence $\alpha(A_i)\leq\alpha(A)$. $\square$

B.11 Game-Theoretic Properties of Frugal LMMF Mechanisms

Theorem 4. *Any frugal LMMF mechanism is RM.*

Proof. The definition of mechanism $\mathcal{M}$ implies that it is sufficient to show that $\mathcal{M}$ is RM. Let $I=(A,B,\alpha,\beta,d)$ and $I'=(A,B,\alpha,\beta',d)$ denote OAFD instances such that $\beta(b)\leq\beta'(b)$ for all objects b in B, let μ belong to $\mathcal{M}(I)$, and let μ' belong to $\mathcal{M}(I')$. We need to prove that $u(\mu,d,a)\leq u(\mu',d,a)$ for all agents a in A. Let Λ and Λ' denote the breakpoint functions for Λ_I and $\Lambda_{I'}$, respectively. We begin by proving the following useful claim.

Claim 1: Let a and a' be agents in A and let b be an object in B such that $\mu'(a,b)<\mu(a,b)$ and $\mu'(a',b)>\mu(a',b)$. Then $\Lambda(a)\leq\Lambda(a')$ and $\Lambda'(a')\leq\Lambda'(a)$.

To prove Claim 1, first observe that $0\leq\mu'(a,b)<\mu(a,b)\leq d(a,b)$ and $0\leq\mu(a',b)<\mu'(a',b)\leq d(a',b)$ by Lemma 4. Since $\mu(a,b)>0$ and $\mu(a',b)<d(a',b)$, Lemma 6 implies that $\Lambda(a)\leq\Lambda(a')$. Similarly, since $\mu'(a',b)>0$ and $\mu'(a,b)<d(a,b)$, Lemma 6 implies that $\Lambda'(a')\leq\Lambda'(a)$. This completes the proof of Claim 1.

Let A' denote $\{a\in A\mid u(\mu,d,a)>u(\mu',d,a)\}$. To establish the lemma, we need to prove that A' is empty. Assume for the sake of contradiction that A' is nonempty. Let λ^* denote $\min_{a\in A'}\Lambda'(a)$, and let A'' denote $\{a\in A'\mid\Lambda'(a)=\lambda^*\}$; thus A'' is nonempty.

Let B' denote $\{b \in B \mid \mu(A'', b) > \mu'(A'', b)\}$. The set B' is nonempty since A'' is a nonempty subset of A'. Let b denote an object in B'.

Let A''' denote $\{a \in A'' \mid \mu(a, b) > \mu'(a, b)\}$. The set A''' is nonempty since $\mu(A'', b) > \mu'(A'', b)$. Let a denote an agent in A'''. Since $u(\mu, d, a) = \alpha(a)\Lambda(a)$ and $u(\mu', d, a) = \alpha(a)\Lambda'(a)$ by Lemma 4, and since a belongs to A', we deduce that $\Lambda(a) > \Lambda'(a) = \lambda^*$.

Since $\mu(A'', b) > \mu'(A'', b)$ and Theorem 1 implies that $\mu(A, b) = \mu'(A, b)$, we deduce that there is an agent in $A \setminus A''$, call it a', such that $\mu(a', b) < \mu'(a', b)$.

Since $\mu(a, b) > \mu'(a, b)$ and $\mu(a', b) < \mu'(a', b)$, Claim 1 implies that $\Lambda(a) \leq \Lambda(a')$ and $\Lambda'(a') \leq \Lambda'(a) = \lambda^*$. Since $\Lambda(a) > \lambda^*$, we have $\Lambda'(a') \leq \lambda^* < \Lambda(a) \leq \Lambda(a')$. Thus a' belongs to A', and hence the definition of λ^* implies $\Lambda'(a') \geq \lambda^*$. Since $\Lambda'(a') \leq \lambda^*$, we conclude that $\Lambda'(a') = \lambda^*$. Since a' belongs to A' and $\Lambda'(a') = \lambda^*$, we deduce that a' belongs to A'', a contradiction. □

Theorem 5. *Any frugal LMMF mechanism is PM.*

Proof. By the definition of mechanism $\mathcal{M}$, it is sufficient to show that $\mathcal{M}$ is PM. Let $P(k)$ denote the predicate "for any OAFD instances $I = (A, B, \alpha, \beta, d)$ and $I' = (A', B, \alpha', \beta, d_{A'})$ such that $|A| = k$ and I' belongs to shrink(I), any allocations μ in $\mathcal{M}(I)$ and μ' in $\mathcal{M}(I')$, and any agent a in A' such that $\alpha'(a) = \alpha(a)$, we have $u(\mu, d, a) \leq u(\mu', d, a)$." We prove by induction that $P(k)$ holds for all $k \geq 0$, which implies that the theorem holds.

Base case: It is easy to see that $P(0)$ holds.

Induction step: Let k be a positive integer and assume that $P(i)$ holds for $0 \leq i < k$. We need to prove that $P(k)$ holds. Let $I = (A, B, \alpha, \beta, d)$ and $I' = (A', B, \alpha', \beta, d_{A'})$ be OAFD instances such that $|A| = k$ and I' belongs to shrink(I). Let allocation μ (resp., μ') belong to $\mathcal{M}(I)$ (resp., $\mathcal{M}(I')$). Let $a^\dagger$ be an agent in A' such that $\alpha'(a^\dagger) = \alpha(a^\dagger)$. We need to prove that $u(\mu, d, a^\dagger) \leq u(\mu', d, a^\dagger)$. Let λ_1 (resp., λ'_1) denote brkpts$(I, 1)$ (resp., brkpts$(I', 1)$), and let A_1 (resp., A'_1) denote agents$(I, 1)$ (resp., agents$(I', 1)$). We consider two cases.

Case 1: $a^\dagger \in A_1$. Since $a^\dagger$ belongs to A_1, Lemma 4 implies that $u(\mu, d, a^\dagger) = \lambda_1\alpha(a^\dagger)$. The definition of λ'_1 implies that $u(\mu', d, a^\dagger) \geq \lambda'_1\alpha(a^\dagger)$. Since $u(\mu, d, a^\dagger) = \lambda_1\alpha(a^\dagger)$ and $u(\mu', d, a^\dagger) \geq \lambda'_1\alpha(a^\dagger)$, it is sufficient to prove that $\lambda'_1 \geq \lambda_1$. Let $\mu_{A'}$ denote the restriction of μ to A'; thus $\mu_{A'}$ belongs to frugal(I'). Since μ' belongs to LMMF(I'), we deduce that $\mathbf{u}(I', \mu')$ is lexicographically at least $\mathbf{u}(I', \mu_{A'})$. Hence $\lambda'_1 \geq \lambda_1$, as required.

Case 2: $a^\dagger \in A' \setminus A_1$. Let $\hat{I} = (\hat{A}, B, \hat{\alpha}, \hat{\beta}, \hat{d})$ and $\hat{I}' = (\hat{A}', B, \hat{\alpha}', \hat{\beta}', \hat{d}')$ denote the OAFD instances such that $\hat{I} = \text{sub}(I, A_1 \cup (A \setminus A'), \mu)$ and $\hat{I}' = \text{sub}(I', A_1 \cap A', \mu')$. Notice that $\hat{A} = A \setminus (A_1 \cup (A \setminus A')) = A' \setminus A_1 = \hat{A}'$ and $\hat{d} = d_{\hat{A}} = d_{\hat{A}'} = \hat{d}'$. Moreover, the case assumption implies that $a^\dagger$ belongs to $A' \setminus A_1 = \hat{A}$. Let $\hat{\mu}$ and $\hat{\mu}'$ be allocations in $\mathcal{M}(\hat{I})$ and $\mathcal{M}(\hat{I}')$, respectively. By Lemma 1, it is sufficient to prove that $u(\hat{\mu}, d, a^\dagger) \leq u(\hat{\mu}', d, a^\dagger)$.

Since $\hat{\alpha}$ (resp., $\hat{\alpha}'$) is the restriction of α (resp., α') to $\hat{A}$, we have $\hat{\alpha}'(a) \leq \hat{\alpha}(a)$ for all agents a in $\hat{A}$. Let $\hat{I}^*$ denote the OAFD instance $(\hat{A}, B, \hat{\alpha}, \hat{\beta}', \hat{d})$, which belongs to shrink$(\hat{I}')$. Let $\hat{\mu}^*$ denote an allocation in $\mathcal{M}(\hat{I}^*)$. The induction hypothesis implies that $u(\hat{\mu}', d, a) \geq u(\hat{\mu}^*, d, a)$ for all agents a in $\hat{A}$ such that $\hat{\alpha}(a) = \hat{\alpha}'(a)$. Since $a^\dagger$ belongs to $\hat{A}$ and $\hat{\alpha}'(a^\dagger) = \hat{\alpha}(a^\dagger)$, we deduce that $u(\hat{\mu}^*, d, a^\dagger) \leq u(\hat{\mu}', d, a^\dagger)$. Below we complete the proof by showing that $u(\hat{\mu}, d, a^\dagger) \leq u(\hat{\mu}^*, d, a^\dagger)$.

Let b be an object in B. We have

$$\mu(A_1 \cup (A \setminus A'), b) \geq \mu(A_1, b) = \beta_I(b) = \min\left(\beta(b), \sum_{a \in A} d(a,b)\right) \geq \mu'(A', b),$$

where the first equality holds by Lemma 3, the second equality holds by the definition of $\beta_I(b)$, and the second inequality holds because μ' belongs to frugal(I'). Therefore, $\mu(A_1 \cup (A \setminus A'), b) \geq \mu'(A', b) \geq \mu'(A' \cap A_1, b)$. Since $\hat{\beta}(b) = \beta(b) - \mu(A_1 \cup (A \setminus A'), b)$ and $\hat{\beta}'(b) = \beta(b) - \mu'(A' \cap A_1, b)$, we deduce that $\hat{\beta}(b) \leq \hat{\beta}'(b)$. Hence Theorem 4 implies that $u(\hat{\mu}, d, a^\dagger) \leq u(\hat{\mu}^*, d, a^\dagger)$, as required. □

Theorem 6. *Any frugal LMMF mechanism is GSP.*

Proof. The definition of mechanism $\mathcal{M}$ implies that it is sufficient to show that mechanism $\mathcal{M}$ is GSP. For any OAFD instances $I = (A, B, \alpha, \beta, d)$ and $I' = (A, B, \alpha, \beta, d')$, any subset A' of A such that $d_{A \setminus A'} = d'_{A \setminus A'}$, and any allocation μ' in $\mathcal{M}(I')$, we define (I, I', A', μ') as a manipulation. For any manipulation $\Phi = (I, I', A', \mu')$ where $I = (A, B, \alpha, \beta, d)$, we define the set of winning agents, denoted $W(\Phi)$, as $\{a \in A \mid u(\mu', d, a) > \alpha(a)\Lambda_I(a)\}$. Similarly, we define the set $L(\Phi)$ of losing agents as $\{a \in A \mid u(\mu', d, a) < \alpha(a)\Lambda_I(a)\}$. Remark: Lemma 4 implies that $u(\mu, d, a) = \alpha(a)\Lambda_I(a)$ for all allocations μ in $\mathcal{M}(I)$ and all agents a in A.

Let $P(k)$ denote the predicate "for any manipulation $\Phi = (I, I', A', \mu')$ where $I = (A, B, \alpha, \beta, d)$, $|A| = k$, and $W(\Phi) \cap A' \neq \emptyset$, we have $L(\Phi) \cap A' \neq \emptyset$." Below we prove by induction on k that $P(k)$ holds for all $k \geq 0$; the claim of the theorem follows immediately.

It is easy to see that $P(0)$ holds. Let k be a positive integer and assume that $P(i)$ holds for $0 \leq i < k$. We need to prove that $P(k)$ holds. Let $\Phi = (I, I', A', \mu')$ be a manipulation where $I = (A, B, \alpha, \beta, d)$, $I' = (A, B, \alpha, \beta, d')$, $|A| = k$, and $W(\Phi) \cap A' \neq \emptyset$. We need to prove that $L(\Phi) \cap A' \neq \emptyset$. Let λ_1 (resp., λ'_1) denote brkpts$(I, 1)$ (resp., brkpts$(I', 1)$), and let A_1 (resp., A'_1) denote agents$(I, 1)$ (resp., agents$(I', 1)$). We consider four cases.

Case 1: $\lambda'_1 < \lambda_1$. Lemma 3 implies that λ'_1 is equal to $\text{cap}(I', A'_1)/\alpha(A'_1)$ and λ_1 is equal to $\min_{X \subseteq A} \text{cap}(I, X)/\alpha(X)$. Thus

$$\text{cap}(I', A'_1)/\alpha(A'_1) < \min_{X \subseteq A} \text{cap}(I, X)/\alpha(X) \leq \text{cap}(I, A'_1)/\alpha(A'_1),$$

where the first inequality follows from the case assumption and the second inequality follows from $A'_1 \subseteq A$. Multiplying by $\alpha(A'_1)$, we obtain $\text{cap}(I', A'_1) < \text{cap}(I, A'_1)$. If $A' \cap A'_1 = \emptyset$, then $d'_a = d_a$ for all agents a in A'_1 and hence $\text{cap}(I', A'_1) = \text{cap}(I, A'_1)$, a contradiction. It remains to consider the case where $A' \cap A'_1 \neq \emptyset$. Let a belong to $A' \cap A'_1$. Thus

$$u(\mu', d, a) \leq \mu'(a, B) = u(\mu', d', a) = \alpha(a)\lambda'_1 < \alpha(a)\lambda_1,$$

where the two equalities follows from Lemma 4. Hence a is in $L(\Phi) \cap A'$.

Case 2: $\lambda'_1 \geq \lambda_1$ and $L(\Phi) \cap A_1 \neq \emptyset$. Let a be an agent in $L(\Phi) \cap A_1$. If a is in A' then $L(\Phi) \cap A' \neq \emptyset$, as required. Thus, in what follows, we assume that a is not in A'. Let i denote the least integer such that a is in agents(I', i). We have

$$\alpha(a)\Lambda_{I'}(a) = u(\mu', d', a) = u(\mu', d, a) < \alpha(a)\lambda(a) = \alpha(a)\lambda_1,$$

where the first equality holds by Lemma 4, the second equality holds since a is not in A' and hence $d'_a = d_a$, the inequality holds since a is in $L(\Phi)$, and the third equality holds since a is in A_1. Thus $\lambda'_1 \leq \Lambda_{I'}(a) < \lambda_1$, contradicting the first condition in the case assumption.

Case 3: $\lambda'_1 \geq \lambda_1$, $L(\Phi) \cap A_1 = \emptyset$, and $W(\Phi) \cap A_1 \neq \emptyset$. Let a denote an agent in $W(\Phi) \cap A_1$. Thus $u(\mu', d, a) > \alpha(a)\Lambda_I(a) = \alpha(a)\lambda_1$. Since $u(\mu', d, A_1) \leq \mathrm{cap}(I, A_1) = \alpha(A_1)\lambda_1$ by Lemma 3, we deduce that $u(\mu', d, A_1 - a) < \alpha(A_1 - a)\lambda_1$. Thus there is an agent a' in $A_1 - a$ such that $u(\mu', d, a') < \alpha(a')\lambda_1 = \alpha(a')\Lambda_I(a')$. Hence $L(\Phi) \cap A_1 \neq \emptyset$, contradicting the second condition in the case assumption.

Case 4: $\lambda'_1 \geq \lambda_1$, $L(\Phi) \cap A_1 = \emptyset$, and $W(\Phi) \cap A_1 = \emptyset$. Let μ denote an allocation in $\mathcal{M}(I)$. Let $\hat{I} = (\hat{A}, B, \hat{\alpha}, \hat{\beta}, \hat{d})$ denote the OAFD instance $\mathrm{sub}(I, A_1, \mu)$; thus $\hat{A} = A \setminus A_1$. Let $\hat{\mu}$ denote the restriction of μ to $\hat{A}$; Lemma 1 implies that $\hat{\mu}$ is in $\mathcal{M}(\hat{I})$. Let $I^* = (\hat{A}, B, \hat{\alpha}, \beta^*, d^*)$ denote the OAFD instance $\mathrm{sub}(I, A_1, \mu')$ and let μ^* denote the restriction of μ' to $\hat{A}$; Lemma 1 implies that μ^* is in $\mathcal{M}(I^*)$. Let $\tilde{I}$ denote the OAFD instance $(\hat{A}, B, \hat{\alpha}, \beta^*, \hat{d})$ and let $\tilde{\mu}$ be in $\mathcal{M}(\tilde{I})$.

Claim 1: $\Lambda_I(a) \geq \Lambda_{\tilde{I}}(a)$ holds for all agents a in $\hat{A}$. The third condition in the case assumption implies that $u(\mu', d, a) \geq \alpha(a)\lambda_1$ for all agents a in A_1. Thus Lemma 7 implies that $\mu(A_1, b) \leq \mu'(A_1, b)$ for all objects b in B. It follows that $\hat{\beta}(b) \geq \beta^*(b)$ for all objects b in B. Hence

$$\alpha(a)\Lambda_I(a) = u(\mu, d, a) = u(\hat{\mu}, \hat{d}, a) \geq u(\tilde{\mu}, \hat{d}, a) = \alpha(a)\Lambda_{\tilde{I}}(a),$$

where the first and last equalities hold by Lemma 4, the second equality holds by the definition of $\hat{\mu}$, and the inequality holds by Theorem 4. Dividing by $\alpha(a)$ yields the claim.

Claim 2: $u(\mu', d, a) = u(\mu^*, \hat{d}, a)$ for all agents a in $\hat{A}$. We have

$$u(\mu', d, a) = \sum_{b \in B} \min(\mu'(a, b), d(a, b)) = \sum_{b \in B} \min(\mu^*(a, b), \hat{d}(a, b)) = u(\mu^*, \hat{d}, a),$$

where the second equality holds by the definition of μ^*. The claim follows.

Let A'' denote $A' \setminus A_1$ and let Φ' denote the manipulation $(\tilde{I}, I^*, A'', \mu^*)$.

Claim 3: $W(\Phi') \cap A'' \neq \emptyset$. Since $W(\Phi) \cap A' \neq \emptyset$, the third condition in the case assumption implies that $W(\Phi) \cap A'' \neq \emptyset$. Let a be an agent in $W(\Phi) \cap A''$. Thus

$$u(\mu^*, \hat{d}, a) = u(\mu', d, a) > \alpha(a)\Lambda_I(a) \geq \alpha(a)\Lambda_{\tilde{I}}(a),$$

where the equality holds by Claim 2, the first inequality holds because a is in $W(\Phi)$, and the second inequality holds by Claim 1. Since $u(\mu^*, \hat{d}, a) > \Lambda_{\tilde{I}}(a)$ and a is in A'', the claim holds.

Since $|\hat{A}| < k$ and Claim 3 holds, the induction hypothesis implies that $L(\Phi') \cap A'' \neq \emptyset$. Let a be in $L(\Phi') \cap A''$. Thus a is in $A'' \subseteq \hat{A}$ and

$$u(\mu', d, a) = u(\mu^*, \hat{d}, a) < \alpha(a)\Lambda_{\tilde{I}}(a) \leq \alpha(a)\Lambda_I(a),$$

where the equality holds by Claim 2, the first inequality holds because a is in $L(\Phi')$, and the second inequality holds by Claim 1. Since $u(\mu', d, a) < \alpha(a)\Lambda_I(a)$ and a is in $A'' \subseteq A'$, we deduce that a is in $W(\Phi) \cap A'$. □

B.12 Proof of Lemma 8

Proof. Let M be an MMF OAFD mechanism. Consider an OAFD instance $I = (A, B, \alpha, \beta, d)$ with n agents $a_1, \ldots, a_n$, each with endowment 1, and two objects b_1 and b_2, each with supply n, and where $d(a_1, b_1) = d(a_1, b_2) = 1$, and $d(a, b_1) = 2$ and $d(a, b_2) = 0$ for all agents a in $A - a_1$. Mechanism M gives a utility of $1 + 1/n$ to each agent in A. If agent a_1 is allocated an $\alpha(a_1)/\alpha(A) = 1/n$ fraction of each object, then a_1 achieves utility 2. Hence M is at most $\frac{1}{2}\left(1 + \frac{1}{n}\right)$-SI. Let z be any value greater than $\frac{1}{2}$. By choosing a sufficiently large n, we deduce that M is not z-SI. $\square$

B.13 Proof of Lemma 9

Proof. Let M be an MMF-SI OAFD mechanism. Consider an OAFD instance $I = (A, B, \alpha, \beta, d)$ with three agents a_1, a_2, and a_3, each with endowment 1, and two objects b_1 and b_2, each with supply 6, and where $d(a_1, b_1) = 3$, $d(a_1, b_2) = 1$, $d(a_2, b_1) = d(a_3, b_1) = 0$, and $d(a_2, b_2) = d(a_3, b_2) = 3$. Let μ belong to $M(I)$. It is easy to verify that $u(\mu, d, a_1) = 3$. Let d' denote (d_{A-a_1}, d''), where d'' belongs to demands$(\{a_1\}, B)$, $d''(a_1, b_1) = 3$, and $d''(a_1, b_2) = 2$. Let I' denote the OAFD instance $(A, B, \alpha, \beta, d')$ and let μ' belong to $M(I')$. It is easy to verify that $u(\mu', d', a_1) = 4$. We conclude that mechanism M is not SP. $\square$