CS311H

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• In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?



Good Morning, Colleagues



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Are there any questions?





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 - Tuesday and Wednesday devoted to review.



Quiz

- If the class has 60 students, how many ways can I divide the class in half?
- If I have 100 Snickers bars, in how many ways can I divide them up among the students in that class?
- In the same class, if 10 of the students are named "Will," 15 are named "William," 5 are named "Bill" and the rest have unique names, how many ways can I write the class members' (first) names in order?
- According to Pascal's identity, what is $\binom{35}{12} + \binom{35}{13}$?



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• Find closed form solution for $a_0 = 5$, $a_1 = 2$, and $a_n = -10a_{n-1} - 25a_{n-2}$.



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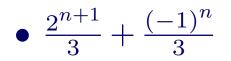
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• Let ABCDEFGH be a regular octagon of side length 1, and O be the center of the octagon. In addition to the sides of the octagon, line segments are drawn from O to each vertex, making a total of 16 line segments. Let a_n be the number of paths (not necessarily simple) of length nalong these line segments that start at O and terminate at O. Give a closed form solution of a_n .



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Solution

Let b_n be the number of path of length n that start at O and terminate at A (b_n also works for BCDEFGH). Since for the first step, we need move to one of the 8 vertices. Then we get the recurrence relationship that

 $a_n = 8b_{n-1}$

For b_n , consider the last step, it can be from its two adjacent vertices or from the center O_1 . Thus we have

$$b_n = 2b_{n-1} + a_{n-1}$$



Substituting b_n by $a_{n+1}/8$ we get

$$a_{n+1} - 2a_n - 8a_{n-1} = 0$$

For initial condition, we have $a_0 = 1$ and $a_1 = 0$. The characteristic polynomial is

$$x^2 - 2x - 8$$

which has roots of x = 4 and x = -2. Thus the solution of the homogeneous recurrence relationship is in form

$$a_n = \alpha(4)^n + \beta(-2)^n$$



Using initial condition, we have

$$a_0 = 1 = \alpha + \beta$$
$$a_1 = 0 = (4)\alpha + (-2)\beta$$

Thus we have and $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$ and the close form solution is

$$a_n = \frac{1}{3}4^n + \frac{2}{3}(-2)^n$$

