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On the Existence of Three-Dimensional Stable Matchings with Cyclic Preferences

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Abstract We study the three-dimensional stable matching problem with cyclic preferences. This model involves three types of agents, with an equal number of agents of each type. The types form a cyclic order such that each agent has a complete preference list over the agents of the next type. We consider the open problem of the existence of three-dimensional matchings in which no triple of agents prefer each other to their partners. Such matchings are said to be weakly stable. We show that contrary to published conjectures, weakly stable three-dimensional matchings need not exist. Furthermore, we show that it is NP-complete to determine whether a weakly stable three-dimensional matching exists. We achieve this by reducing from the variant of the problem where preference lists are allowed to be incomplete. We generalize our results to the k-dimensional stable matching problem with cyclic preferences for $k \geq 3$.

Keywords Stable matching · Three-dimensional matching · NP-completeness .

1 Introduction

The study of stable matchings was started by Gale and Shapley [9], who investigated a market with two types of agents. The two-dimensional stable matching problem involves an equal number of men and women, each of whom has a complete preference list over the agents of the opposite sex. The goal is to find a matching between the men and the women such that no man and woman prefer each other to their partners.

A preliminary version of this work appears in [14].

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Matchings satisfying this property are said to be stable. Gale and Shapley showed that a solution for the two-dimensional stable matching problem always exists and can be computed in polynomial time. Their result also applies to the variant where preference lists may be incomplete due to unacceptable partners, and the number of men may be different from the number of women.

The problem of generalizing stable matchings to markets with three types of agents was posed by Knuth [13]. In pursuit of an existence theorem and an elegant theory analogous to those of the Gale-Shapley model, the three-dimensional stable matching problem has been studied with respect to a number of preference structures. When each agent has preferences over pairs of agents from the other two types, stable matchings need not exist [1, 16]. Furthermore, it is NP-complete to determine whether a stable matching exists [16, 18], even if the preferences are consistent with product orders [11]. When two types of agents care primarily about each other and secondarily about the remaining type, a stable matching always exists and can be obtained by computing two-dimensional stable matchings using the Gale-Shapley algorithm in a hierarchical manner [5]. When the types form a cyclic order such that each type of agent cares primarily about the next type and secondarily about the other type, stable matchings need not exist [3].

A prominent problem mentioned in several of the aforementioned papers [3, 11, 16] is the three-dimensional stable matching problem for the case where the types form a cyclic order such that each type of agent cares only about the next type and not the other type. Following the terminology of the survey of Manlove [15], we call this the three-dimensional stable matching problem with cyclic preferences (3-DSM-CYC), and refer to the three types of agents as men, women, and dogs. A number of stability notions [11] can be considered in 3-DSM-CYC. In this paper, we focus on weak stability, which is the most permissive one and has received the most attention in the literature. It is known that determining whether a 3-DSM-CYC instance has a strongly stable matching is NP-complete [2]. For the variant where ties are allowed, determining the existence of a super-stable matching is also NP-complete [12]. However, it remained an open problem for weakly stable matchings in 3-DSM-CYC.

In 3-DSM-CYC, there are an equal number of men, women, and dogs. Each man has a complete preference list over the women, each woman has a complete preference list over the dogs, and each dog has a complete preference list over the men. A family is a triple consisting of a man, a woman, and a dog. A matching is a set of agent-disjoint families. A family is strongly blocking if every agent in the family prefers each other to their partners in the matching. A matching is weakly stable if it admits no strongly blocking family. This problem is related to applications such as kidney exchange [2] and three-sided network services [4].

The formulation of 3-DSM-CYC first appeared in the paper of Ng and Hirschberg [16], where it is attributed to Knuth. Using a greedy approach, Boros et al. [3] showed that every 3-DSM-CYC instance with at most three agents per type has a weakly stable matching. Their result also applies to the k-dimensional generalization of the problem, which we call k-DSM-CYC. For $k \geq 3$, they showed that every k-DSM-CYC instance with at most k agents per type has a weakly stable matching. Using a case analysis, Eriksson et al. [6] showed that every 3-DSM-CYC instance with at most four agents per type has a weakly stable matching, and they conjectured that every

3-DSM-CYC instance has a weakly stable matching. In fact, they posed the stronger conjecture that for a certain “strongest link” generalization of 3-DSM-CYC, every instance with at least two agents per type has at least two weakly stable matchings. Eriksson et al. also investigated and ruled out the use of certain arguments based on “effectivity functions” and “balanced games” for proving the 3-DSM-CYC conjecture. Using an efficient greedy procedure, Hofbauer [10] showed that for $k \geq 3$, every k-DSM-CYC instance with at most $k+1$ agents per type has a weakly stable matching. Using a satisfiability problem formulation and an extensive computer-assisted search, Pashkovich and Poirrier [17] showed that every 3-DSM-CYC instance with exactly five agents per type has at least two weakly stable matchings. Escamocher and O’Sullivan [7] showed that the number of weakly stable matchings is exponential in the size of the 3-DSM-CYC instance if agents of the same type are restricted to have the same preferences. They also conjectured that for unrestricted 3-DSM-CYC instances, there are exponentially many weakly stable matchings.

Hardness results are known for some related problems. For the variant of 3-DSM-CYC where preference lists are allowed to be incomplete, which we refer to as 3-DSMI-CYC, Biró and McDermid [2] showed that determining whether a weakly stable matching exists is NP-complete. Farczadi et al. [8] showed that determining whether a given perfect two-dimensional matching can be extended to a three-dimensional weakly stable matching in 3-DSM-CYC is also NP-complete. However, the existence of weakly stable matchings in 3-DSM-CYC remained unresolved. Manlove [15] described it as an “intriguing open problem”, and Woeginger [19] classified it as “hard and outstanding”.

Our Techniques and Contributions. In this paper, we show that there exists a 3-DSM-CYC instance that has no weakly stable matching. This disproves the conjectures of Eriksson et al. [6] and Escamocher and O’Sullivan [7]. Furthermore, we show that determining whether a 3-DSM-CYC instance has a weakly stable matching is NP-complete. We achieve this by reducing from the problem of determining whether a 3-DSMI-CYC instance has a weakly stable matching. We generalize our results to k-DSM-CYC for $k \geq 3$.

Our main technique involves converting each agent in 3-DSMI-CYC to a gadget consisting of one non-dummy agent and many dummy agents. The dummy agents in our gadget give rise to chains of “admirers”. (See Remark 5 in Section 4.3.) By applying the weak stability condition to the chains of admirers, we are able to obtain some control over the partner of the non-dummy agent.

Organization of This Paper. In Section 2, we present the formal definitions of k-DSM-CYC and k-DSMI-CYC. In Section 3, we show that the NP-completeness result of Biró and McDermid [2] can be extended to k-DSMI-CYC. In Section 4, we show that k-DSM-CYC is NP-complete by a reduction from k-DSMI-CYC. In Section 5, we conclude by mentioning some potential future work.

2 Preliminaries

In this paper, we use $\langle z \in Z \mid \mathscr{P}(z)\rangle$ to denote the list of all tuples $z \in Z$ satisfying predicate $\mathscr{P}(z)$, where the tuples are sorted in increasing lexicographical order. Given two lists Y and Z, we denote their concatenation as $Y \cdot Z$. For any $k \geq 1$, we use $\oplus_k$ to denote addition modulo k.

2.1 The Models

Let $k \geq 2$. The k-dimensional stable matching problem with incomplete lists and cyclic preferences (k-DSMI-CYC) involves a finite set $A = I \times \{0,\ldots,k-1\}$ of agents, where I is a finite set of identifiers and each agent $\alpha = (i,t) \in A$ is associated with an identifier i and a type t.

Remark 1 When $k = 3$, we can think of the sets $I \times \{0\}$, $I \times \{1\}$, and $I \times \{2\}$ as the sets of men, women, and dogs, respectively. The identifiers in the set I serve only to distinguish the various agents of a given type; there is no special relationship between a man and a woman who share the same identifier.

Each agent $\alpha = (i,t) \in A$ has a strict preference list P_α over a subset of agents of type $t' = t \oplus_k 1$. In other words, every agent in $I \times \{t \oplus_k 1\}$ appears in P_α at most once, and every element in P_α belongs to $I \times \{t \oplus_k 1\}$. For every $\alpha, \alpha', \alpha'' \in A$, we say that α prefers α' to α'' if α' appears in P_α and either agent α'' appears in P_α after α' or agent α'' does not appear in P_α. We denote this k-DSMI-CYC instance as $X = (A, \{P_\alpha\}_{\alpha \in A})$.

Given a k-DSMI-CYC instance $X = (A, \{P_\alpha\}_{\alpha \in A})$, a *family* is a tuple

$$(\alpha_0, \ldots, \alpha_{k-1}) \in A^k$$

such that $\alpha_t \in I \times \{t\}$ and $\alpha_{t \oplus_k 1}$ appears in P_{α_t} for every $t \in \{0,\ldots,k-1\}$. A *matching* μ is a set of agent-disjoint families. In other words, for every $t, t' \in \{0,\ldots,k-1\}$ and $(\alpha_0,\ldots,\alpha_{k-1}), (\alpha'_0,\ldots,\alpha'_{k-1}) \in \mu$, if $\alpha_t = \alpha'_t$, then $\alpha_{t'} = \alpha'_{t'}$. Given a matching μ and an agent $\alpha \in A$, if $\alpha = \alpha_t$ for some $(\alpha_0,\ldots,\alpha_{k-1}) \in \mu$ and $t \in \{0,\ldots,k-1\}$, we say that α is matched to $\alpha_{t \oplus_k 1}$, and we write $\mu(\alpha) = \alpha_{t \oplus_k 1}$. Otherwise, we say that α is unmatched, and we write $\mu(\alpha) = \alpha$.

Given a matching μ, we say that a family $(\alpha_0,\ldots,\alpha_{k-1})$ is *strongly blocking* if α_t prefers $\alpha_{t \oplus_k 1}$ to $\mu(\alpha_t)$ for every $t \in \{0,\ldots,k-1\}$. A matching μ is *weakly stable* if it does not admit any strongly blocking family.

The k-dimensional stable matching problem with cyclic preferences (k-DSM-CYC) is defined as the special case of k-DSMI-CYC in which every agent in $I \times \{t \oplus_k 1\}$ appears exactly once in P_α for every agent $\alpha = (i,t) \in A$.

Notice that when incomplete lists are allowed, the case of an unequal number of agents of each type can be handled within our k-DSMI-CYC model by padding with dummy agents whose preference lists are empty. Hence, the results of Biró and McDermid [2] apply to our 3-DSMI-CYC model. When preference lists are complete, we follow the literature and focus on the case where each type has an equal number of agents. Our result shows that even when restricted to the case of an equal number of agents of each type, a given k-DSM-CYC instance need not admit a weakly stable matching, and determining the existence of a weakly stable matching is NP-complete.

Remark 2 Since our formulation of k-DSM-CYC assumes an equal number of agents of each type, any weakly stable matching for a k-DSMI-CYC instance necessarily matches every agent: If an agent is unmatched, then there is an unmatched agent $\hat{\alpha}_t$ of each type $t \in \{0,\ldots,k-1\}$, and $(\hat{\alpha}_0,\ldots,\hat{\alpha}_{k-1})$ forms a strongly blocking family.

2.2 Polynomial-Time Verification

Given a matching μ of a k-DSMI-CYC instance with n agents per type, it is straightforward to determine whether μ is weakly stable in $O(n^k)$ time by checking that none of the $O(n^k)$ families is strongly blocking. The following theorem shows that when k is large, there is a more efficient method to determine whether a given matching is weakly stable.

Theorem 1 *There exists a* poly(n,k)*-time algorithm to determine whether a given matching μ is weakly stable for a k-*DSMI-CYC *instance, where n is the number of agents per type.*

Proof Given μ, consider the directed graph G with vertex set A such that there exists an edge from α to α' if and only if α prefers α' to $\mu(\alpha)$. Then cycles in G of length k correspond to strongly blocking families of μ. Notice that if a cycle in G contains an agent in $I \times \{t\}$, then it also contains an agent in $I \times \{t \oplus_k 1\}$. Hence no cycle in G has length less than k. Thus determining whether μ is weakly stable is equivalent to determining whether the directed graph G has a cycle of length at most k, which can be done in poly(n,k) time. □

3 NP-Completeness of k-DSMI-CYC

Biró and McDermid [2] have shown that, for $k = 3$, it is NP-complete to determine whether a given k-DSMI-CYC instance has a weakly stable matching. In this section, we show that this problem remains NP-complete for $k > 3$. Our NP-completeness proof uses a reduction from 3-DSMI-CYC.

3.1 The Reduction

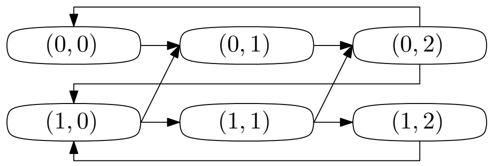
Let $k \geq 4$. Consider an input 3-DSMI-CYC instance $X = (A, \{P_\alpha\}_{\alpha\in A})$ where $A = I \times \{0,1,2\}$. Our reduction constructs a k-DSMI-CYC instance $\hat{X} = (\hat{A}, \{\hat{P}_{\hat{\alpha}}\}_{\hat{\alpha}\in\hat{A}})$ as follows. Let $\hat{I} = I \times I$ and $\hat{A} = \hat{I} \times \{0,\ldots,k-1\}$. For every agent $(i,t) \in A$, we call $\hat{\alpha} = (i,i,t) \in \hat{A}$ the non-dummy agent corresponding to (i,t). We refer to the agents in the set

$$\{(i,j,t) \in \hat{A} \mid t \notin \{0,1,2\} \text{ or } i \neq j\}$$

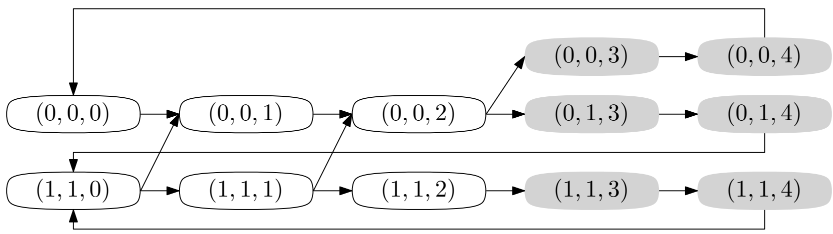
as dummy agents. For each agent $\hat{\alpha} = (i, j, t) \in \hat{A}$, we construct the preference list $\hat{P}_{\hat{\alpha}}$ according to the cases discussed below. The remarks accompanying each case are intended to convey the intuition underlying the formal construction.

- If $t \in \{0,1\}$ and $i = j$, we list in $\hat{P}_{\hat{\alpha}}$ the non-dummy agents
$$\{(i', j', t') \in \hat{I} \times \{t+1\} \mid i' = j' \text{ and } (i', t') \text{ is in } P_{(i,t)}\}$$
in the order in which the corresponding agent (i', t') appears in $P_{(i,t)}$.
Remarks: Agents in this category are non-dummy agents of type 0 or 1. The preferences of each such agent mimic the preferences of the corresponding agent in A.
- If $t = 2$ and $i = j$, we list in $\hat{P}_{\hat{\alpha}}$ the agents
$$\{(i', j', t') \in \hat{I} \times \{3\} \mid i' = i \text{ and } (j', 0) \text{ is in } P_{(i,2)}\}$$
in the order in which the corresponding agent $(j', 0)$ appears in $P_{(i,2)}$.
Remarks: Any agent $\hat{\alpha}$ in this category is a non-dummy agent of type 2. The preferences of $\hat{\alpha}$ cannot directly mimic the preferences of the corresponding type-2 agent α in A, since $\hat{\alpha}$ has preferences over agents of type 3, while α has preferences over agents of type 0. Instead, our plan is to construct a dedicated chain of $k-3$ dummy agents of types $3, \ldots, k-1$ to simulate each entry in the preference list of α.
- If $0 \le t \le 2$ and $i \ne j$, we define $\hat{P}_{\hat{\alpha}}$ as the empty list.
Remarks: Agents in this category are dummy agents of type 0, 1, or 2. The sole purpose of these agents is to ensure that there are $|\hat{I}|$ agents of each type in $\hat{A}$. Note that an agent with an empty preference list is unmatched in any weakly stable matching.
- If $3 \le t \le k-2$ and $(j, 0)$ is in $P_{(i,2)}$, we define $\hat{P}_{\hat{\alpha}}$ as $\langle (i, j, t+1) \rangle$.
Remarks: For any i and j in I such that $(j, 0)$ is in $P_{(i,2)}$, the $k-4$ dummy agents in this category form the prefix of the length-$(k-3)$ chain dedicated to simulating the preference ranking of agent $(i, 2)$ for agent $(j, 0)$. Each of these dummy agents is only interested in being matched to its successor in the chain.
- If $t = k-1$ and $(j, 0)$ is in $P_{(i,2)}$, we define $\hat{P}_{\hat{\alpha}}$ as $\langle (j, j, 0) \rangle$.
Remarks: For any i and j in I such that $(j, 0)$ is in $P_{(i,2)}$, an agent $\hat{\alpha}$ in this category is the last dummy agent in the length-$(k-3)$ chain dedicated to simulating the preference ranking of agent $(i, 2)$ for agent $(j, 0)$. Agent $\hat{\alpha}$ is only interested in being matched to the non-dummy agent $(j, j, 0)$ corresponding to agent $(j, 0)$.
- If $3 \le t \le k-1$ and $(j, 0)$ is not in $P_{(i,2)}$, we define $\hat{P}_{\hat{\alpha}}$ as the empty list.
Remarks: For any i and j in I such that $(j, 0)$ is not in $P_{(i,2)}$, we do not need to create an associated chain. Instead, the $k-3$ dummy agents reserved for this purpose are assigned empty preference lists, effectively removing them from the problem instance.

Figure 1 shows an example of the reduction when $k = 5$ and $I = \{0, 1\}$. In the input 3-DSMI-CYC instance, the two type-2 agents have preferences lists of length 2 and 1, for a total length of 3. Accordingly, there are three chains of dummy agents, each of length $5 - 3 = 2$: The chain consisting of dummy agents $(0, 0, 3)$ and $(0, 0, 4)$



(a) The input 3-DSMI-CYC instance.



(b) The output 5-DSMI-CYC instance. Agents with empty preference lists are omitted.

Fig. 1 Example of a reduction from 3-DSMI-CYC to 5-DSMI-CYC. An arrow indicates that the target agent appears in the preference list of the source agent.

(resp., $(0,1,3)$ and $(0,1,4)$, $(1,1,3)$ and $(1,1,4)$) simulates the preference ranking of agent $(0,2)$ (resp., $(0,2)$, $(1,2)$) for agent $(0,0)$ (resp., $(1,0)$, $(1,0)$). If we were to increase k to some value $k' > 5$, then $k' - 5$ additional dummy agents would be inserted into each of these three chains.

3.2 Correctness of the Reduction

Lemma 1 *Let $k \geq 4$. Consider the reduction given in Section 3.1. The output k-DSMI-CYC instance $\hat{X}$ has a weakly stable matching if and only if the input 3-DSMI-CYC instance X has a weakly stable matching.*

Proof Since every family in $\hat{X}$ has the form

$$((i_0,i_0,0),(i_1,i_1,1),(i_2,i_2,2),(i_2,i_0,3),\ldots,(i_2,i_0,k-1)),$$

where (i_0,i_1,i_2) is a family in X, there is a-one-to-one correspondence between families in $\hat{X}$ and families in X. This induces a one-to-one correspondence between matchings in $\hat{X}$ and matchings in X. It is straightforward to see that a family in $\hat{X}$ is a strongly blocking family of a matching in $\hat{X}$ if and only if the corresponding family in X is a strong blocking family of the corresponding matching in X. Hence $\hat{X}$ has a weakly stable matching if and only if X has a weakly stable matching. □

Theorem 2 *Let $k \geq 3$. Then there exists a k-DSMI-CYC instance that has no weakly stable matching.*

Proof Biró and McDermid [2, Lemma 1] show that there exists a 3-DSMI-CYC instance X that has no weakly stable matching. So we may assume that $k \geq 4$. Then Lemma 1 implies that given X as an input, the reduction in Section 3.1 produces a k-DSMI-CYC instance $\hat{X}$ that has no weakly stable matching. □

Theorem 3 *Let $k \geq 3$. Then it is NP-complete to determine whether a k-DSMI-CYC instance has a weakly stable matching.*

Proof Biró and McDermid [2, Theorem 1] show that it is NP-complete to determine whether a 3-DSMI-CYC instance has a weakly stable matching. So we may assume that $k \geq 4$. Then Lemma 1 implies the correctness of the reduction from 3-DSMI-CYC to k-DSMI-CYC presented in Section 3.1. Moreover, the reduction can be implemented in $\text{poly}(n,k)$ time, where n is the number of agents of each type. Theorem 1 implies that the problem of determining whether a k-DSMI-CYC instance has a weakly stable matching is in NP. Thus the problem of determining whether a k-DSMI-CYC instance has a weakly stable matching is NP-complete. □

4 NP-Completeness of k-DSM-CYC

In this section, we show that for every $k \geq 3$, it is NP-complete to determine whether a k-DSM-CYC instance has a weakly stable matching. (Recall that, as indicated in Remark 2, any weakly stable matching for a k-DSM-CYC instance matches all of the agents.) We achieve this by reducing from the problem of determining whether a k-DSMI-CYC instance has a weakly stable matching. Since the dimensions of both the input instance and the output instance of the reduction are equal to k, throughout this section, we write $\oplus$ instead of $\oplus_k$ for better readability.

Remark 3 While the results in this section are presented for any $k \geq 3$, the reader may find it more convenient to focus on the special case $k = 3$ in an initial reading.

4.1 The Reduction

Let $k \geq 3$. Consider an input k-DSMI-CYC instance $X = (A, \{P_\alpha\}_{\alpha \in A})$ where $A = I \times \{0, \dots k-1\}$. We may assume that $I = \{0, \dots, |I|-1\}$, so agents in A can be compared lexicographically. Our reduction constructs a k-DSM-CYC instance $\hat{X} = (\hat{A}, \{\hat{P}_{\hat{\alpha}}\}_{\hat{\alpha} \in \hat{A}})$ as follows.

- Let $J = \{0, \dots, (k-1)^2\}$. Let $\hat{I} = J \times A$ and $\hat{A} = J \times A \times \{0, \dots, k-1\}$. For every agent $\alpha \in A$, we call $J \times \{\alpha\} \times \{0, \dots, k-1\}$ the gadget corresponding to α.
- For every agent $\hat{\alpha} = (j, \alpha, t) \in \hat{A}$ such that $j = 0$ and $\alpha \in I \times \{t\}$, we call $\hat{\alpha}$ the non-dummy agent corresponding to α. Let $\hat{P}'_\alpha$ be the list obtained by replacing every α' in P_α by $(0, \alpha', t \oplus 1)$. We define the preference list $\hat{P}_{\hat{\alpha}}$ as $\hat{P}'_\alpha \cdot \langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid \alpha' = \alpha \rangle$ followed by the remaining agents in $J \times A \times \{t \oplus 1\}$ in an arbitrary order.
- For every agent $\hat{\alpha} = (j, \alpha, t) \in \hat{A}$ such that $j = (k-1)^2$, we call $\hat{\alpha}$ a boundary dummy agent, and we define the preference list $\hat{P}_{\hat{\alpha}}$ as

$$\langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid \alpha' = \alpha \text{ and } j' < (k-1)^2 \rangle \cdot \langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid j' = (k-1)^2 \rangle$$

  followed by the remaining agents in $J \times A \times \{t \oplus 1\}$ in an arbitrary order.

– For every agent $\hat{\alpha} = (j, \alpha, t) \in \hat{A}$ such that $(j, \alpha, t) \notin \{0\} \times (I \times \{t\}) \times \{t\}$ and $j < (k-1)^2$, we call $\hat{\alpha}$ a non-boundary dummy agent, and we define the preference list $\hat{P}_{\hat{\alpha}}$ as $\langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid \alpha' = \alpha \rangle$ followed by the remaining agents in $J \times A \times \{t \oplus 1\}$ in an arbitrary order.

As shown in Figure 2(a), the gadget corresponding to $\alpha \in I \times \{t\}$ can be visualized as a grid of agents with k rows and $(k-1)^2 + 1$ columns. The non-boundary dummy agents in the same row have essentially the same preferences, which begin with the agents in the next row from left to right. The preferences of the boundary dummy agents are similar to those of the non-boundary dummy agents, except that they incorporate the other boundary dummy agents in a special manner. Meanwhile, the preferences of the non-dummy agent $(0, \alpha, t)$ reflect the preferences of agent α by starting with $\hat{P}'_{\alpha}$.

Remark 4 The reason our gadget has $(k-1)^2 + 1$ columns will become clearer when we present Lemmas 4 and 5 below. At a high level, Lemma 4 is invoked $k-1$ times within the proof of Lemma 5, and each such invocation leads to an increase of $k-1$ in the number of columns.

4.2 Correctness of the Reduction

Lemma 2 below implies that we can use the reduction of Section 4.1 to transform any instance of k-DSMI-CYC that has no weakly stable matching into an instance of k-DSM-CYC that has no weakly stable matching. The proof of Lemma 2 is presented in Section 4.3.

Lemma 2 *Let $k \geq 3$. Consider the reduction given in Section 4.1. If the input k-DSMI-CYC instance X has no weakly stable matching, then the output k-DSM-CYC instance $\hat{X}$ has no weakly stable matching.*

The proof of Theorem 4 below uses Theorem 2 and Lemma 2.

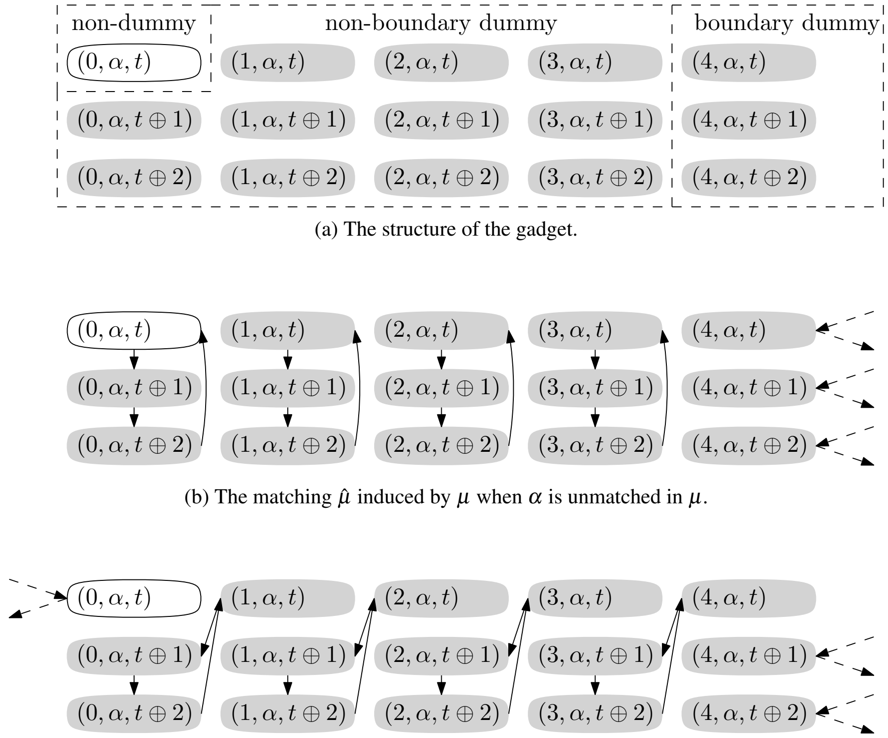
Theorem 4 *Let $k \geq 3$. Then there exists a k-DSM-CYC instance that has no weakly stable matching.*

Proof By Theorem 2, there exists a k-DSMI-CYC instance X that has no weakly stable matching. Then Lemma 2 implies that given X as an input, the reduction in Section 4.1 produces a k-DSM-CYC instance $\hat{X}$ that has no weakly stable matching. □

Lemma 3 below establishes the converse of Lemma 2. Note that Lemma 3 is not needed to establish Theorem 4. The proof of Lemma 3 is presented in Section 4.4.

Lemma 3 *Let $k \geq 3$. Consider the reduction in Section 4.1. If the input k-DSMI-CYC instance X has a weakly stable matching, then the output k-DSM-CYC instance $\hat{X}$ has a weakly stable matching.*

Lemmas 2 and 3 are the key ingredients in the proof of Theorem 5 below.



(a) The structure of the gadget.

(b) The matching $\hat{\mu}$ induced by μ when α is unmatched in μ.

(c) The matching $\hat{\mu}$ induced by μ when α is matched in μ.

Fig. 2 Example of a gadget corresponding to $\alpha \in I \times \{t\}$ when $k = 3$. An arrow indicates that the source agent is matched to the target agent.

Theorem 5 *Let $k \geq 3$. Then it is NP-complete to determine whether a k-DSM-CYC instance has a weakly stable matching.*

Proof By Theorem 3, it is NP-complete to determine whether a k-DSMI-CYC instance has a weakly stable matching. Lemmas 2 and 3 imply the correctness of the reduction from k-DSMI-CYC to k-DSM-CYC presented in Section 4.1. Moreover, the reduction can be implemented in poly(n,k) time, where n is the number of agents of each type. Theorem 1 implies that the problem of determining whether a k-DSM-CYC instance has a weakly stable matching is in NP. Thus the problem of determining whether a k-DSM-CYC instance has a weakly stable matching is NP-complete. □

4.3 Proof of Lemma 2

To establish Lemma 2, it suffices to show that every weakly stable matching in $\hat{X}$ induces a weakly stable matching in X. Throughout this subsection, let $\hat{\mu}$ denote a weakly stable matching in $\hat{X}$. We will use $\hat{\mu}$ to construct a weakly stable matching in X.

We color each agent in $\hat{A}$ green or red as follows. For any agent $\alpha \in A$, we color the non-dummy agent $\hat{\alpha}$ corresponding to α green if $\hat{\mu}(\hat{\alpha}) \in \hat{P}'_{\alpha}$, and red otherwise. All dummy agents in $\hat{A}$ are colored red.

In Lemma 4 below, we show that if the non-dummy agent in a given gadget is red, then many of the agents in the gadget are matched to agents in the same gadget.

Remark 5 In the proof of Lemma 4, we can think of $\hat{\alpha}_0, \ldots, \hat{\alpha}_{k-1}$ as a chain of "admirers" in the gadget corresponding to α^*, where $\hat{\alpha}_s$ prefers $\hat{\alpha}_{s+1}$ to $\hat{\mu}(\hat{\alpha}_s)$. By applying the weak stability condition to this chain of admirers, we show that $\hat{\alpha}_{k-1}$ is matched to a partner no worse than $\hat{\alpha}_0$.

Lemma 4 *Let $t^* \in \{0, \ldots, k-1\}$ and $\alpha^* \in I \times \{t^*\}$ such that the non-dummy agent $(0, \alpha^*, t^*)$ corresponding to α^* is red. Let $t \in \{0, \ldots, k-1\}$ and $j \in J$ such that $j \leq (k-1) \cdot (k-2)$. Then $\hat{\mu}(j, \alpha^*, t) \in \{0, \ldots, j+k-1\} \times \{\alpha^*\} \times \{t \oplus 1\}$.*

Proof For any $s \in \{0, \ldots, k-1\}$, let $\hat{A}_s$ denote

$$\langle (j', \alpha', t') \in J \times \{\alpha^*\} \times \{t \oplus s \oplus 1\} \mid j' \leq j+k-s-1 \rangle.$$

Note that $j+k-s-1$ is nonnegative and at most $(k-1)(k-2)+k-1 = (k-1)^2$, so the list $\hat{A}_s$ is well-defined and has length $j+k-s$. (In the terminology of Figure 2, $\hat{A}_s$ is the ordered list of the leftmost $j+k-s$ agents in the row of $(k-1)^2+1$ type-$(t \oplus s \oplus 1)$ agents in the gadget corresponding to α^*.) Before proceeding with the proof of the lemma, we establish the following claim.

Claim: Let $s \in \{0, \ldots, k-1\}$ and let $\hat{\alpha}$ be an agent in $\hat{A}_s$ such that $\hat{\mu}(\hat{\alpha})$ is not in $\hat{A}_{s \oplus 1}$. Then $\hat{\alpha}$ prefers every agent in $\hat{A}_{s \oplus 1}$ to $\hat{\mu}(\hat{\alpha})$.

We prove the claim by considering three cases.

Case 1: $\hat{\alpha}$ is a non-dummy agent. Since $\hat{\alpha}$ is in $\hat{A}_s$, we have $\hat{\alpha} = (0, \alpha^*, t^*)$. Hence $\hat{\alpha}$ is red. The definition of the preference list $\hat{P}_{\hat{\alpha}}$ (for a non-dummy agent $\hat{\alpha}$) implies that $\hat{P}'_{\alpha^*} \cdot \hat{A}_{s \oplus 1}$ is a prefix of $\hat{P}_{\hat{\alpha}}$. Since $\hat{\alpha}$ is red, we deduce that $\hat{\mu}(\hat{\alpha})$ is not in $\hat{P}'_{\alpha^*}$, and the claim follows.

Case 2: $\hat{\alpha}$ is a non-boundary dummy agent. The definition of the preference list $\hat{P}_{\hat{\alpha}}$ (for a non-boundary dummy agent $\hat{\alpha}$) implies that $\hat{A}_{s \oplus 1}$ is a prefix of $\hat{P}_{\hat{\alpha}}$. The claim follows.

Case 3: $\hat{\alpha}$ is a boundary dummy agent. In this case, $\hat{A}_s$ is of length $(k-1)^2+1$. Hence $j+k-s = (k-1)^2+1$, from which it follows that $j = (k-1)(k-2)$ and $s = 0$. Thus $\hat{A}_{s \oplus 1} = \hat{A}_1$ is of length $(k-1)^2$, and hence the definition of the preference list $\hat{P}_{\hat{\alpha}}$ (for a boundary dummy agent $\hat{\alpha}$) implies that $\hat{A}_{s \oplus 1}$ is a prefix of $\hat{P}_{\hat{\alpha}}$. The claim follows.

Having established the claim, we now proceed with the proof of the lemma. Assume for the sake of contradiction that $\hat{\mu}(j, \alpha^*, t)$ is not in $\hat{A}_0$.

For every $s \in \{0, \ldots, k-2\}$, since the length of $\hat{A}_s$ is greater than the length of $\hat{A}_{s+1}$, there exists $\hat{\alpha}_s$ in $\hat{A}_s$ such that $\hat{\mu}(\hat{\alpha}_s)$ is not in $\hat{A}_{s+1}$. Let $\hat{\alpha}_{k-1}$ denote (j, α^*, t). Then $\hat{\alpha}_{k-1}$ is in $\hat{A}_{k-1}$ and $\hat{\mu}(\hat{\alpha}_{k-1})$ is not in $\hat{A}_0$.

Since $\hat{\mu}$ is a weakly stable matching of $\hat{X}$, the family $(\hat{\alpha}_{k-t-1}, \ldots, \hat{\alpha}_{(k-t-1) \oplus (k-1)})$ is not strongly blocking. So there exists $s \in \{0, \ldots, k-1\}$ such that $\hat{\alpha}_s$ does not prefer $\hat{\alpha}_{s \oplus 1}$ to $\hat{\mu}(\hat{\alpha}_s)$. This contradicts the claim, since $\hat{\alpha}_s$ is in $\hat{A}_s$, $\hat{\mu}(\hat{\alpha}_s)$ is not in $\hat{A}_{s \oplus 1}$, and $\hat{\alpha}_{s \oplus 1}$ is in $\hat{A}_{s \oplus 1}$. □

In Lemma 5 below, we apply Lemma 4 inductively to show that if the non-dummy agent in a given gadget is red, then every agent in its family belongs to the same gadget.

Lemma 5 *Let $j_0,\ldots,j_{k-1} \in J$ and $\alpha_0,\ldots,\alpha_{k-1} \in A$ such that*

$$((j_0,\alpha_0,0),\ldots,(j_{k-1},\alpha_{k-1},k-1)) \in \hat{\mu}.$$

Let $t^* \in \{0,\ldots,k-1\}$ such that $(j_{t^*},\alpha_{t^*},t^*) = (0,\alpha_{t^*},t^*)$ is the non-dummy agent corresponding to α_{t^*} and is red. Then, for every $s \in \{0,\ldots,k-1\}$, we have $\alpha_{t^*\oplus s} = \alpha_{t^*}$ and $j_{t^*\oplus s} \le (k-1)\cdot s$.

Proof We prove the claim by induction on s. When $s = 0$, we have $\alpha_{t^*\oplus s} = \alpha_{t^*\oplus 0} = \alpha_{t^*}$ and $j_{t^*\oplus s} = j_{t^*} = 0 \le (k-1)\cdot s$.

Suppose $\alpha_{t^*\oplus(s-1)} = \alpha_{t^*}$ and $j_{t^*\oplus(s-1)} \le (k-1)\cdot(s-1)$, where $s \in \{1,\ldots,k-1\}$. Let $t = t^*\oplus(s-1)$. Then $\alpha_t = \alpha_{t^*\oplus(s-1)} = \alpha_{t^*}$ and

$$\begin{aligned} j_t &= j_{t^*\oplus(s-1)} \\ &\le (k-1)\cdot(s-1) \\ &\le (k-1)\cdot(k-2). \end{aligned}$$

So Lemma 4 implies that $\hat{\mu}(j_t,\alpha_{t^*},t) \in \{0,\ldots,j_t+k-1\}\times\{\alpha_{t^*}\}\times\{t\oplus 1\}$. Hence $j_{t\oplus 1} \le j_t+k-1$ and $\alpha_{t\oplus 1} = \alpha_{t^*}$, since $\hat{\mu}(j_t,\alpha_{t^*},t) = \hat{\mu}(j_t,\alpha_t,t) = (j_{t\oplus 1},\alpha_{t\oplus 1},t\oplus 1)$. Thus $\alpha_{t^*\oplus s} = \alpha_{t\oplus 1} = \alpha_{t^*}$ and

$$\begin{aligned} j_{t^*\oplus s} &= j_{t\oplus 1} \\ &\le j_t+k-1 \\ &= j_{t^*\oplus(s-1)}+k-1 \\ &\le (k-1)\cdot(s-1)+k-1 \\ &= (k-1)\cdot s, \end{aligned}$$

as required. □

We say that a family in $\hat{\mu}$ is monochromatic if all of the agents in the family have the same color (green or red).

Lemma 6 *Every family in $\hat{\mu}$ is monochromatic.*

Proof Assume for the sake of contradiction that $((j_0,\alpha_0,0),\ldots,(j_{k-1},\alpha_{k-1},k-1))$ is a non-monochromatic family in $\hat{\mu}$. Then there is a $t \in \{0,\ldots,k-1\}$ such that agent (j_t,α_t,t) is green and agent $(j_{t\oplus 1},\alpha_{t\oplus 1},t\oplus 1)$ is red. Since (j_t,α_t,t) is a green agent, we deduce that $(j_{t\oplus 1},\alpha_{t\oplus 1},t\oplus 1)$ is a red non-dummy agent. Hence Lemma 5 implies that $\alpha_t = \alpha_{t\oplus 1}$. Thus $(j_t,\alpha_t,t) = (j_t,\alpha_{t\oplus 1},t)$ is a dummy agent in the gadget corresponding to $\alpha_{t\oplus 1}$, and hence is red, a contradiction. □

We say that a family in $\hat{\mu}$ is green if all of the agents in the family are green. Since every green agent is a non-dummy, any green family in $\hat{\mu}$ is of the form $((0,\alpha_0,0),\ldots,(0,\alpha_{k-1},k-1))$ where $\alpha_t \in I\times\{t\}$ for each $t \in \{0,\ldots,k-1\}$.

Lemma 7 *Let*

$$((0,\alpha_0,0),\ldots,(0,\alpha_{k-1},k-1))$$

be a green family in $\hat{\mu}$. *Then* $(\alpha_0,\ldots,\alpha_{k-1})$ *is a family in* X.

Proof Let $t \in \{0,\ldots,k-1\}$. As remarked above, $\alpha_t \in I \times \{t\}$. It remains to prove that $\alpha_{t\oplus 1} \in P_{\alpha_t}$. Since agent $(0,\alpha_t,t)$ is green, we have $\hat{\mu}(0,\alpha_t,t) = (0,\alpha_{t\oplus 1},t\oplus 1) \in \hat{P}'_{\alpha_t}$, and hence $\alpha_{t\oplus 1} \in P_{\alpha_t}$, as required. □

For any green family $\hat{F} = ((0,\alpha_0,0),\ldots,(0,\alpha_{k-1},k-1))$ in $\hat{\mu}$, we refer to the family $(\alpha_0,\ldots,\alpha_{k-1})$ of Lemma 7 as the family in X induced by $\hat{F}$. The green families in $\hat{\mu}$ induce agent-disjoint families in X. Thus the set of families in X induced by all of the green families in $\hat{\mu}$ is a matching in X; we refer to this matching as the matching in X induced by $\hat{\mu}$. We are now ready to complete the proof of Lemma 2.

Proof of Lemma 2 Let μ denote the matching in X induced by $\hat{\mu}$. It is sufficient to prove that μ is a weakly stable matching of X. Assume for the sake of contradiction that μ admits a strongly blocking family $(\alpha_0,\ldots,\alpha_{k-1})$. Since $\hat{\mu}$ is a weakly stable matching of $\hat{X}$, the family of non-dummy agents

$$((0,\alpha_0,0),\ldots,(0,\alpha_{k-1},k-1))$$

is not strongly blocking. So there exists $t \in \{0,\ldots,k-1\}$ such that $(0,\alpha_t,t)$ does not prefer $(0,\alpha_{t\oplus 1},t\oplus 1)$ to $\hat{\mu}(0,\alpha_t,t)$. Since $(\alpha_0,\ldots,\alpha_{k-1})$ is a family in X, agent $\alpha_{t\oplus 1}$ is in P_{α_t}. So $(0,\alpha_{t\oplus 1},t\oplus 1)$ is in $\hat{P}'_{\alpha_t}$. Hence $\hat{\mu}(0,\alpha_t,t)$ appears in $\hat{P}'_{\alpha_t}$ no later than $(0,\alpha_{t\oplus 1},t\oplus 1)$, since $\hat{P}'_{\alpha_t}$ is a prefix of the preference list $\hat{P}_{(0,\alpha_t,t)}$.

Since $\hat{\mu}(0,\alpha_t,t)$ is in $\hat{P}'_{\alpha_t}$, the non-dummy agent $(0,\alpha_t,t)$ is green. Thus Lemma 6 implies that the family in $\hat{\mu}$ containing $(0,\alpha_t,t)$ is green. Since μ is the matching in X induced by $\hat{\mu}$, we deduce that $\hat{\mu}(0,\alpha_t,t) = (0,\mu(\alpha_t),t\oplus 1)$. Since $(0,\mu(\alpha_t),t\oplus 1)$ appears in $\hat{P}'_{\alpha_t}$ no later than $(0,\alpha_{t\oplus 1},t\oplus 1)$, agent $\mu(\alpha_t)$ appears in P_{α_t} no later than $\alpha_{t\oplus 1}$. Hence α_t does not prefer $\alpha_{t\oplus 1}$ to $\mu(\alpha_t)$. So $(\alpha_0,\ldots,\alpha_{k-1})$ is not a strongly blocking family of μ, a contradiction. □

4.4 Proof of Lemma 3

The goal of this subsection is to prove Lemma 3. It suffices to show that every weakly stable matching μ in X induces a weakly stable matching $\hat{\mu}$ in $\hat{X}$. We construct the matching $\hat{\mu}$ induced by μ as follows.

- For every $(\alpha_0,\ldots,\alpha_{k-1}) \in \mu$, we include in $\hat{\mu}$ the family

$$((0,\alpha_0,0),\ldots,(0,\alpha_{k-1},k-1)).$$

- For every agent $\alpha \in A$ and $j \in J$ such that $j < (k-1)^2$, we include in $\hat{\mu}$ the family $((j+\delta_0(\alpha),\alpha,0),\ldots,(j+\delta_{k-1}(\alpha),\alpha,k-1))$, where

$$\delta_t(\alpha) = \begin{cases} 1 & \text{if } \mu(\alpha) \neq \alpha \text{ and } \alpha \in I \times \{t\} \\ 0 & \text{otherwise} \end{cases}$$

– For every $t \in \{0,\dots,k-1\}$, let R_t be the list

$$\langle (j',\alpha',t') \in \{(k-1)^2\} \times A \times \{t\} \mid \delta_{t'}(\alpha') = 0 \rangle.$$

We include in $\hat{\mu}$ the family $(R_0[s],\dots,R_{k-1}[s])$ for every $0 \le s < |A| - |\mu|$, where $R_t[s]$ denotes the $(s+1)$th element of R_t.

Figures 2(b) and 2(c) show the gadget under the matching $\hat{\mu}$.

It is straightforward to check that the families in $\hat{\mu}$ induced by a matching μ are agent-disjoint. Hence $\hat{\mu}$ is a valid matching in $\hat{X}$.

Lemma 8 *Let $\hat{\mu}$ be the matching in $\hat{X}$ induced by a matching μ in X. Let $t \in \{0,\dots,k-1\}$ and $\alpha \in A$ such that $\alpha \in I \times \{t\}$. Let $j' \in J$ and $\alpha' \in A$ such that non-dummy agent $(0,\alpha,t)$ prefers $(j',\alpha',t \oplus 1)$ to $\hat{\mu}(0,\alpha,t)$. Then $(j',\alpha',t \oplus 1)$ is in $\hat{P}'_\alpha$ and α prefers α' to $\mu(\alpha)$.*

Proof Notice that $\hat{P}'_\alpha \cdot \langle (0,\alpha,t \oplus 1) \rangle$ is a prefix of the preference list $\hat{P}_{(0,\alpha,t)}$ of non-dummy agent $(0,\alpha,t)$. We consider two cases.

Case 1: $\mu(\alpha) \neq \alpha$. Then $\hat{\mu}(0,\alpha,t) = (0,\mu(\alpha),t \oplus 1)$. Since agent $(0,\alpha,t)$ prefers $(j',\alpha',t \oplus 1)$ to $(0,\mu(\alpha),t \oplus 1)$, we deduce that $(j',\alpha',t \oplus 1)$ appears in $\hat{P}'_\alpha$ before $(0,\mu(\alpha),t \oplus 1)$. Hence α prefers α' to $\mu(\alpha)$.

Case 2: $\mu(\alpha) = \alpha$. Then $\hat{\mu}(0,\alpha,t) = (0,\alpha,t \oplus 1)$. Since agent $(0,\alpha,t)$ prefers $(j',\alpha',t \oplus 1)$ to $(0,\alpha,t \oplus 1)$, we deduce that $(j',\alpha',t \oplus 1)$ is in $\hat{P}'_\alpha$. Thus α' is in P_α, and hence α prefers α' to $\mu(\alpha)$. □

Lemma 9 *Let $\hat{\mu}$ be the matching in $\hat{X}$ induced by a weakly stable matching μ in X. Let $j_0,\dots,j_{k-1} \in J$ and $\alpha_0,\dots,\alpha_{k-1} \in A$ such that*

$$((j_0,\alpha_0,0),\dots,(j_{k-1},\alpha_{k-1},k-1))$$

is a strongly blocking family of $\hat{\mu}$. Then $j_t - \delta_t(\alpha_t) \ge (k-1)^2$ for every $t \in \{0,\dots,k-1\}$.

Proof Let $t^* \in \{0,\dots,k-1\}$ such that

$$j_{t^*} - \delta_{t^*}(\alpha_{t^*}) = \min_{t \in \{0,\dots,k-1\}} (j_t - \delta_t(\alpha_t)).$$

For the sake of contradiction, suppose $j_{t^*} - \delta_{t^*}(\alpha_{t^*}) < (k-1)^2$. We consider two cases.

Case 1: $j_{t^*} = 0$ and $\alpha_{t^*} \in I \times \{t^*\}$. Let $T = \{t \mid j_t = 0 \text{ and } \alpha_t \in I \times \{t\}\}$. Then $t^* \in T$. We consider two subcases.

Case 1.1: $T = \{0,\dots,k-1\}$. Then for every $t \in \{0,\dots,k-1\} = T$, since $(0,\alpha_t,t)$ prefers $(0,\alpha_{t\oplus 1},t \oplus 1)$ to $\hat{\mu}(0,\alpha_t,t)$, Lemma 8 implies that α_t prefers $\alpha_{t\oplus 1}$ to $\mu(\alpha_t)$. Hence $(\alpha_0,\dots,\alpha_{k-1})$ is a strongly blocking family of μ, which contradicts the stability of μ.

Case 1.2: $\{t^*\} \subseteq T \subsetneq \{0,\dots,k-1\}$. Then there exists s^* such that $s^* \in T$ and $s^* \oplus 1 \notin T$. Since $s^* \in T$, we have $j_{s^*} = 0$ and $\alpha_{s^*} \in I \times \{s^*\}$. Since $(0,\alpha_{s^*},s^*)$ prefers $(j_{s^*\oplus 1},\alpha_{s^*\oplus 1},s^* \oplus 1)$ to $\hat{\mu}(0,\alpha_{s^*},s^*)$, Lemma 8 implies that $(j_{s^*\oplus 1},\alpha_{s^*\oplus 1},s^* \oplus 1)$ is in $\hat{P}'_{\alpha_{s^*}}$. Hence $j_{s^*\oplus 1} = 0$ and $\alpha_{s^*\oplus 1} \in I \times \{s^* \oplus 1\}$, which contradicts $s^* \oplus 1 \notin T$.

Case 2: Either $j_{t^*} \neq 0$ or $\alpha_{t^*} \notin I \times \{t^*\}$. Thus $(j_{t^*}, \alpha_{t^*}, t^*)$ is a dummy agent. We consider two subcases.

Case 2.1: $j_{t^*} < (k-1)^2$. Since the non-boundary dummy agent $(j_{t^*}, \alpha_{t^*}, t^*)$ prefers $(j_{t^*\oplus 1}, \alpha_{t^*\oplus 1}, t^* \oplus 1)$ to $\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*)$ and

$$\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*) = (j_{t^*} + \delta_{t^*\oplus 1}(\alpha_{t^*}) - \delta_{t^*}(\alpha_{t^*}), \alpha_{t^*}, t^* \oplus 1),$$

we have $j_{t^*\oplus 1} < j_{t^*} + \delta_{t^*\oplus 1}(\alpha_{t^*}) - \delta_{t^*}(\alpha_{t^*})$, which contradicts the definition of t^*.

Case 2.2: $j_{t^*} = (k-1)^2$. Then $\delta_{t^*}(\alpha_{t^*}) = 1$ since $j_{t^*} - \delta_{t^*}(\alpha_{t^*}) < (k-1)^2$. So $\alpha_{t^*} \in I \times \{t^*\}$, and hence $\delta_{t^*\oplus 1}(\alpha_{t^*}) = 0$.

Since the boundary dummy agent $(j_{t^*}, \alpha_{t^*}, t^*)$ prefers $(j_{t^*\oplus 1}, \alpha_{t^*\oplus 1}, t^* \oplus 1)$ to $\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*)$ and

$$\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*) = (j_{t^*} - 1, \alpha_{t^*}, t^* \oplus 1),$$

we have $j_{t^*\oplus 1} < j_{t^*} - 1 = j_{t^*} + \delta_{t^*\oplus 1}(\alpha_{t^*}) - \delta_{t^*}(\alpha_{t^*})$, which contradicts the definition of t^*. □

Proof of Lemma 3 Suppose X has a weakly stable matching μ. Let $\hat{\mu}$ be the matching in $\hat{X}$ induced by μ. It suffices to show that $\hat{\mu}$ does not admit a strongly blocking family.

For the sake of contradiction, suppose $\hat{\mu}$ admits a strongly blocking family

$$((j_0, \alpha_0, 0), \ldots, (j_{k-1}, \alpha_{k-1}, k-1)).$$

Lemma 9 implies that for every $t \in \{0, \ldots, k-1\}$, we have $j_t - \delta_t(\alpha_t) \geq (k-1)^2$. Since $j_t \leq (k-1)^2$ and $\delta_t(\alpha_t) \geq 0$, we deduce that $j_t = (k-1)^2$ and $\delta_t(\alpha_t) = 0$ for every $t \in \{0, \ldots, k-1\}$. Hence for every $t \in \{0, \ldots, k-1\}$, there exists s_t such that $(j_t, \alpha_t, t) = R_t[s_t]$.

Let $t^* \in \{0, \ldots, k-1\}$ such that

$$s_{t^*} = \min_{t \in \{0, \ldots, k-1\}} s_t.$$

Since $\hat{\mu}(R_{t^*}[s_{t^*}]) = R_{t^*\oplus 1}[s_{t^*}]$ and the boundary dummy agent $R_{t^*}[s_{t^*}]$ prefers boundary dummy agent $R_{t^*\oplus 1}[s_{t^*\oplus 1}]$ to boundary dummy agent $\hat{\mu}(R_{t^*}[s_{t^*}])$, we deduce that $R_{t^*\oplus 1}[s_{t^*\oplus 1}]$ is lexicographically smaller than $R_{t^*\oplus 1}[s_{t^*}]$. Hence $s_{t^*\oplus 1} < s_{t^*}$, which contradicts the definition of t^*. □

5 Concluding Remarks

We have shown that a 3-DSM-CYC instance need not admit a weakly stable matching, and that it is NP-complete to determine whether a given 3-DSM-CYC instance admits a weakly stable matching. It seems that for the three-dimensional stable matching problem, none of the preference structures studied in the literature admits a non-trivial generalization of the existence theorem of Gale and Shapley. (The existence result in Danilov's model [5] follows from applying the Gale-Shapley algorithm in a straightforward manner.) It would be interesting to consider solution concepts such as popular matchings instead of stable matchings in the multi-dimensional matching context.

The 3-DSMI-CYC instance with no weakly stable matching presented by Biró and McDermid [2, Lemma 1] has six agents of each type. The reduction of Section 4.1 blows up the number of agents by a factor of $k[(k-1)^2+1]$. Thus, for $k=3$, we obtain an explicit construction of a 3-DSM-CYC instance with no weakly stable matching and $6 \cdot 15 = 90$ agents of each type. It would be interesting to identify smaller 3-DSM-CYC instances with no weakly stable matching.

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