arXiv:2607.18634v3 [cs.GT] 2 Sep 2026

Hospitals/Residents with Inseparable Couples: Finding a Coalition-Stable Assignment Is NP-Hard

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Abstract. In recent work on course allocation, Rodríguez and Manlove consider the complexity of finding a stable assignment under four notions of stability, including two coalitional notions. In one case, which they call pair-size stability, they show that a stable assignment always exists and they provide a polynomial-time algorithm to find one. In a second case, called pair stability, they observe that an earlier NP-hardness result of McDermid and Manlove holds for a special case of course allocation called *Hospitals/Residents with Sizes* (HRS). In a third case, called first-coalition stability, they use a reduction from HRS to show it is NP-hard to find a stable assignment. They leave open the complexity of finding a stable assignment under so-called coalition stability. Building on ideas from McDermid and Manlove, we resolve the open problem of Rodríguez and Manlove by showing that it is NP-hard to find a coalition-stable assignment for HRS. Indeed, our proof shows that the problem remains NP-hard when the hospital capacities and resident sizes are at most two. Accordingly, our NP-hardness result applies to the special case of HRS known as *Hospitals/Residents with Inseparable Couples* (HRIC). Finally, we introduce a novel and natural notion of coalitional stability for both HRS and course allocation, and we show that our NP-hardness result extends to this notion, which we call unitwise-coalition stability.

Keywords: stable matching · hospitals/residents · course allocation · coalition stability · NP-hardness

1 Introduction

We begin by defining a well-known "hospitals and residents" variant of the classic stable marriage problem [8], which we denote HR.[3] We are given a set of hospitals and a set of residents; the goal is to determine a "good" assignment of residents to hospitals. Each resident (resp., hospital) has strict preferences over the set of hospitals (resp., residents) that it finds acceptable.[4] Each hospital h has a

[3] In [8], this is referred to as the college admissions problem.

[4] Since we will only consider assigning resident r to hospital h if h is acceptable to r and r is acceptable to h, we can assume without loss of generality that h appears on the preference list of r if and only if r appears on the preference list of h.

positive integer capacity that represents the maximum number of residents that can be assigned to h. An assignment for a given HR instance is a set μ of resident-hospital pairs (if (r, h) belongs to μ, then μ is said to assign resident r to hospital h) subject to the following constraints.

1. Acceptability constraints. If (r, h) belongs to μ, then h is acceptable to r and r is acceptable to h.
2. Resident degree constraints. Any resident appears in at most one pair in μ.
3. Hospital capacity constraints. The number of pairs in μ involving a hospital h is at most the capacity of h.

A resident r is said to be unassigned under μ if no pair in μ involves r. For any assignment μ and any hospital h, we define $\text{res}(\mu, h)$ as $\{r \mid (r, h) \in \mu\}$. A hospital h is said to be under-subscribed under μ if $|\text{res}(\mu, h)|$ is less than the capacity of h. A resident-hospital pair (r, h) is said to be a blocking pair for an assignment μ if (1) r is unassigned under μ or r prefers h to its assigned hospital under μ and (2) h is under-subscribed under μ or h prefers r to its least preferred assigned resident under μ. An assignment is said to be stable if it does not admit a blocking pair.

In this paper we are concerned with a generalization of the HR problem called *Hospitals/Residents with Sizes* (HRS) introduced by McDermid and Manlove [9]. In this generalization, each resident has a positive integer size and the hospital capacity constraints are modified to require that the total size of the residents assigned to a hospital h is at most the capacity of h. McDermid and Manlove consider a resident-hospital pair (r, h) to be a blocking pair for an assignment μ if (1) r is unassigned under μ or r prefers h to its assigned hospital under μ and (2) the size of $\{r' \in \text{res}(\mu, h) \mid h \text{ prefers } r \text{ to } r'\}$ is at least the size of r minus the unused capacity of h under μ. As for the HR problem, an assignment is said to be stable if it does not admit a blocking pair.

Alternative notions of stability have been proposed for HRS. Adopting the terminology of Rodríguez and Manlove [10], we refer to the stability notion investigated by McDermid and Manlove [9] as pair stability. In the concluding remarks of their paper, McDermid and Manlove propose a second notion of stability for HRS in which condition (2) above is replaced by "there exists a subset of $\{r' \in \text{res}(\mu, h) \mid h \text{ prefers } r \text{ to } r'\}$ with total size at most the size of r and at least the size of r minus the unused capacity of h under μ." Rodríguez and Manlove refer to a resident-hospital pair (r, h) satisfying (1) and the modified version of (2) as a size-blocking pair. The associated notion of stability (i.e., absence of size-blocking pair) has subsequently been referred to as pair-size stability [10], occupancy-stability [4], and size-stability [7].[5] In the present paper, we use the term pair-size stability.

Rodríguez and Manlove [10] study a generalization of HRS to the setting of course allocation, where the goal is to assign students to courses. Unlike HRS, where each resident can be assigned to at most one hospital, a student can

[5] Delacrétaz studies the special case where the resident sizes belong to $\{1, 2\}$.

take many courses and a course can be taken by many students.[6] Rodríguez and Manlove prove that a pair-size stable assignment always exists in the course allocation setting, and they provide a polynomial-time algorithm for finding such an assignment. Independently, Balasundaram et al. [4] obtain the same result for pair-size stability (which they call occupancy stability) in the HRS setting.

Rodríguez and Manlove propose and investigate two "coalitional" notions of stability for course allocation, which they call first-coalition stability and coalition stability. For these notions a blocking coalition is given by a set of courses and a student, as opposed to a course-student pair. These coalitional notions are also applicable to the HRS special case of course allocation. Using notation similar to that introduced by Rodríguez and Manlove, we define HRS-C (resp., HRS-P, HRS-PS, and HRS-FC) as the problem of finding a coalition-stable (resp., pair-stable, pair-size-stable, first-coalition-stable) assignment for a given HRS instance, or reporting correctly that no such assignment exists. McDermid and Manlove show that HRS-P is NP-hard [9]. As discussed earlier, Rodríguez and Manlove provide a polynomial-time algorithm for finding a pair-size stable assignment in the course allocation setting (which generalizes HRS-PS). Rodríguez and Manlove show that HRS-FC is NP-hard, and leave open the complexity of HRS-C.

We now define the two coalitional notions of stability introduced by Rodríguez and Manlove [10]. Given the NP-hardness results discussed in the previous paragraph, we find it sufficient to define these notions in the restricted setting of HRS, as opposed to the more general course allocation setting. Given an assignment μ for an HRS instance, Rodríguez and Manlove define a blocking (resp., first-blocking) coalition for μ as a pair (R, h) where R is a set of residents and h is a hospital such that (1) for all r in R, r is unassigned under μ or r prefers h to its assigned hospital under μ, and (2) there exists a subset of $\{r' \in \mathrm{res}(\mu, h) \mid h \text{ prefers } r \text{ to } r' \text{ for all (resp., some) } r \text{ in } R\}$ with total size at most the total size of R and at least the total size of R minus the unused capacity of h under μ.[7]

Rodríguez and Manlove [10] consider six different problems (e.g., testing whether a given assignment is stable, finding a stable assignment) with respect to each of the four notions of stability discussed above. For some of the resulting 24 problem combinations, there is both an existence question (i.e., whether a stable assignment is guaranteed to exist) and a complexity question (i.e., what is the complexity of finding a stable assignment if one exists) to resolve. Rodríguez and Manlove resolve the questions associated with all 24 problem combinations, except they leave open the question of whether a coalition-stable assignment is

[6] We can model an HRS instance I as a course allocation instance I' as follows: for each size-k resident in I, we add a unit-capacity k-credit course to I'; for each capacity-k hospital in I, we add a student to I' with credit limit k.

[7] Our definitions of the terms "blocking coalition" and "first-blocking coalition", as well as our earlier definitions of the terms "blocking pair" and "size-blocking pair", are formulated somewhat differently than in earlier work, but are equivalent to the original definitions.

guaranteed to exist for any course allocation instance, along with the associated complexity question.[8] In the present paper, we resolve these questions by exhibiting an HRS instance with four residents and three hospitals that does not admit a coalition-stable assignment, and by proving that HRS-C is NP-hard.[9]

We show that HRS-C is NP-hard using the same high-level approach as McDermid and Manlove [9, Theorem 3.8, erratum] used to show that HRS-P is NP-hard. More specifically, we also use a reduction from a special case of 3-SAT called $(2, 2)$-E3-SAT; this satisfiability variant is formally defined in Section 2. The transformation of [9] from $(2, 2)$-E3-SAT to HRS-P is presented in a somewhat monolithic manner: parameterized sets of agents (residents and hospitals) are listed along with the associated preference lists [9, Figure 1, erratum]. At the same time, like many transformations from 3-SAT variants, the transformation admits a gadget-based interpretation: for each variable (resp., clause) in the given $(2, 2)$-E3-SAT instance, the resulting HRS-P instance includes a set of agents inducing a copy of a variable (resp., clause) gadget. Moreover, the variable and the clause gadgets are implicitly constructed from smaller gadgets. The variable gadget is constructed from a smaller gadget that admits exactly two stable assignments; in the present paper, we refer to this smaller gadget as a bistable gadget. The clause gadget is constructed using three copies of a smaller gadget, one for each literal in the clause; in the present paper, we refer to this smaller gadget as a literal gadget. The literal gadget is in turn constructed from a smaller gadget that does not admit a stable assignment; in the present paper, we refer to this smaller gadget as an unstable gadget.

We find it useful to present our transformation in a modular and explicitly gadget-based style. To aid the reader's intuition, we present a diagram of each gadget using the digraph representation proposed by Balinski and Ratier [5] and subsequently employed in various works (see, e.g., [1,2,3]); in the present paper, we refer to such a diagram as a preference digraph. To better illustrate the relationship between our gadgets and those of [9], in Appendix B, we present preference digraphs for the gadgets in [9].

We now highlight the main new ideas underlying our gadget-based transformation from $(2, 2)$-E3-SAT to HRS-C. A key difference between our setting and that of [9] is that a blocking coalition involving a hospital h cannot propose to reduce the total size of the set of residents assigned to h. This seems to necessitate a more complicated construction than in [9] for the unstable, literal, and bistable gadgets. For the unstable (resp., literal, bistable) gadget, one may compare the preference digraphs of Figures 1(a) and 7 (resp., Figures 2(a) and 8, Figures 3(a) and 9).

[8] Regarding the problem of testing whether a given assignment is coalition-stable, Rodríguez and Manlove establish co-NP-completeness in the context of course allocation; their proof also applies to the HRS special case.

[9] In [10], Rodríguez and Manlove also consider problem variants with so-called "downward-feasible constraints". Our hardness results hold even in the absence of such constraints.

In developing our variable gadget from our bistable gadget as in the preference digraph of Figure 4(a), we noticed that the corresponding construction in [9] can be streamlined via the same approach. The preference digraph of Figure 10 in Appendix B shows the non-streamlined variable gadget implicit in [9], which has two more residents and two more hospitals than the streamlined version.

While our literal gadget differs from that of [9], it satisfies similar abstract properties such that the construction of our clause gadget from three copies of the literal gadget can follow precisely the same pattern as is implicit in [9]. Likewise, our variable and clause gadgets satisfy similar abstract properties to those satisfied by the corresponding gadgets of [9], such that our overall construction of the HRS-C instance corresponding to a given $(2,2)$-E3-SAT instance follows the pattern implicit in [9]. An advantage of our modular gadget-based presentation is that individual modules may be reusable in proofs of other similar results. In Section 5, we discuss a natural coalitional notion of stability that is different from the two notions (first-coalition stability and coalition stability) defined earlier, and we explain why our NP-hardness result related to coalition stability also applies to this third notion, which we call unitwise-coalition stability.

Like the NP-hardness proof of McDermid and Manlove for HRS-P, our NP-hardness proof for HRS-C involves HRS instances where the resident sizes and hospital capacities belong to $\{1,2\}$. Because the resident sizes belong to $\{1,2\}$, these NP-hardness results apply to the important special case of HRS called *Hospitals/Residents with Inseparable Couples* (HRIC).

As in McDermid and Manlove [9], our NP-hardness proof involves HRS-C instances with incomplete preferences, that is, where only a subset of the residents (resp., hospitals) are acceptable to a given hospital (resp., resident). Indeed, our construction only requires preference lists of length 3 (resp., 4) for the residents (resp., hospitals). The reader may wonder whether our NP-hardness result continues to hold for HRS-C instances with complete preferences. In Section 4.4, we give a simple reduction showing that HRS-C remains NP-hard in the special case of complete preferences, even when the resident sizes and hospital capacities continue to belong to $\{1,2\}$.

In the foregoing, we have focused our discussion of the prior literature on results that are directly relevant to our NP-hardness proof. For a broader overview of two-sided assignment problems with preferences, we refer the reader to Rodríguez and Manlove [10].

2 Preliminaries

We now give a formal definition of the HRS-C problem discussed in Section 1. The input is an HRS instance, which we now describe. An HRS instance I consists of a set of residents, where each resident r has a positive integer size, denoted $\text{size}(r)$, and a set of hospitals, denoted $\text{hosps}(I)$, where each hospital h has a positive integer capacity, denoted $\text{cap}(h)$. For any set of residents R, we define $\text{size}(R)$ as $\sum_{r\in R}\text{size}(r)$. Each resident (resp., hospital) has a strict preference list over the set of hospitals (resp., residents) that it considers to be acceptable. An assignment μ for such an instance I is a set of resident-hospital pairs subject to certain constraints to be defined below. If the resident-hospital pair (r, h) belongs to an assignment μ, we say that r is assigned to h under μ. As in Section 1, we define $\text{res}(\mu, h)$ as $\{r \mid (r, h) \in \mu\}$. An assignment μ is required to satisfy the following constraints.

1. Acceptability constraints. If (r, h) belongs to μ, then h is acceptable to r and r is acceptable to h.
2. Resident degree constraints. Any resident appears in at most one pair in μ.
3. Hospital capacity constraints. For any hospital h, we have $\text{size}(\text{res}(\mu, h)) \leq \text{cap}(h)$.

We assume without loss of generality that for any resident r and hospital h, r considers h to be acceptable if and only if h considers r to be acceptable.

Given such an HRS instance I and an assignment μ for I, we say that a set of residents R and a hospital h form a blocking coalition (R, h) for μ in I if the following conditions hold: (1) for all r in R, r is unassigned under μ or r prefers h to its assigned hospital under μ, and (2) there exists a subset R' of $\{r' \in \text{res}(\mu, h) \mid h \text{ prefers } r \text{ to } r' \text{ for all } r \text{ in } R\}$ such that

$$\text{size}(R) - \text{cap}(h) + \text{size}(\text{res}(\mu, h)) \leq \text{size}(R') \leq \text{size}(R).$$

For any HRS instance I and any assignment μ for I, we define $\text{block}(I, \mu)$ as the set of all blocking coalitions for μ in I, and we say that μ is coalition-stable for I if $\text{block}(I, \mu) = \emptyset$. Given an HRS instance I as input, the HRS-C problem asks us to either find a coalition-stable assignment for I, or to report correctly that no such assignment exists. The decision version of HRS-C asks whether a given HRS instance admits a coalition-stable assignment.

Our main result is to show that HRS-C is NP-hard. We obtain this result by presenting a polynomial-time transformation from the known NP-complete problem $(2, 2)$-E3-SAT to the decision version of HRS-C. The problem $(2, 2)$-E3-SAT is a special case of the well-known 3-SAT problem that was shown to be NP-complete by Berman et al. [6, Theorem 1] (where $(2, 2)$-E3-SAT is referred to as (3,B2)-SAT). The input is a 3-CNF formula, that is, a propositional formula that is the conjunction of a number of clauses, where each clause is a disjunction of three distinct literals, and each literal is a variable or the negation of a variable. In addition, we require that each variable occurs exactly four times, twice in negated form and twice in positive (i.e., not negated) form. Given this requirement, any instance of $(2, 2)$-E3-SAT involves $3n$ variables and $4n$ clauses for some positive integer n.

Let ϕ be an instance of $(2, 2)$-E3-SAT and let n be a positive integer such that ϕ has $3n$ variables and $4n$ clauses. We index the variables from 1 to $3n$ and the clauses from 1 to $4n$. For any j in $\{1, \dots, 3n\}$ and any k in $\{1, 2, 3, 4\}$, we define literal (j, k) as follows: if $k = 1$ (resp., $k = 2$) then it is the first (resp., second) positive instance of variable j in ϕ; if $k = 3$ (resp., $k = 4$) then it is the first (resp., second) negative instance of variable j in ϕ.

Our NP-hardness proof involves the construction of various HRS-C "gadgets", some of which are defined in terms of smaller "sub-gadgets". The following additional definitions, along with Lemmas 2.1 and 2.2 below, are useful for reasoning about the interface between a gadget and one of its sub-gadgets. For any HRS-C instance I and any subset H of the hospitals in I, we define $I\big|_H$ as the HRS-C instance obtained from I by removing all residents who do not have at least one acceptable hospital in H, removing all hospitals not in H, and pruning the removed hospitals from the preference lists of the non-removed residents. For any HRS-C instance I, any assignment μ of I, and any subset H of the hospitals in I, we define $\mu\big|_H$ as the assignment $\{(r,h) \in \mu \mid h \in H\}$ for $I\big|_H$, and we define $B\big|_H$ as $\{(R,h) \in B \mid h \in H\}$ where B denotes $\text{block}(I,\mu)$.

Lemma 2.1. *Let I be an HRS-C instance, let μ be an assignment of I, let B denote* $\text{block}(I,\mu)$*, let H be a subset of* $\text{hosps}(I)$*, let I' denote $I\big|_H$, let B' denote $B\big|_H$, let μ' denote $\mu\big|_H$, and let B'' denote* $\text{block}(I',\mu')$*. Then $B' \subseteq B''$.*

Proof. Let (R,h) be a blocking coalition in B'. Hence h is a hospital in H and R is a set of residents in I'. Below we establish the claim of the lemma by proving that (R,h) belongs to B''.

We begin by arguing that condition (1) in the definition of a blocking coalition in B'' holds for (R,h). Let r belong to R. If r is unassigned in μ or (r,h') belongs to μ where h' belongs to $\text{hosps}(I) \setminus H$, then r is unassigned in μ' and hence r prefers h to its assignment in μ', as required by condition (1). If (r,h') belongs to μ where h' belongs to H, then (r,h') belongs to μ', and since (R,h) belongs to B', we conclude that r prefers h to h', as required by condition (1).

It remains to argue that condition (2) in the definition of a blocking coalition in B'' holds for (R,h). This follows because the set of acceptable residents of h is the same in I and I', and $\text{res}(\mu,h) = \text{res}(\mu',h)$. ☐

For any HRS-C instance I, any subset H of $\text{hosps}(I)$, and any assignment μ of I, we define $\text{bdry}(I,H,\mu)$ as the set of all blocking coalitions (R,h) in $\text{block}(I\big|_H,\mu\big|_H)$ for which there is a resident r in R and a hospital h' in $\text{hosps}(I) \setminus H$ such that r prefers h' to h.

Lemma 2.2. *Let I be an HRS-C instance, let μ be an assignment of I, let B denote* $\text{block}(I,\mu)$*, let H be a subset of* $\text{hosps}(I)$*, let I' denote $I\big|_H$, let B' denote $B\big|_H$, let μ' denote $\mu\big|_H$, and let B'' denote* $\text{block}(I',\mu')$*. Then $B'' \subseteq B' \cup \text{bdry}(I,H,\mu)$.*

Proof. Let (R,h) be a blocking coalition in $B'' \setminus \text{bdry}(I,H,\mu)$ where h is a hospital in H and R is a set of residents in I'. Below we establish the claim of the lemma by proving that (R,h) belongs to B'. Since h belongs to H, this is equivalent to proving that (R,h) belongs to B.

We begin by arguing that condition (1) in the definition of a blocking coalition in B holds for (R,h). Let r belong to R. We consider three cases.

Case 1: r is unassigned in μ. Hence r prefers h to its assignment in μ, as required by condition (1).

Case 2: (r, h') belongs to μ where h' belongs to $\text{hosps}(I) \setminus H$. Since (R, h) does not belong to $\text{bdry}(I, H, \mu)$, we deduce that r prefers h to h', as required by condition (1).

Case 3: (r, h') belongs to μ where h' belongs to H. Hence (r, h') belongs to μ'. Since (R, h) belongs to B'', we conclude that r prefers h to h', as required by condition (1).

It remains to argue that condition (2) in the definition of a blocking coalition in B holds for (R, h). This follows because the set of acceptable residents of h is the same in I and I', and $\text{res}(\mu, h) = \text{res}(\mu', h)$. ◻

3 Building Blocks

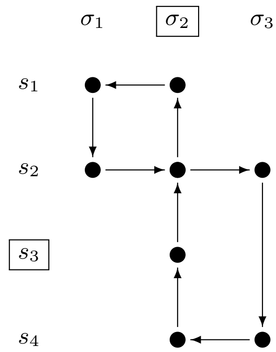
In this section, we present three gadgets. The first of these, which we call the unstable gadget, is an HRS-C instance with four residents and three hospitals that does not admit a coalition-stable assignment, thereby providing a negative answer to the question of Rodríguez and Manlove [10] regarding the existence of coalition-stable assignments. The second gadget that we present, which we call the literal gadget, contains a copy of the unstable gadget along with three additional residents and two additional hospitals. (One resident in the unstable gadget has its preference list extended to include one of the two additional hospitals.) In Section 4.2, we define a clause gadget that includes three copies of the literal gadget, one for each of the associated literals. The third gadget that we present, which we call the bistable gadget, has five residents and three hospitals and admits exactly two coalition-stable assignments. In Section 4.1, we define a variable gadget that includes a copy of the bistable gadget.

Our unstable, literal, and bistable gadgets are analogous to fragments of the monolithic construction presented by McDermid and Manlove [9] in the context of pair stability. Each of these gadgets differs from the corresponding fragment in its detailed design.[10] Throughout the remainder of this section, the only notion of stability we are concerned with is coalition stability. Accordingly, we find it convenient to use the term "stability" (resp., "stable") to mean "coalition stability" (resp., "coalition-stable"). For any HRS-C instance I, any assignment μ of I, and any subset H of $\text{hosps}(I)$, we say that μ is H-stable if no blocking coalition in $\text{block}(I, \mu)$ involves a hospital in H.

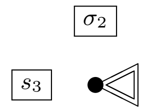
3.1 Unstable Gadget

In this section, we present an HRS-C instance that does not admit a stable assignment, thereby settling (in the negative) the open question of Rodríguez and Manlove [10] concerning whether a stable assignment always exists.

[10] Using the notation of [9, Figure 1, erratum], the correspondence is as follows: the subinstance induced by the set of residents $\{q^k_{j,1}, q^k_{j,2}, q^k_{j,3}\}$ and the set of hospitals $\{p^k_{j,1}, p^k_{j,2}\}$ is analogous to our unstable gadget; the subinstance induced by the set of residents $\{q^k_{j,1}, q^k_{j,2}, q^k_{j,3}, v(p^k_{j,3})\}$ and the set of hospitals $\{p^k_{j,1}, p^k_{j,2}, p^k_{j,3}\}$ is analogous to our literal gadget; the subinstance induced by the set of residents $\{r^1_j, r^2_j, r^3_j, r^4_j\}$ and the set of hospitals $\{h^1_j, h^2_j\}$ is analogous to our bistable gadget.



(a) The unstable gadget.



(b) The associated icon.

Figure 1: The preference digraph for the unstable gadget has four residents $s_1, \ldots, s_4$ and three hospitals $\sigma_1, \sigma_2, \sigma_3$ with sizes and capacities as indicated. The unstable gadget icon is used in the preference digraph of Figure 2(a); the dot in this icon corresponds to the resident-hospital pair (s_3, σ_2).

As discussed in Section 1, we find it useful to present all of our gadgets using a preference digraph. Figure 1(a) presents the preference digraph for our unstable gadget: each dot represents a mutually acceptable resident-hospital pair; dots in the same row (resp., column) have the same associated resident (resp., hospital); the symbol associated with each row (resp., column) represents the associated resident (resp., hospital); if the symbol is enclosed in a box then the size (resp., capacity) of the resident (resp., hospital) is 2, otherwise it is 1; the directed edges within a row (resp., column) are used to represent the preference list of the associated resident (resp., hospital). To understand how the directed edges determine preference order, it is sufficient to consider the example of hospital σ_2 in Figure 1(a). Hospital σ_2 has four acceptable residents s_1, s_2, s_3, and s_4. These four residents are linked together (within column σ_2) by three directed edges forming a simple path from s_4 to s_3 to s_2 to s_1. This path signifies that s_1 (resp., s_2, s_3, s_4) is the first (resp., second, third, fourth) preference of σ_2. Similarly, resident s_2 has three acceptable hospitals σ_1, σ_2, and σ_3, where σ_3 (resp., σ_2, σ_1) is the first (resp., second, third) preference of s_2.

Throughout the remainder of Section 3.1, let I denote the unstable gadget of Figure 1(a). We define the *critical assignment of I* as $\{(s_1, \sigma_1), (s_2, \sigma_3), (s_4, \sigma_2)\}$.

Lemma 3.1. *Instance I does not admit a stable assignment, and the critical assignment of I is the unique assignment μ of I such that* $\text{block}(I, \mu) = \{(s_3, \sigma_2)\}$.

Proof. Let μ^* denote the critical assignment of I and let B^* denote $\text{block}(I, \mu^*)$. We first argue that $B^* = \{(s_3, \sigma_2)\}$. It is easy to see that (s_3, σ_2) belongs to B^* and is the unique blocking coalition in B^* that involves resident s_3. Each resident in the set $\{s_1, s_2, s_4\}$ gets its most preferred hospital and so cannot participate in any blocking coalition. Thus $B^* = \{(s_3, \sigma_2)\}$, as required.

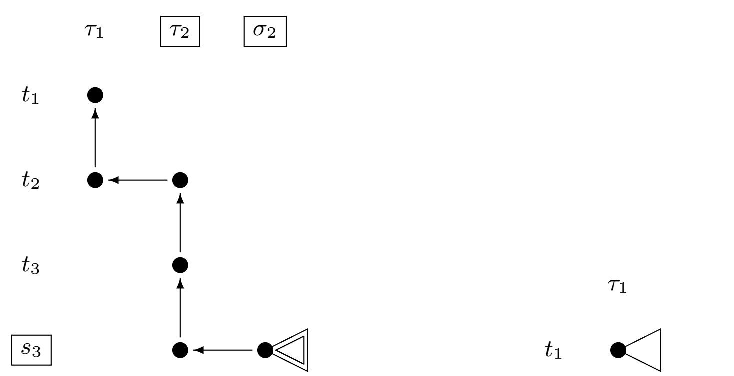
Let μ be an assignment for I such that $\text{block}(I, \mu) \subseteq \{(s_3, \sigma_2)\}$. Below we complete the proof by proving that $\mu = \mu^*$. Let B denote $\text{block}(I, \mu)$.

We begin by arguing that (s_1, σ_2) does not belong to μ. Assume for the sake of contradiction that (s_1, σ_2) belongs to μ. Hence (s_3, σ_2) does not belong to μ. In addition, since (s_1, σ_1) does not belong to B, we deduce that (s_2, σ_1) belongs to μ. Hence (s_2, σ_2) does not belong to μ and we conclude that $\text{res}(\mu, \sigma_2) \subseteq \{s_1, s_4\}$. It follows that (s_2, σ_2) belongs to B, a contradiction.

We now argue that (s_3, σ_2) does not belong to μ. Assume for the sake of contradiction that (s_3, σ_2) belongs to μ. Hence $\text{res}(\mu, \sigma_2) = \{s_3\}$. Since (s_4, σ_3) does not belong to B, we deduce that (s_4, σ_3) belongs to μ. Since (s_2, σ_1) does not belong to B, we deduce that (s_2, σ_1) belongs to μ. It follows that s_1 is unassigned in μ and hence $(\{s_1, s_2\}, \sigma_2)$ belongs to B, a contradiction.

Since (s_1, σ_2) and (s_3, σ_2) do not belong to μ, we conclude that $\text{res}(\mu, \sigma_2) \subseteq \{s_2, s_4\}$. Since (s_4, σ_2) does not belong to B, we deduce that (s_4, σ_2) belongs to μ. Hence $\text{res}(\mu, \sigma_3) \subseteq \{s_2\}$. Since (s_2, σ_3) does not belong to B, we deduce that (s_2, σ_3) belongs to μ. Hence $\text{res}(\mu, \sigma_1) \subseteq \{s_1\}$. Since (s_1, σ_1) does not belong to B, we deduce that (s_1, σ_1) belongs to μ. In summary, $\mu = \mu^*$, as required. □

3.2 Literal Gadget



(a) The literal gadget. (b) The associated icon.

Figure 2: The preference digraph for the literal gadget includes a copy of the unstable gadget and has a total of $3 + 4 = 7$ residents and $2 + 3 = 5$ hospitals. Within the literal gadget, resident s_3 has two acceptable hospitals τ_2 and σ_2, with τ_2 being the first preference of s_3 as indicated by the directed edge in row s_3. The literal gadget icon is used in the preference digraphs of Figures 5 and 6; the dot in this icon refers to the resident-hospital pair (t_1, τ_1).

Throughout the remainder of Section 3.2, let I denote the literal gadget of Figure 2(a) and let I' denote the unstable gadget of I. We define the *critical assignment of* I as the union of the critical assignment of I' and $\{(s_3, \tau_2), (t_2, \tau_1)\}$.

Lemma 3.2. *Instance I does not admit a stable assignment, and the critical assignment of I is the unique assignment μ of I such that* $\text{block}(I, \mu) = \{(t_1, \tau_1)\}$.

Proof. Let μ^* denote the critical assignment of I, let B^* denote $\text{block}(I, \mu^*)$, let H denote the set of hospitals in I', let μ' denote $\mu^*\big|_H$, and let B' denote $\text{block}(I', \mu')$. Thus μ' is the critical assignment of I', and hence Lemma 3.1 implies $B' = \{(s_3, \sigma_2)\}$.

We claim that $B^* = \{(t_1, \tau_1)\}$. It is easy to see that (t_1, τ_1) belongs to B^*. Since resident t_1 has only one acceptable hospital τ_1, and since t_1 has size 1 and τ_1 has capacity 1, resident t_1 cannot belong to any other blocking coalition in B^* apart from (t_1, τ_1). Since residents s_3 and t_2 are each assigned to their most preferred hospital under μ^*, neither s_3 nor t_2 participates in any blocking coalition in B^*. Since resident t_3 has size 1, resident s_3 has size 2, and (s_3, τ_2) belongs to μ^*, we deduce that resident t_3 does not participate in any blocking coalition in B^*. Given the foregoing observations, it is sufficient to argue that μ^* is H-stable. Lemma 2.1 implies $B^*\big|_H \subseteq B'$. Since $B' = \{(s_3, \sigma_2)\}$ and since resident s_3 does not participate in any blocking coalition in B^*, we deduce that μ^* is H-stable.

Let μ_1 be an assignment of I such that $\text{block}(I, \mu_1) \subseteq \{(t_1, \tau_1)\}$ and let B denote $\text{block}(I, \mu_1)$. Below we complete the proof by establishing that $\mu_1 = \mu^*$.

We claim that $\text{bdry}(I, H, \mu_1) \subseteq \{(s_3, \sigma_2)\}$. Let (R, h) belong to $\text{bdry}(I, H, \mu_1)$. Since s_3 is the only resident in I' with an acceptable hospital in $\text{hosps}(I) \setminus H$, we deduce that s_3 belongs to R. Since σ_2 is the only acceptable hospital of s_3 in I', we deduce that $h = \sigma_2$. Since $\text{size}(s_3) = 2 = \text{cap}(\sigma_2)$, the claim follows.

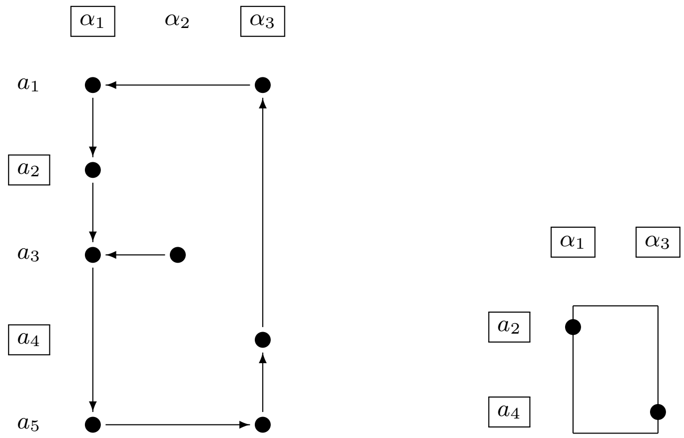
Let B'' denote $\text{block}(I', \mu_1\big|_H)$. Since $\text{bdry}(I, H, \mu_1) \subseteq \{(s_3, \sigma_2)\}$, Lemma 2.2 implies $B'' \setminus \{(s_3, \sigma_2)\} \subseteq B\big|_H \subseteq B$. Since $B \subseteq \{(t_1, \tau_1)\}$, we conclude from Lemma 3.1 that $\mu_1\big|_H = \mu'$. Hence $B'' = \{(s_3, \sigma_2)\}$. Since (s_3, σ_2) does not belong to B, we deduce that (s_3, τ_2) belongs to μ_1. It follows that $\text{res}(\mu_1, \tau_2) = \{s_3\}$. Since $(\{t_2, t_3\}, \tau_2)$ does not belong to B, we deduce that (t_2, τ_1) belongs to μ_1. Thus $\mu_1 = \mu' \cup \{(s_3, \tau_2), (t_2, \tau_1)\} = \mu^*$. $\square$

3.3 Bistable Gadget

Throughout the remainder of Section 3.3, let I denote the bistable gadget depicted in Figure 3(a). We define the *true (resp., false) assignment of I* as $\{(a_1, \alpha_3), (a_2, \alpha_1), (a_3, \alpha_2), (a_5, \alpha_3)\}$ (resp., $\{(a_3, \alpha_1), (a_4, \alpha_3), (a_5, \alpha_1)\}$).

Lemma 3.3. *The true and false assignments of I are stable, and I does not admit any other stable assignments.*

Proof. Let μ_1 (resp., μ_2) denote the true (resp., false) assignment of I. We begin by arguing that μ_1 is stable for I. Residents a_2 and a_5 are each assigned to their most preferred hospital, and hence cannot belong to a blocking coalition for μ_1. Resident a_1 cannot belong to a blocking coalition for μ_1 because hospital α_1 prefers a_2 to a_1. Because a_2 has size 2, it is straightforward to argue that any blocking coalition involving a_3 and α_1 has to be $(\{a_3, a_5\}, \alpha_1)$. Since resident a_5



(a) The bistable gadget.

(b) The associated icon.

Figure 3: The preference digraph for the bistable gadget has five residents $a_1, \ldots, a_5$ and three hospitals $\alpha_1, \alpha_2, \alpha_3$. The bistable gadget icon is used in the preference digraph of Figure 4(a); the left (resp., right) dot in this icon corresponds to the resident-hospital pair (a_2, α_1) (resp., (a_4, α_3)).

cannot belong to a blocking coalition for μ_1, we conclude that resident a_3 cannot belong to a blocking coalition for μ_1. Resident a_4 cannot belong to a blocking coalition for μ_1 because hospital α_3 prefers a_1 to a_4.

Now we argue that μ_2 is stable for I. Hospital α_1 cannot belong to a blocking coalition for μ_2 because $\text{res}(\mu_2, \alpha_1) = \{a_3, a_5\}$ contains the two most preferred residents of α_1. Hospital α_2 cannot belong to a blocking coalition for μ_2 because its lone acceptable resident a_3 is assigned to its most preferred hospital α_1. Hospital α_3 cannot belong to a blocking coalition for μ_2 because resident a_4 has size two and the total size of all residents that α_3 prefers to a_4 is only one.

It remains to argue that μ_1 and μ_2 are the only stable assignments for I. Let μ be a stable assignment for I and let B denote $\text{block}(I, \mu)$. Since μ is stable, B is empty. We consider two cases.

Case 1: (a_3, α_2) belongs to μ. Since (a_3, α_1) does not belong to B, the Case 1 condition implies that $\text{res}(\mu, \alpha_1)$ is either $\{a_2\}$ or $\{a_1, a_5\}$. Since (a_3, α_1) does not belong to B, the Case 1 condition implies (a_1, α_1) does not belong to μ and hence $\text{res}(\mu, \alpha_1) = \{a_2\}$. Since $(\{a_3, a_5\}, \alpha_1)$ does not belong to B, we deduce that (a_5, α_3) belongs to μ. Since (a_1, α_3) does not belong to B, we deduce that (a_1, α_3) belongs to μ. In summary, under the Case 1 condition, $\mu_1 \subseteq \mu$. Since hospitals α_1, α_2, and α_3 are full under μ_1, we conclude that $\mu = \mu_1$.

Case 2: (a_3, α_2) does not belong to μ. Since (a_3, α_2) does not belong to B, we deduce that (a_3, α_1) belongs to μ and hence (a_2, α_1) does not belong to μ. Thus $\text{res}(\mu, \alpha_1) \subseteq \{a_1, a_3, a_5\}$.

We first argue that (a_5, α_1) belongs to μ. Assume for the sake of contradiction that (a_5, α_1) does not belong to μ. Hence $\text{res}(\mu, \alpha_1) \subseteq \{a_1, a_3\}$. Since (a_1, α_1)

does not belong to B, we deduce that (a_1, α_1) belongs to μ. Since (a_4, α_3) does not belong to B, we deduce that (a_4, α_3) belongs to μ. Thus (a_5, α_3) does not belong to μ. Together with our assumption that (a_5, α_1) does not belong to μ, this implies a_5 is unassigned in μ. Since $\text{res}(\mu, \alpha_1) \subseteq \{a_1, a_3\}$, we conclude that (a_5, α_1) belongs to B, a contradiction.

Since (a_3, α_1) and (a_5, α_1) belong to μ, we conclude that $\text{res}(\mu, \alpha_1) = \{a_3, a_5\}$.

We claim that (a_1, α_3) does not belong to μ. Assume for the sake of contradiction that (a_1, α_3) belongs to μ. Thus (a_4, α_3) does not belong to μ. Hence (a_5, α_3) belongs to B, a contradiction.

Since (a_1, α_3) does not belong to μ and (a_4, α_3) does not belong to B, we deduce that (a_4, α_3) belongs to μ. In summary, under the Case 2 condition, $\mu_2 \subseteq \mu$ and $\text{res}(\mu, \alpha_2) = \emptyset$. Since hospitals α_1 and α_3 are full under μ_2, we conclude that $\mu = \mu_2$. $\square$

4 NP-Hardness

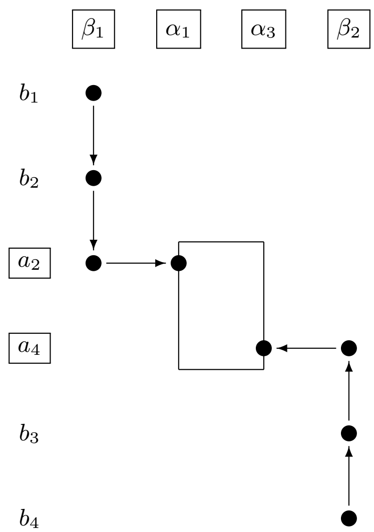
In the present section, we use a reduction from $(2, 2)$-E3-SAT to prove that the decision version of HRS-C is NP-hard. Since course allocation generalizes HRS, we conclude that the problem of finding a coalition-stable assignment for a given instance of course allocation is NP-hard, thereby resolving an open problem of Rodríguez and Manlove [10].

As discussed earlier, the literal gadget of Section 3.2 satisfies similar abstract properties to those satisfied by the corresponding fragment of the construction of McDermid and Manlove [9]. Consequently, we can use the approach implicit in [9] to build a clause gadget from three copies of the literal gadget. Similarly, the bistable gadget can be used to build a variable gadget using the approach implicit in [9].[11] Finally, again following the approach implicit in [9], we can build a suitable overall HRS-C instance for a given $(2, 2)$-E3-SAT formula ϕ using a copy of the variable gadget for each variable in ϕ and a copy of the clause gadget for each clause in ϕ.

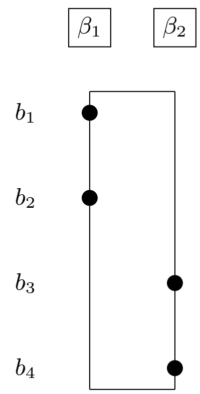
For the sake of completeness, below we provide all of the details concerning the construction of the variable gadget, the clause gadget, and the overall HRS-C instance corresponding to a given $(2, 2)$-E3-SAT formula ϕ. We emphasize that all of the constructions and arguments in this section precisely mirror elements implicit in the work of McDermid and Manlove [9]. As in Section 3, throughout this section we use the term "stability" (resp., "stable") to mean "coalition stability" (resp., "coalition-stable").

[11] In studying the variable gadget implicit in [9], which is induced by the residents in the set $\{r_j^1, r_j^2, r_j^3, r_j^4, r_j^5, r_j^6, x_j^1, x_j^2, y_j^1, y_j^2\}$ and the hospitals in the set $\{h_j^1, h_j^2, h_j^3, h_j^4, h_j^T, h_j^F\}$, we observed that their construction can be simplified. Specifically, the set of residents can be reduced to $\{r_j^1, r_j^2, r_j^3, r_j^4, x_j^1, x_j^2, y_j^1, y_j^2\}$ and the set of hospitals can be reduced to $\{h_j^1, h_j^2, h_j^T, h_j^F\}$. Figure 4(a) shows how we construct our variable gadget from our bistable gadget, and the same approach can be used to construct a streamlined version of the variable gadget implicit in [9] from the bistable gadget implicit in [9].

4.1 Variable Gadget



(a) The variable gadget.



(b) The associated icon.

Figure 4: The preference digraph for the variable gadget includes a copy of the bistable gadget and has a total of $4+5=9$ residents and $2+3=5$ hospitals. The variable gadget icon is used in the preference digraph of Figure 6; the upper-left (resp., lower-left, upper-right, lower-right) dot in this icon refers to the resident-hospital pair (b_1, β_1) (resp., (b_2, β_1), (b_3, β_2), (b_4, β_2)).

Throughout the remainder of Section 4.1, let I denote the variable gadget depicted in Figure 4(a) and let I' denote the bistable gadget of I. We define the *true (resp., false) assignment of* I as the union of the true (resp., false) assignment of I' and $\{(b_1, \beta_1), (b_2, \beta_1), (a_4, \beta_2)\}$ (resp., $\{(a_2, \beta_1), (b_3, \beta_2), (b_4, \beta_2)\}$).

Lemma 4.1. *The true and false assignments of I are stable, and I does not admit any other stable assignments.*

Proof. Let μ_1 (resp., μ_2) denote the true (resp., false) assignment of I, let B_1 (resp., B_2) denote $\text{block}(I, \mu_1)$ (resp., $\text{block}(I, \mu_2)$), let H denote $\text{hosps}(I')$, let μ_1' (resp., μ_2') denote $\mu_1\big|_H$ (resp., $\mu_2\big|_H$), and let B_1' (resp., B_2') denote $\text{block}(I', \mu_1')$ (resp., $\text{block}(I', \mu_2')$). Thus μ_1' (resp., μ_2') is the true (resp., false) assignment of I' and $B_1' = B_2' = \emptyset$.

We begin by proving that μ_1 is stable; a symmetric argument establishes that μ_2 is stable. Since μ_1' is H-stable, Lemma 2.1 implies μ_1 is H-stable. Since hospital β_2 is assigned to its most preferred resident a_4 and is full under μ_1, we conclude that μ_1 is $\{\beta_2\}$-stable. Since residents a_2, b_1, and b_2 are each assigned to their most preferred hospital in μ_1, no blocking coalition in B_1 involves a_2,

b_1, or b_2. It follows that μ_1 is $\{\beta_1\}$-stable. Since μ_1 is $(H \cup \{\beta_1, \beta_2\})$-stable, we conclude that μ_1 is stable for I.

It remains to prove that μ_1 and μ_2 are the only stable assignments for I. Let μ be a stable assignment for I, let B denote $\text{block}(I, \mu) = \emptyset$, and let μ' denote $\mu\big|_H$. Since $\text{bdry}(I, H, \mu) = \emptyset$ and μ is H-stable, Lemma 2.2 implies μ' is H-stable. Hence μ' is a stable assignment for I'. By Lemma 3.3, there are two cases to consider.

Case 1: $\mu' = \mu'_1$. Since (a_2, α_1) belongs to μ'_1, the Case 1 condition implies (a_2, β_1) does not belong to μ. Since (b_1, β_1) (resp., (b_2, β_1)) does not belong to B, we deduce that (b_1, β_1) (resp., (b_2, β_1)) belongs to μ. Since (a_4, α_3) does not belong to μ'_1, the Case 1 condition implies (a_4, α_3) does not belong to μ. Since (a_4, β_2) does not belong to B, we deduce that (a_4, β_2) belongs to μ. Thus residents b_3 and b_4 are unassigned in μ. In summary, $\mu = \mu_1$.

Case 2: $\mu' = \mu'_2$. By a symmetric argument to that used in Case 1, we find that $\mu = \mu_2$. $\square$

4.2 Clause Gadget

In this section, we describe a clause gadget that contains three copies of the literal gadget. The three copies of the literal gadget have pairwise disjoint sets of agents (7 residents and 5 hospitals each); we use a superscript ℓ in $\{1, 2, 3\}$ to identify the agents in copy ℓ. For example, we write t_1^ℓ to refer to the resident t_1 in copy ℓ. See Figure 5 for a depiction of the clause gadget.

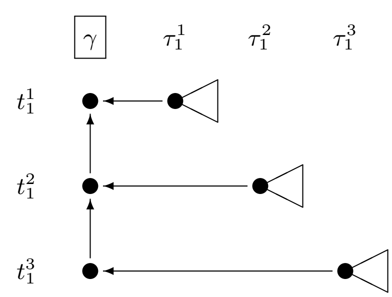


Figure 5: The preference digraph for the clause gadget includes three copies of the literal gadget and has $3 \times 7 = 21$ residents and $1 + 3 \times 5 = 16$ hospitals.

Throughout the remainder of Section 4.2, let I denote the clause gadget of Figure 5 and for any ℓ in $\{1, 2, 3\}$, let I^ℓ denote literal gadget ℓ of I and let H^ℓ denote $\text{hosps}(I^\ell)$. We say that an assignment μ of I is critical if for all ℓ in $\{1, 2, 3\}$, $\mu\big|_{H^\ell}$ is the critical assignment of I^ℓ. We say that an assignment μ of I is nice if every blocking coalition in $\text{block}(I, \mu)$ either involves hospital γ or belongs to $\{(t_1^1, \tau_1^1), (t_1^2, \tau_1^2), (t_1^3, \tau_1^3)\}$.

Lemma 4.2. *Let μ be a critical assignment of I, let ℓ belong to $\{1, 2, 3\}$, let B denote* $\text{block}(I, \mu)$*, and let B' denote $B\big|_{H^\ell}$. Then $B' \subseteq \{(t_1^\ell, \tau_1^\ell)\}$.*

Proof. Let μ' denote $\mu\big|_{H^\ell}$ and let B'' denote block(I^ℓ, μ'). Since μ is critical, μ' is the critical assignment of literal gadget I^ℓ. Hence Lemma 3.2 implies $B'' = \{(t_1^\ell, \tau_1^\ell)\}$. By Lemma 2.1, we conclude that $B' \subseteq \{(t_1^\ell, \tau_1^\ell)\}$. $\square$

Lemma 4.3. *Let μ be a nice assignment of I, let ℓ belong to $\{1,2,3\}$, and let μ' denote $\mu\big|_{H^\ell}$. Then μ' is the critical assignment of I^ℓ.*

Proof. Let B denote block(I, μ), let B' denote $B\big|_{H^\ell}$, let B'' denote block(I^ℓ, μ'), and assume for the sake of contradiction that μ' is not the critical assignment of I^ℓ. Lemma 3.2 implies B'' includes a blocking coalition $(R, h) \neq (t_1^\ell, \tau_1^\ell)$ such that h belongs to H^ℓ. Since bdry$(I^\ell, H^\ell, \mu') = \{(t_1^\ell, \tau_1^\ell)\}$ and $(R, h) \neq (t_1^\ell, \tau_1^\ell)$, Lemma 2.2 implies (R, h) belongs to B' and hence also to B. Since h belongs to H^ℓ, we deduce that μ is not a nice assignment of I, a contradiction. $\square$

Lemma 4.4. *Let μ be an assignment of I. Then μ is a nice assignment of I if and only if μ is a critical assignment of I.*

Proof. We first address the if direction. Assume that μ is a critical assignment of I. Let ℓ belong to $\{1,2,3\}$, let B denote block(I, μ), and let B' denote $B\big|_{H^\ell}$. Lemma 4.2 implies $B' \subseteq \{(t_1^\ell, \tau_1^\ell)\}$. It follows that μ is a nice assignment of I.

We now address the only if direction. Assume that μ is a nice assignment of I. Lemma 4.3 implies that for any ℓ in $\{1,2,3\}$, $\mu\big|_{H^\ell}$ is the critical assignment of I^ℓ. Thus μ is a critical assignment of I, as required. $\square$

4.3 Transformation

Throughout this section, we fix a $(2,2)$-E3-SAT instance ϕ. Below we describe how to transform ϕ into an instance I of HRS-C so that I admits a stable assignment if and only if ϕ is satisfiable. It will be evident that the transformation is polynomial-time computable.

Let n be a positive integer such that ϕ has $4n$ clauses and $3n$ variables. We index the clauses (resp., variables) from 1 to $4n$ (resp., $3n$). We use two schemes for identifying the $12n$ literals of ϕ. For any i in $\{1,\ldots,4n\}$ and any ℓ in $\{1,2,3\}$, we say that the literal in position ℓ of clause i has A-identifier (i, ℓ). For any j in $\{1,\ldots,3n\}$, we say that a literal has B-identifier $(j,1)$ (resp., $(j,2)$, $(j,3)$, $(j,4)$) if it is the first positive (resp., second positive, first negative, second negative) occurrence of variable j in ϕ. In order to translate between these two schemes, we define the bijection $\varphi : \{1,\ldots,4n\} \times \{1,2,3\} \to \{1,\ldots,3n\} \times \{1,2,3,4\}$ to map any given A-identifier to the corresponding B-identifier. Our instance I will be constructed from $3n$ variable gadgets and $4n$ clause gadgets. Below we discuss these components in greater detail.

As discussed in Section 4.1, our variable gadget has 9 residents and 5 hospitals. We create $3n$ copies of this gadget, one for each variable in ϕ; each copy has its own set of 9 residents and 5 hospitals. For any j in $\{1,\ldots,3n\}$, we define V^j as copy j of the variable gadget. We use a superscript j to identify the agents in V^j. For example, we write a_1^j to refer to the resident corresponding to a_1 in V^j.

In our overall instance I, the preference list of each hospital in V^j is the same as in the isolated gadget V^j; likewise, the preference list of each resident in V^j that does not belong to $\{b_1^j, \ldots, b_4^j\}$ is the same as in the isolated gadget V^j. Each resident in $\{b_1^j, \ldots, b_4^j\}$ has a preference list of length 1 in V^j and of length 3 in I: the most preferred hospital of b_k^j in I is its lone acceptable hospital in V^j; the two remaining hospitals on the preference list of b_k^j do not belong to V^j and are specified below.

As discussed in Section 4.2, our clause gadget has 21 residents and 16 hospitals. We create $4n$ copies of this gadget, one for each clause in ϕ; each copy has its own set of 21 residents and 16 hospitals. For any i in $\{1, \ldots, 4n\}$, we define C^i as copy i of the clause gadget. We use a superscript i to identify the agents in C^i. For example, we write $t_1^{i,1}$ to refer to the resident corresponding to t_1^1 in C^i. In our overall instance I, the preference list of each hospital in C^i is the same as in the isolated gadget C^i; likewise, the preference list of each resident in C^i that does not belong to $\{t_1^{i,1}, t_1^{i,2}, t_1^{i,3}\}$ is the same as in the isolated gadget C^i.

It remains to describe the preference lists of the residents b_k^j and $t_1^{i,\ell}$ in I. For each A-identifier (i, ℓ), we identify resident $t_1^{i,\ell}$ with the resident b_k^j such that B-identifier (j, k) is equal to $\varphi(i, \ell)$. The preference list of length 3 of resident $t_1^{i,\ell} = b_k^j$ in I is obtained by merging the preference list of b_k^j in V^j with the preference list of $t_1^{i,\ell}$ in C^i, as follows: the top preference is the lone acceptable hospital of b_k^j in V^j; the second and third preferences are the first and second preferences of $t_1^{i,\ell}$ in C^i.

The above completes our description of the HRS-C instance I corresponding to $(2,2)$-E3-SAT instance ϕ. We remark that the total number of residents in I is $3n \cdot 9 + 4n \cdot (21 - 3) = 99n$ and the total number of hospitals in I is $3n \cdot 5 + 4n \cdot 16 = 79n$. The resident sizes and hospital capacities all belong to $\{1, 2\}$, and the longest preference list of any resident (resp., hospital) in I is 3 (resp., 4). Figure 6 gives an example of the transformation for a specific $(2,2)$-E3-SAT instance ϕ with $n = 1$ (i.e., three variables and four clauses).

In the remainder of this section, we establish the correctness of the polynomial-time transformation described above. That is, we prove that I admits a stable assignment if and only if ϕ is satisfiable.

If Direction Assume that ϕ is satisfiable, and fix a satisfying truth assignment $\pi = (\pi_1, \ldots, \pi_{3n})$ for ϕ; in other words, π_j is the truth value of variable j under π. We need to prove that HRS-C instance I admits a stable assignment.

We now describe how to use π to construct an assignment μ for I. After describing the construction, we prove that μ is a valid assignment for I (i.e., no resident or hospital is over-subscribed), and then we prove that μ is stable for I.

For each variable j such that π_j is true (resp., false), we include the true (resp., false) assignment of V^j in μ. For each clause i, we include in μ the critical assignment of C^i that, for each ℓ in $\{1, 2, 3\}$, assigns resident $t_1^{i,\ell}$ to hospital γ^i if and only if the literal with A-identifier (i, ℓ) evaluates to false under truth assignment π. This completes the definition of assignment μ.

We now argue that μ is a valid assignment of I. It is easy to see that for any j in $\{1, \dots, 3n\}$, no hospital in V^j is over-subscribed and that no resident in V^j outside the set $\{b_1^j, \dots, b_4^j\}$ is over-subscribed. Similarly, it is easy to see that for any i in $\{1, \dots, 4n\}$, no hospital in C^i apart from γ^i is over-subscribed and no resident in C^i outside the set $\{t_1^{i,1}, t_1^{i,2}, t_1^{i,3}\}$ is over-subscribed. Thus Lemmas 4.5 and 4.7 below imply μ is a valid assignment of I.

Lemma 4.5. *Let i belong to $\{1, \dots, 4n\}$. Then hospital γ^i is not over-subscribed.*

Proof. Since π is a satisfying truth assignment, γ^i has capacity 2, the set of acceptable residents of γ^i is $\{t_1^{i,\ell} \mid \ell \in \{1,2,3\}\}$, and each resident in this set has size 1, we deduce that γ^i is not over-subscribed. $\square$

Lemma 4.6. *Let i belong to $\{1, \dots, 4n\}$ and let ℓ belong to $\{1,2,3\}$. Resident $t_1^{i,\ell}$ is assigned under μ to its most (resp., second most) preferred hospital if and only if literal ℓ of clause i evaluates to true (resp., false) under π.*

Proof. Let (j,k) denote the B-identifier $\varphi(i,\ell)$. Observe that the true (resp., false) assignment for variable gadget j assigns (resp., does not assign) residents b_1^j and b_2^j to β_1^j, their most preferred hospital in I, and does not assign (resp., assigns) residents b_3^j and b_4^j to β_2^j, their most preferred hospital in I. The claim of the lemma follows straightforwardly from the definition of μ. $\square$

Lemma 4.7. *Let i belong to $\{1, \dots, 4n\}$, let ℓ belong to $\{1,2,3\}$, and let (j,k) denote the B-identifier $\varphi(i,\ell)$. Then resident $t_1^{i,\ell} = b_k^j$ is not over-subscribed.*

Proof. Since $\mu\big|_{\text{hosps}(C^i)}$ is a critical assignment of C^i, resident $t_1^{i,\ell}$ is not assigned to its third preference in I (i.e., $\tau_1^{i,\ell}$) under μ. By Lemma 4.6, we conclude that resident $t_1^{i,\ell}$ is not over-subscribed. $\square$

It remains to prove that μ is stable for I.

Lemma 4.8. *Let i belong to $\{1, \dots, 4n\}$ and let H denote* hosps(C^i)*. Then μ is an H-stable assignment of I.*

Proof. Let B denote block(I, μ). The construction of μ implies $\mu' = \mu\big|_H$ is a critical assignment of C^i. Thus Lemma 4.4 implies μ' is a nice assignment of C^i. Hence Lemma 2.1 implies that for every blocking coalition (R,h) in B such that h belongs to hosps(C^i), either $h = \gamma^i$ or (R,h) belongs to

$$\{(t_1^{i,1}, \tau_1^{i,1}), (t_1^{i,2}, \tau_1^{i,2}), (t_1^{i,3}, \tau_1^{i,3})\}.$$

Thus Claims 1 and 2 below imply μ is an H-stable assignment of I, as required.

Claim 1: The hospital γ^i is not involved in any blocking coalition in B. The only residents who could participate in a blocking coalition in B with γ^i are $t_1^{i,1}$, $t_1^{i,2}$, and $t_1^{i,3}$. Lemma 4.6 implies that for any ℓ in $\{1,2,3\}$, resident $t_1^{i,\ell}$ is assigned to one of its two most preferred hospitals. Since γ^i is the second preference of

$t_1^{i,\ell}$, we deduce that $t_1^{i,\ell}$ does not participate in a blocking coalition in B with γ^i. This completes the proof of Claim 1.

Claim 2: For any ℓ in $\{1,2,3\}$, hospital $\tau_1^{i,\ell}$ is not involved in any blocking coalition in B. Let ℓ belong to $\{1,2,3\}$. Size and capacity considerations imply that the only possible blocking coalitions in B involving $\tau_1^{i,\ell}$ are $(t_1^{i,\ell}, \tau_1^{i,\ell})$ and $(t_2^{i,\ell}, \tau_1^{i,\ell})$. Lemma 4.6 implies resident $t_1^{i,\ell}$ is assigned to one of its two most preferred hospitals. Since $\tau_1^{i,\ell}$ is its third choice, $(t_1^{i,\ell}, \tau_1^{i,\ell})$ does not belong to B. Since μ assigns $t_2^{i,\ell}$ to $\tau_1^{i,\ell}$, $(t_2^{i,\ell}, \tau_1^{i,\ell})$ does not belong to B. This completes the proof of Claim 2. □

Lemma 4.9. *Let j belong to $\{1, \ldots, 3n\}$ and let H denote* hosps(V^j)*. Then μ is an H-stable assignment of I.*

Proof. Let B denote block(I, μ), let B' denote $B\big|_H$, let μ' denote $\mu\big|_H$, and let B'' denote block(V^j, μ'). By the definition of μ, μ' is the true or the false assignment for V^j. Thus Lemma 4.1 implies μ' is stable for V^j and hence $B'' = \emptyset$. Hence Lemma 2.1 implies $B' = \emptyset$; the claim of the lemma follows. □

Lemma 4.10. *Assignment μ is stable for I.*

Proof. Since Lemma 4.8 holds for all i in $\{1, \ldots, 4n\}$ and Lemma 4.9 holds for all j in $\{1, \ldots, 3n\}$, we conclude that μ is a stable assignment for I. □

Only If Direction Assume that I admits a stable assignment, and fix a stable assignment μ of I. We need to prove that ϕ is satisfiable.

Lemma 4.11. *Let j belong to $\{1, \ldots, 3n\}$, let H denote* hosps(V^j)*, and let μ' denote $\mu\big|_H$. Then μ' is the true or the false assignment for V^j.*

Proof. Since bdry$(I, H, \mu) = \emptyset$, Lemma 2.2 implies block(V^j, μ') is equal to block$(I, \mu)\big|_H = \emptyset\big|_H = \emptyset$. Thus μ' is a stable assignment for V^j and the claim of the lemma follows from Lemma 4.1. □

Let $\pi = (\pi_1, \ldots, \pi_{3n})$ be the truth assignment for ϕ such that π_j is true if and only if $\mu\big|_{\text{hosps}(V^j)}$ is the true assignment for V^j. Below we complete the "only if" proof by showing that π is a satisfying truth assignment for ϕ.

Lemma 4.12. *Let i belong to $\{1, \ldots, 4n\}$ and let ℓ belong to $\{1,2,3\}$. Then the literal of ϕ with A-identifier (i, ℓ) evaluates to true under π if and only if μ assigns resident $t_1^{i,\ell}$ to its most preferred hospital.*

Proof. Let (j, k) denote the B-identifier $\varphi(i, \ell)$, let H denote hosps(V^j), and let μ' denote $\mu\big|_H$. The most preferred hospital of resident $t_1^{i,\ell} = b_k^j$ is $\beta^j_{\lceil k/2 \rceil}$. By Lemma 4.11, μ' is the true or the false assignment for V^j. If k belongs to $\{1,2\}$, then resident $t_1^{i,\ell}$ is (resp., is not) assigned to its most preferred hospital under the true (resp., false) assignment for V^j. If k belongs to $\{3,4\}$, then resident $t_1^{i,\ell}$ is (resp., is not) assigned to its most preferred hospital under the false (resp., true) assignment for V^j. The claim of the lemma follows. □

Lemma 4.13. *Let i belong to $\{1, \ldots, 4n\}$ and let ℓ belong to $\{1, 2, 3\}$. Then μ assigns resident $t_1^{i,\ell}$ to one of its two most preferred hospitals.*

Proof. Let B denote block(I, μ), let H denote hosps(C^i), let B' denote $B\big|_H$, let I' denote $I\big|_H$, let μ' denote $\mu\big|_H$, and let B'' denote block(I', μ'). Since $B = B' = \emptyset$ and every blocking coalition in bdry(I, H, μ) either involves hospital γ^i or is contained in $\{(t_1^{i,1}, \tau_1^{i,1}), (t_1^{i,2}, \tau_1^{i,2}), (t_1^{i,3}, \tau_1^{i,3})\}$, Lemma 2.2 implies μ' is a nice assignment for I'. Thus Lemma 4.4 implies μ' is a critical assignment for I'. It follows that μ does not assign resident $t_1^{i,\ell}$ to its least preferred hospital $\tau_1^{i,\ell}$ (i.e., its third choice).

It remains to argue that $t_1^{i,\ell}$ cannot be unassigned in μ. Suppose $t_1^{i,\ell}$ is unassigned in μ. Then $(t_1^{i,\ell}, \tau_1^{i,\ell})$ is a blocking coalition in B, a contradiction. □

Lemma 4.14. *Let i belong to $\{1, \ldots, 4n\}$ and let ℓ belong to $\{1, 2, 3\}$. Then the literal of ϕ with A-identifier (i, ℓ) evaluates to false under truth assignment π if and only if μ assigns $t_1^{i,\ell}$ to γ^i.*

Proof. Immediate from Lemmas 4.12 and 4.13, since hospital γ^i is the second most preferred hospital of resident $t_1^{i,\ell}$. □

Lemma 4.15. *Truth assignment π satisfies ϕ.*

Proof. Let i belong to $\{1, \ldots, 4n\}$. We need to prove that truth assignment π satisfies clause i of ϕ. Assume for the sake of contradiction that π does not satisfy clause i. By Lemma 4.14, μ assigns the three unit-size residents $t_1^{i,1}$, $t_1^{i,2}$, and $t_1^{i,3}$ to γ^i. Since the capacity of γ^i is 2, we conclude that μ is not a valid assignment for I, a contradiction. □

4.4 Complete Preferences

Our proof that the decision version of HRS-C is NP-hard makes use of HRS-C instances with incomplete preferences, that is, where a resident (resp., hospital) only provides a preference order over its acceptable hospitals (resp., residents). In this section, we use a simple reduction to show that the problem remains NP-hard in the special case where all preference lists are complete.

Let I denote an instance of HRS-C with incomplete preferences and where the resident sizes and hospital capacities all belong to $\{1, 2\}$. For any resident r in I, let A_r denote the set of hospitals that are acceptable to r. Similarly, for any hospital h in I, let A_h denote the set of residents that are acceptable to h.

We now describe how to transform an arbitrary HRS-C instance I into an "equivalent" instance I' of HRS-C with complete preferences and resident sizes and hospital capacities in $\{1, 2\}$. The set of hospitals in I' is the same as in I. The set of residents in I' consists of those in I plus the following "dummy" residents: For each capacity-k hospital h in I, we introduce a set of k size-1 dummy residents $\{r_h^1, \ldots, r_h^k\}$. Each agent in instance I' has complete preferences, determined as follows.

– For any resident r in I, the preference list of r in I' consists of its preference list in I (which contains exactly the hospitals in A_r) followed by the remaining hospitals in arbitrary order.
– For any dummy resident r_h^i, the associated preference list in I' consists of h followed by all other hospitals in arbitrary order.
– For any hospital h in I, the preference list of h in I' consists of its preference list in I (which contains exactly the residents in A_h) followed by dummy residents r_h^1 through r_h^k (in that order), followed by the remaining (non-dummy and dummy) residents in arbitrary order.

We define an assignment μ' in I' to be *special* if the following conditions hold for any hospital h, where R_h denotes the set of all residents assigned to h under μ': (1) $\text{size}(R_h) = \text{cap}(h)$; (2) there is an ℓ in $\{0, \ldots, \text{cap}(h)\}$ such that R_h is the union of $\{r_h^1, \ldots, r_h^\ell\}$ and some subset of A_h.

There is a natural bijection between the set of all assignments for I and the set of all special assignments for I'. Specifically, given an assignment μ for I, we can obtain the corresponding special assignment μ' for I' by assigning, for each hospital h with unused capacity ℓ under μ, dummy residents $r_h^1, \ldots, r_h^\ell$ to hospital h. And given a special assignment μ' for I', we can obtain the corresponding assignment μ for I by removing any resident-hospital pairs involving dummy residents.

The following two claims are straightforward to prove: (1) if μ' is stable for I', then μ' is special; (2) if μ is an assignment for I and μ' is the corresponding special assignment for I', then $\text{block}(I, \mu) = \text{block}(I', \mu')$. It follows that I admits a stable matching if and only if I' admits a stable matching.

5 Unitwise-Coalition Stability

As discussed in Section 1, Rodríguez and Manlove have considered two coalitional notions of stability in the context of course allocation, namely first-coalition stability and coalition stability, which correspond to HRS-FC and HRS-C in the special-case setting of HRS. Rodríguez and Manlove have shown that HRS-FC is NP-hard and we have shown that HRS-C is NP-hard. In this section, we discuss a third notion of coalitional stability in the context of HRS that is arguably more natural than either first-coalition stability or coalition stability, and we indicate why our NP-hardness result for HRS-C extends to this third notion, which we call unitwise-coalition stability.

To motivate this third coalitional notion of stability, consider the following example. Suppose hospital h has $\text{cap}(h) = 4$ and preference list r_1 (most preferred), r_2, r_3, r_4 (least preferred) where $\text{size}(r_1) = 2$, $\text{size}(r_2) = 1$, $\text{size}(r_3) = 2$, and $\text{size}(r_4) = 3$. Further assume that assignment μ assigns residents r_2 and r_4 to hospital h. In this scenario, it is straightforward to verify that there is no blocking coalition (in the sense of HRS-C) for μ involving hospital h. Under our proposed coalitional notion of stability, we consider $(\{r_1, r_3\}, h)$ to be a blocking coalition because h "should" prefer $\{r_1, r_3\}$ to $\{r_2, r_4\}$: the two units of r_1 are preferred to the single unit of r_2 and one unit of r_4; the two units of r_3 are preferred to the two remaining units of r_4. More generally, the idea is that a hospital h prefers a set of residents R to a set of residents R' (disjoint from R) if $\text{size}(R) \geq \text{size}(R')$ and for all i in $\{1, \ldots, \text{size}(R')\}$, h prefers its ith most preferred unit in R to its ith most preferred unit in R'.

Unitwise-coalition stability admits an equivalent definition in terms of additive utility. Assume that the utility hospital h derives from being assigned a set of residents R is equal to the sum of its utility for each resident r in R, where the utility for each resident is a positive real number. Further assume that the utilities respect the preference list of h, in the following sense: if h prefers r to r', then the per-unit utility of r (i.e., the utility of h for r divided by $\text{size}(r)$) is greater than the per-unit utility of r' (i.e., the utility of h for r' divided by $\text{size}(r')$). We consider hospital h to prefer a set of residents R to a set of residents R' (disjoint from R) if $\text{size}(R) \geq \text{size}(R')$ and the utility of h for R exceeds the utility of h for R' under any utilities that respect the preference list of h. It is straightforward to prove that this utility-based formulation is equivalent to unitwise-coalition stability.

We now indicate why the NP-hardness proof of Section 4 also applies to unitwise-coalition stability. Our proof establishes that HRS-C is NP-hard even in the special case where the resident sizes and the hospital capacities belong to $\{1, 2\}$. It is straightforward to verify that in this special case, an assignment is coalition-stable (i.e., stable under HRS-C) if and only if it is unitwise-coalition-stable. [12]

6 Concluding Remarks

In this paper, we presented an instance of HRIC that does not admit a coalition-stable assignment. In addition, we established that it is NP-hard to find a coalition-stable assignment for a given HRIC instance (or correctly report that no such assignment exists), even if every hospital has capacity at most two and the length of the preference list of each resident (resp., hospital) is at most three (resp., four). As discussed in Section 5, this hardness result also holds for unitwise-coalition stability. Since HRS generalizes HRIC and course allocation generalizes HRS, our results resolve the open questions of Rodríguez and Manlove [10] related to finding a coalition-stable assignment in the course allocation setting.

As remarked in Section 1, Rodríguez and Manlove prove that the problem of testing whether a given assignment is coalition-stable is co-NP-complete for course allocation, and their argument also applies to the HRS special case. On the other hand, we claim that the testing problem for HRIC under pair-size, first-coalition, coalition, and unitwise-coalition stability belongs to P. To show this, we first argue that if there is a blocking coalition involving some hospital h, then there is a blocking coalition (R, h) such that $\text{size}(R) \leq 2$. Furthermore, for any

[12] The set of blocking coalitions under unitwise-coalition stability can be strictly larger than under coalition stability, but is nonempty if and only if it is nonempty under coalition stability.

hospital h and any set of residents R such that $\mathrm{size}(R) \leq 2$, it is straightforward to check in polynomial time whether (R, h) is a blocking coalition. Since there are only polynomially many candidate blocking coalitions (R, h) with $\mathrm{size}(R) \leq 2$, the claim follows.

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A Tabular Description of the Transformation

Let ϕ be a $(2,2)$-E3-SAT instance, let n be a positive integer such that ϕ has $4n$ clauses, indexed from 1 to $4n$, and $3n$ variables, indexed from 1 to $3n$. As in Section 4.3, we define both an A-identifier and a B-identifier for each literal in ϕ and we define the bijection $\varphi : \{1, \ldots, 4n\} \times \{1, 2, 3\} \to \{1, \ldots, 3n\} \times \{1, 2, 3, 4\}$ to map any given A-identifier to the corresponding B-identifier.

B Preference Digraphs for the Gadgets Implicit in [9]

The NP-hardness proof for HRS-P given in [9, erratum] is based on a tabular representation of the construction (see Figure 1 of [9, erratum]) that is analogous to Table 1 of Appendix A for our construction. In this section, we present

resident	size	preferences			hospital	capacity	preferences			
a_1^j	1	α_1^j	α_3^j		α_1^j	2	a_5^j	a_3^j	a_2^j	a_1^j
a_2^j	2	α_1^j	β_1^j		α_2^j	1	a_3^j			
a_3^j	1	α_1^j	α_2^j		α_3^j	2	a_1^j	a_4^j	a_5^j	
a_4^j	2	α_3^j	β_2^j		β_1^j	2	a_2^j	b_2^j	b_1^j	
a_5^j	1	α_3^j	α_1^j		β_2^j	2	a_4^j	b_3^j	b_4^j	
$b_k^j = t_1^{i,\ell}$	1	$\beta_{\lceil k/2 \rceil}^j$	γ^i	$\tau_1^{i,\ell}$	γ^i	2	$t_1^{i,1}$	$t_1^{i,2}$	$t_1^{i,3}$	
$s_1^{i,\ell}$	1	$\sigma_1^{i,\ell}$	$\sigma_2^{i,\ell}$		$\sigma_1^{i,\ell}$	1	$s_2^{i,\ell}$	$s_1^{i,\ell}$		
$s_2^{i,\ell}$	1	$\sigma_3^{i,\ell}$	$\sigma_2^{i,\ell}$	$\sigma_1^{i,\ell}$	$\sigma_2^{i,\ell}$	2	$s_1^{i,\ell}$	$s_2^{i,\ell}$	$s_3^{i,\ell}$	$s_4^{i,\ell}$
$s_3^{i,\ell}$	2	$\tau_2^{i,\ell}$	$\sigma_2^{i,\ell}$		$\sigma_3^{i,\ell}$	1	$s_4^{i,\ell}$	$s_2^{i,\ell}$		
$s_4^{i,\ell}$	1	$\sigma_2^{i,\ell}$	$\sigma_3^{i,\ell}$		$\tau_1^{i,\ell}$	1	$t_1^{i,\ell}$	$t_2^{i,\ell}$		
$t_2^{i,\ell}$	1	$\tau_1^{i,\ell}$	$\tau_2^{i,\ell}$		$\tau_2^{i,\ell}$	2	$t_2^{i,\ell}$	$t_3^{i,\ell}$	$s_3^{i,\ell}$	
$t_3^{i,\ell}$	1	$\tau_2^{i,\ell}$								

Table 1: A tabular representation of the construction of Section 4.3 (see also Figure 6). For each resident (resp., hospital), the acceptable hospitals (resp., residents) are listed in decreasing order of preference.

preference digraphs for the gadgets implicit in the construction of [9], to allow the reader to better understand the relationship between our NP-hardness proof for HRS-C and their NP-hardness proof for HRS-P. We do not provide a preference digraph for the clause gadget implicit in [9] because it is the same as ours (see Figure 5), with the understanding that the literal gadget icons correspond to Figure 8 rather than Figure 2(a). Similarly, we do not provide the preference digraph for the global construction of [9] because it is the same as ours (see Figure 6), with the understanding that the variable and literal gadget icons correspond to Figures 4(a) and 2(a) rather than Figures 10 and 8.

Figure 10 depicts the preference digraph for the variable gadget implicit in [9]. This construction can be streamlined by instead applying the construction of Figure 4(a), with the understanding that the bistable gadget icon corresponds to Figure 9 rather than Figure 3(a).

The symbols used to denote specific residents and hospitals in Figures 7 through 10 are based on the notation in [9]; the reader is referred to [9] for further details.

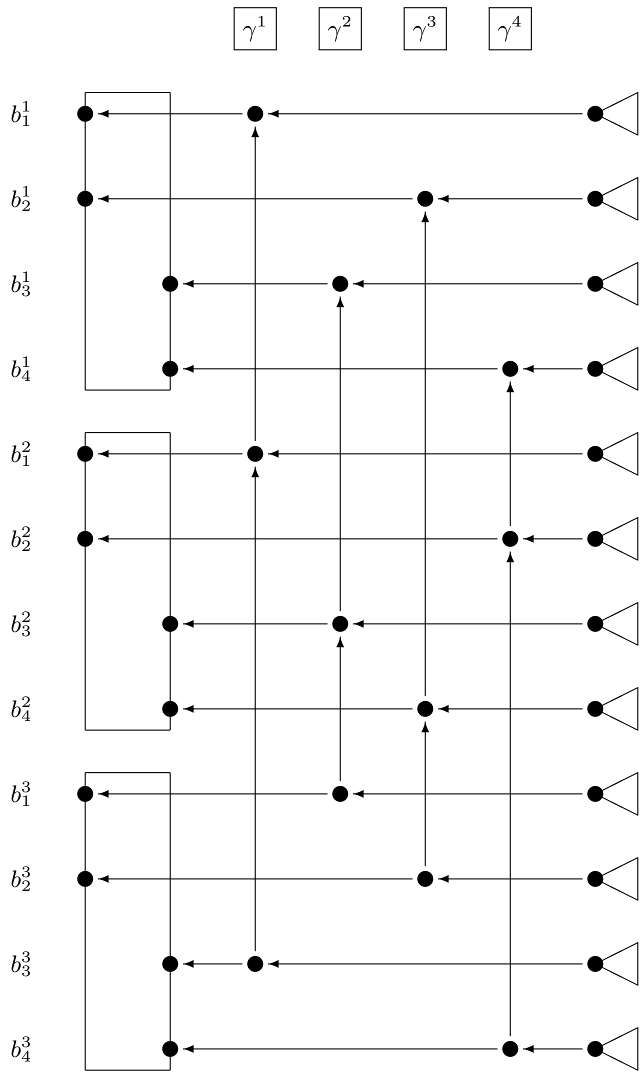

Figure 6: The preference digraph for the HRS-C instance corresponding to the $(2,2)$-E3-SAT formula $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)$. Notice that while the dots associated with the literal icons all lie in the same column, there are no directed edges within this column. This is because each dot in this column is associated with a different copy of the literal gadget hospital τ_1. For example, the topmost three dots in this column are associated with hospitals $\tau_1^{1,1}$, $\tau_1^{3,1}$, and $\tau_1^{2,1}$, respectively.

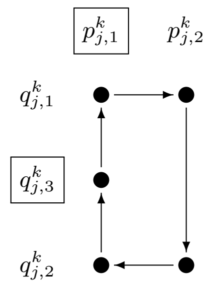

Figure 7: The preference digraph for copy (j, k) of the unstable gadget implicit in [9] has three residents $q^k_{j,1}, q^k_{j,2}, q^k_{j,3}$ and two hospitals $p^k_{j,1}, p^k_{j,2}$.

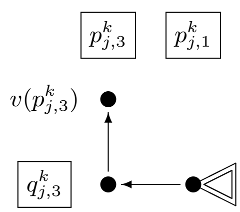

Figure 8: The preference digraph for copy (j, k) of the literal gadget implicit in [9] includes an icon representing a copy of the unstable gadget of Figure 7 and has a total of $1 + 3 = 4$ residents and $1 + 2 = 3$ hospitals.

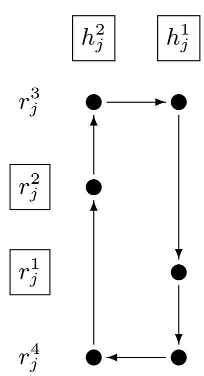

Figure 9: The preference digraph for copy j of the bistable gadget implicit in [9] has four residents $r^1_j, \ldots, r^4_j$ and two hospitals h^1_j, h^2_j.

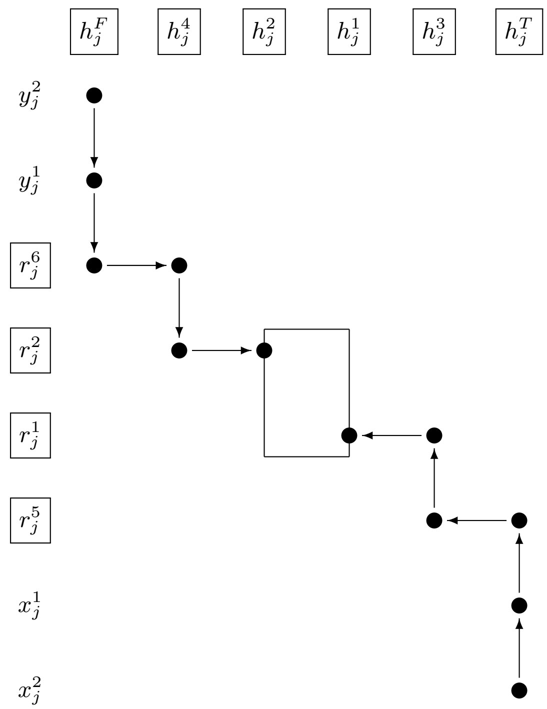

Figure 10: The preference digraph for copy j of the variable gadget implicit in [9] includes an icon representing a copy of the bistable gadget of Figure 9 and has a total of $6+4=10$ residents and $4+2=6$ hospitals. This preference digraph can be streamlined as in Figure 4(a) to reduce the number of residents to $4+4=8$ and the number of hospitals to $2+2=4$.