Batch RL Via
Least Squares Policy Iteration

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* Based on slides by Ronald Parr

Overview

- Motivation

- LSPI
  - Derivation from LSTD
  - Experimental results

Online versus Batch RL

- Online RL: integrates data collection and optimization
  - Select actions in environment and at the same time update parameters based on each observed (s,a,s',r)
- Batch RL: decouples data collection and optimization
  - First generate experience in the environment giving a data set of state-action-reward-state pairs $\{(s_i,a_i,r_i,s_i')\}$
  - Use the fixed set of experience to optimize/learn a policy
- Online vs. Batch:
  - Batch algorithms are often more "data efficient" and stable
  - Batch algorithms ignore the exploration-exploitation problem, and do their best with the data they have

Batch RL Motivation

- There are many applications that naturally fit the batch RL model
- **Medical Treatment Optimization:**
  - Input: collection of treatment episodes for an ailment giving sequence of observations and actions including outcomes
  - Ouput: a treatment policy, ideally better than current practice
- **Emergency Response Optimization:**
  - Input: collection of emergency response episodes giving movement of emergency resources before, during, and after 911 calls
  - Output: emergency response policy

LSPI

- LSPI is a model-free batch RL algorithm
  - Learns a linear approximation of Q-function
  - stable and efficient
  - Never diverges or gives meaningless answers
- LSPI can be applied to a dataset regardless of how it was collected

Terminology

- S: state space, s: individual states
- R(s,a): reward for taking action a in state s
- γ: discount factor
- V: state value
- P(s' | s,a) = T(s,a,s'): transition function
- Q: state-action value
- Policy: $\pi(s) \rightarrow a$

Objective: *Maximize expected, discounted return*

$$E\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

Projection Approach to Approximation

- Recall the standard Bellman equation:

$$V^*(s) = \max_a R(s,a) + \gamma \sum_{s'} P(s'|s,a) V^*(s')$$

  or equivalently $V^* = T[V^*]$ where *T*[.] is the Bellman operator

- Recall from value iteration, the sub-optimality of a value function can be bounded in terms of the Bellman error:

$$\left\| V - T[V] \right\|_\infty$$

- This motivates trying to find an approximate value function with small Bellman error

Projection Approach to Approximation

- Suppose that we have a space of representable value functions
  - E.g. the space of linear functions over given features
- Let Π be a *projection* operator for that space
  - Projects any value function (in or outside of the space) to "closest" value function in the space
- "Fixed Point" Bellman Equation with approximation

$$\hat{V}^* = \prod\left(T[\hat{V}^*]\right)$$

  - Depending on space this will have a small Bellman error
- LSPI will attempt to arrive at such a value function
  - Assumes linear approximation and least-squares projection

Projected Value Iteration

- **Naïve Idea:** try computing projected fixed point using VI
- Exact VI: (iterate Bellman backups)

$$V^{i+1} = T[V^i]$$

- Projected VI: (iterated projected Bellman backups):

$$\hat{V}^{i+1} = \prod\left(T[\hat{V}^i]\right)$$

Projects exact Bellman backup to closest function in our restricted function space

exact Bellman backup (produced value function)

Example: Projected Bellman Backup

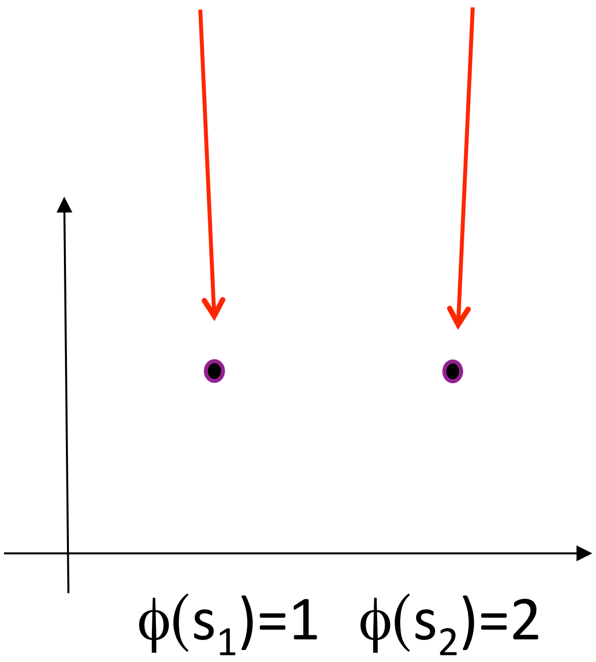
Restrict space to linear functions over a single feature ϕ:

$$\hat{V}(s) = w \cdot \phi(s)$$

Suppose just two states s_1 and s_2 with: $\phi(s_1)=1$, $\phi(s_2)=2$

Suppose exact back of V^i gives:

$$T[\hat{V}^i](s_1) = 2, \;\; T[\hat{V}^i](s_2) = 2$$



Can we represent this exact backup in our linear space?

Example: Projected Bellman Backup

Restrict space to linear functions over a single feature ϕ:

$$\hat{V}(s) = w \cdot \phi(s)$$

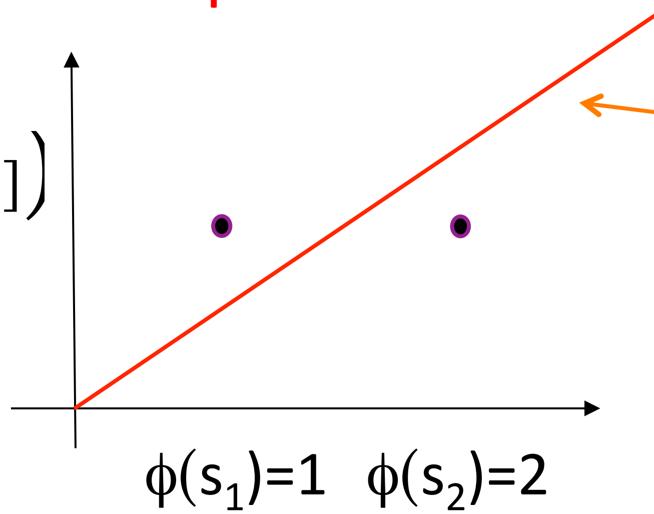
Suppose just two states s_1 and s_2 with: $\phi(s_1)=1$, $\phi(s_2)=2$

Suppose exact back of V^i gives:

$$T[\hat{V}^i](s_1) = 2, \ T[\hat{V}^i](s_2) = 2$$

The backup can't be represented via our linear function:

$$\hat{V}^{i+1} = \prod \left(T[\hat{V}^i] \right)$$



$$\hat{V}^{i+1}(s) = 1.333 \cdot \phi(s)$$

projected backup is just least-squares fit to exact backup

Problem: Stability

- Exact value iteration stability ensured by contraction property of Bellman backups:

$$V^{i+1} = T[V^i]$$

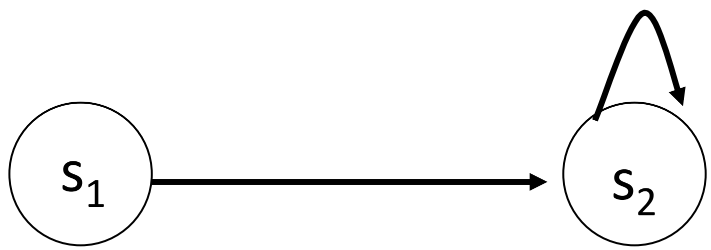
- Is the "projected" Bellman backup a contraction:

$$\hat{V}^{i+1} = \prod\left(T[\hat{V}^i]\right)$$

?

Example: Stability Problem [Bertsekas & Tsitsiklis 1996]

Problem: Most projections lead to backups that are not contractions and unstable



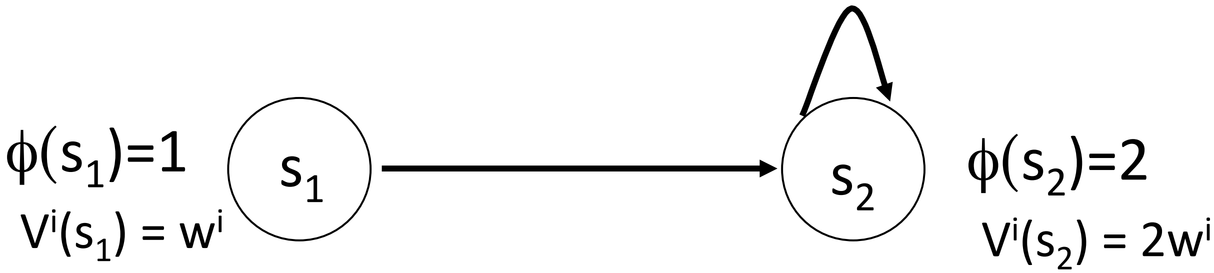
Rewards all zero, $\gamma = 0.9$: $V^* = 0$

Consider linear approx. w/ single feature ϕ with weight w.

$$\hat{V}(s) = w \cdot \phi(s)$$

Optimal w = 0
since V*=0

Example: Stability Problem



From V^i perform projected backup for each state

$$T[\hat{V}^i](s_1) = \gamma \hat{V}^i(s_2) = 1.8w^i$$

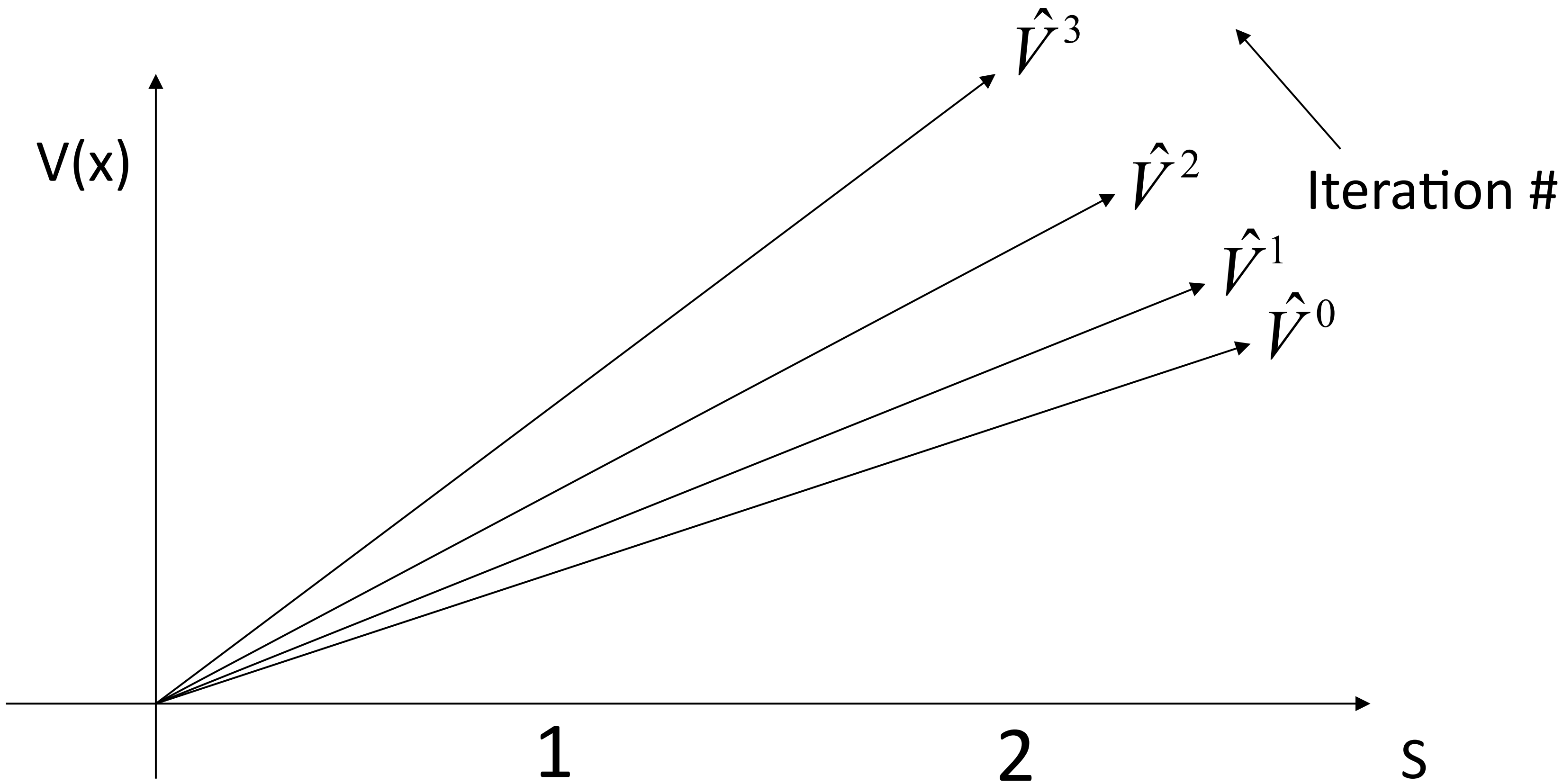
$$T[\hat{V}^i](s_2) = \gamma \hat{V}^i(s_2) = 1.8w^i$$

Can't be represented in our space so find w^{i+1} that gives least-squares approx. to exact backup

After some math we can get: **$w^{i+1} = 1.2\, w^i$**

What does this mean?

Example: Stability Problem



Each iteration of Bellman backup makes approximation worse!

Even for this simple problem "projected" VI diverges.

Understanding the Problem

- What went wrong?
  - Exact Bellman backups reduces error in maximum norm
  - Least squares (= projection) non-expansive in L_2 norm
  - May increase maximum norm distance

- **Conclusion:** Alternating value iteration and function approximation is risky business

Overview

- Motivation
- LSPI
  - Derivation from Least-Squares Temporal Difference Learning
  - Experimental results

How does LSPI fix these?

- LSPI performs approximate policy iteration
  - PI involves policy evaluation and policy improvement
  - Uses a variant of least-squares temporal difference learning (LSTD) for approx. policy evaluation [Bratdke & Barto '96]
- Stability:
  - LSTD directly solves for the fixed point of the approximate Bellman equation for policy values
  - With singular-value decomposition (SVD), this is always well defined
- Data efficiency
  - LSTD finds best approximation for any finite data set
  - Makes a single pass over the data for each policy
  - Can be implemented incrementally

OK, What's LSTD?

- Least Squares Temporal Difference Learning
- Assumes linear value function approximation of K features

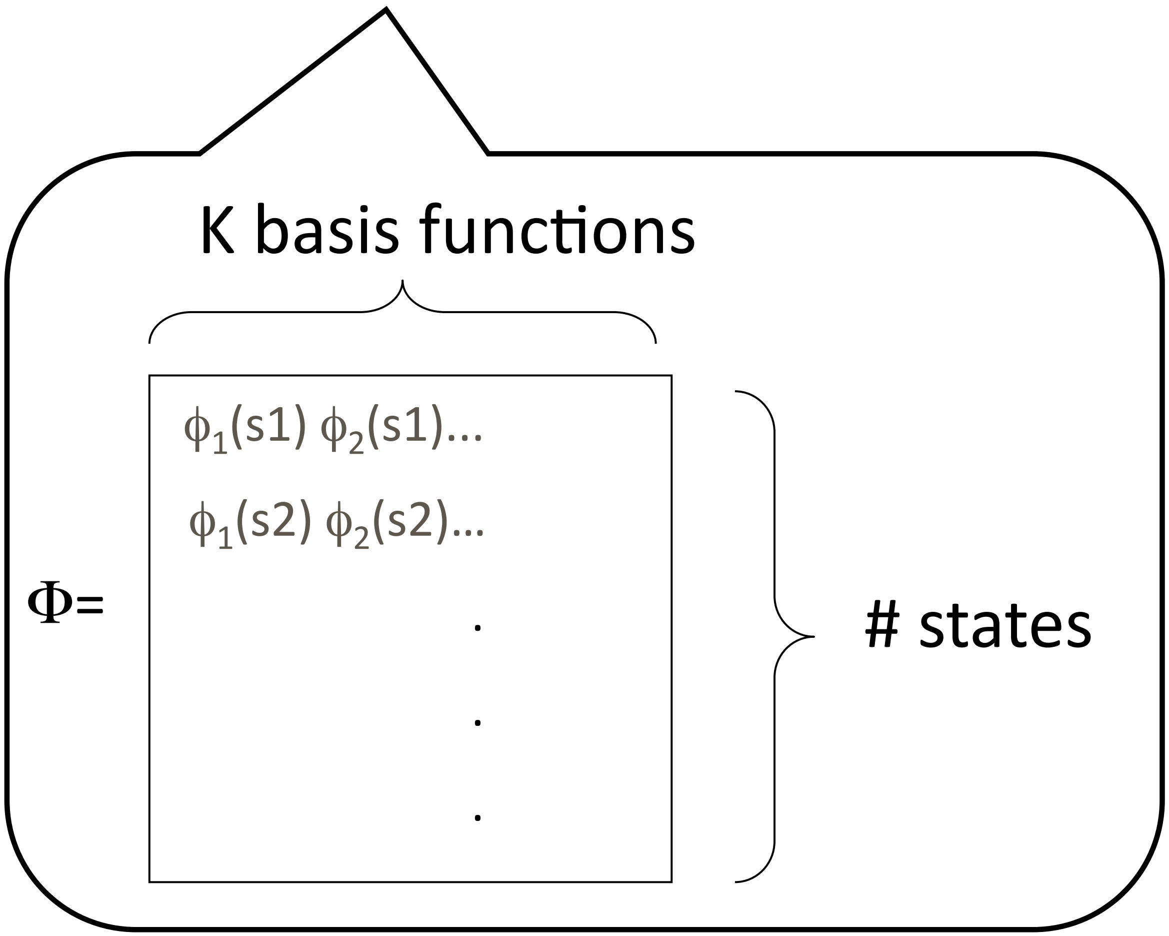
$$\hat{V}(s) = \sum_k w_k \phi_k(s)$$

- The ϕ_k are arbitrary feature functions of states
- Some vector notation

$$\hat{V} = \begin{bmatrix} \hat{V}(s_1) \\ \vdots \\ \hat{V}(s_n) \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} \quad \phi_k = \begin{bmatrix} \phi_k(s_1) \\ \vdots \\ \phi_k(s_n) \end{bmatrix} \quad \Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_K \end{bmatrix}$$

Deriving LSTD

$$\hat{V} = \Phi w$$ assigns a value to every state



$\hat{V}$ is a linear function in the column space of $\phi_1 \ldots \phi_k$, that is,

$$\hat{V} = w_1 \cdot \phi_1 + \cdots + w_K \cdot \phi_K$$

Suppose we know value of policy

- Want: $\Phi w \approx V^{\pi}$

- Least squares weights minimizes squared error

$$w = \underbrace{(\Phi^T \Phi)^{-1} \Phi^T}_{\text{Sometimes called pseudoinverse}} V^{\pi}$$

- Least squares projection is then

$$\hat{V} = \Phi w = \underbrace{\Phi(\Phi^T \Phi)^{-1} \Phi^T}_{\text{Textbook least squares projection operator}} V^{\pi}$$

But we don't know V…

- Recall fixed-point equation for policies

$$V^\pi(s) = R(s,\pi(s)) + \gamma \sum_{s'} P(s'|s,\pi(s)) V^\pi(s')$$

- Will solve a projected fixed-point equation:

$$\hat{V}^\pi = \prod \left(R + \gamma P \hat{V}^\pi \right)$$

$$R = \begin{bmatrix} R(s_1,\pi(s_1)) \\ \vdots \\ R(s_n,\pi(s_n)) \end{bmatrix}, \quad P = \begin{bmatrix} P(s_1 \mid s_1,\pi(s_1)) & \cdots & P(s_n \mid s_1,\pi(s_1)) \\ \vdots & \vdots & \vdots \\ P(s_1 \mid s_n,\pi(s_n)) & \cdots & P(s_1 \mid s_n,\pi(s_n)) \end{bmatrix}$$

- Substituting least squares projection into this gives:

$$\Phi w = \Phi(\Phi^T \Phi)^{-1} \Phi^T \left(R + \gamma P \Phi w \right)$$

- Solving for w: $w = (\Phi^T \Phi - \gamma \Phi^T P \Phi)^{-1} \Phi^T R$

Almost there…

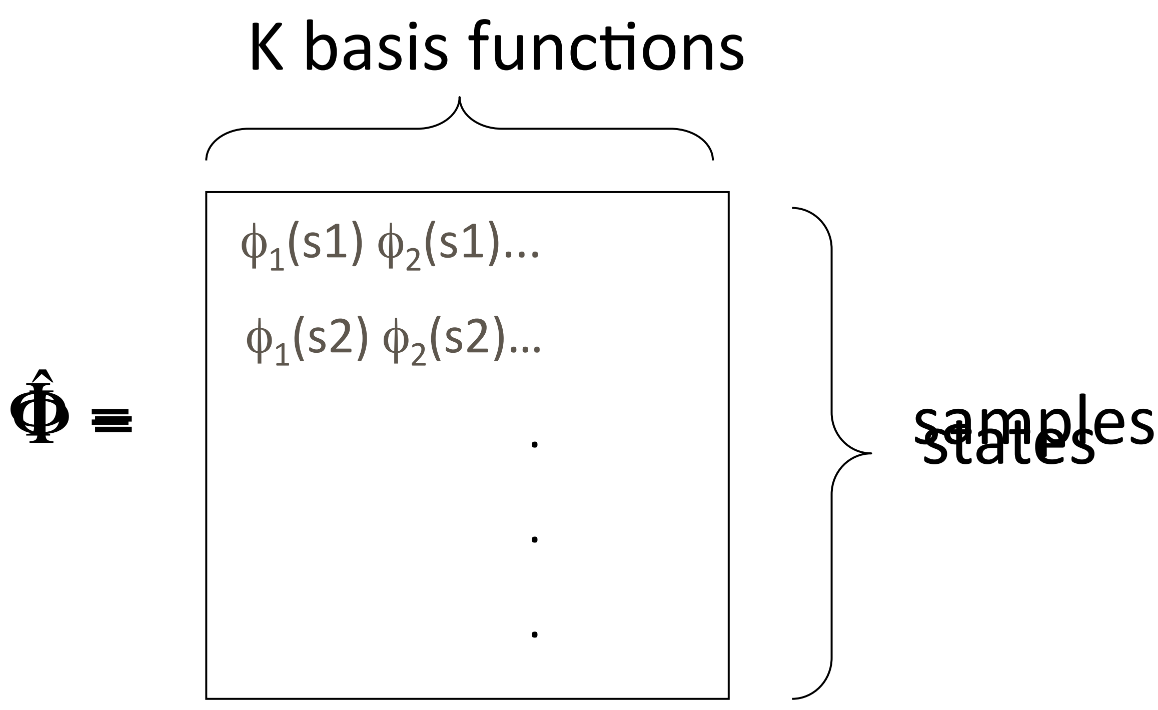
$$w = (\Phi^T \Phi - \gamma \Phi^T P \Phi)^{-1} \Phi^T R$$

- Matrix to invert is only K x K
- But…
  - Expensive to construct matrix (e.g. P is |S|x|S|)
  - We don’t know P
  - We don’t know R

Using Samples for Φ

Suppose we have state transition samples of the policy running in the MDP: $\{(s_i, a_i, r_i, s_i')\}$

Idea: Replace enumeration of states with sampled states



Using Samples for R

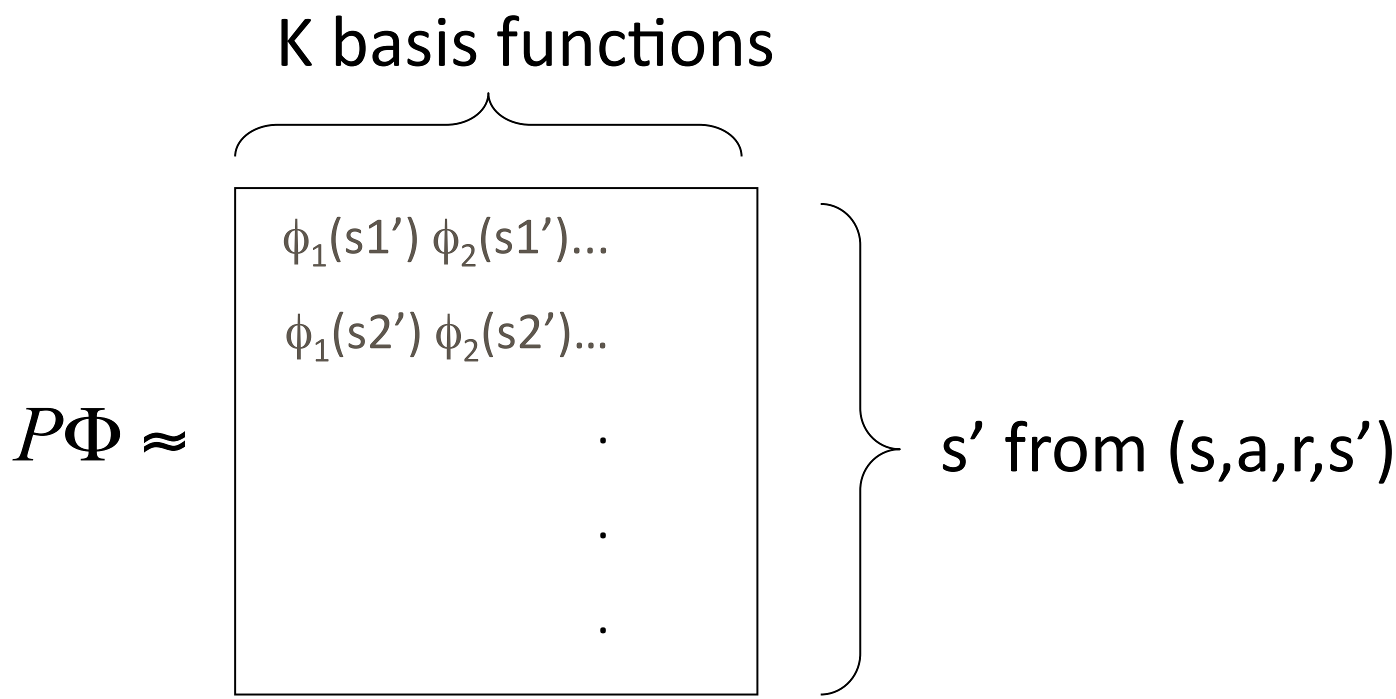
Suppose we have state transition samples of the policy running in the MDP: $\{(s_i, a_i, r_i, s_i')\}$

Idea: Replace enumeration of reward with sampled rewards

$$R = \left.\begin{bmatrix} r_1 \\ r_2 \\ . \\ . \\ . \end{bmatrix}\right\} \text{samples}$$

Using Samples for PΦ

Idea: Replace expectation over next states with sampled next states.



Putting it Together

- LSTD needs to compute:

$$w = (\Phi^T \Phi - \gamma \Phi^T P \Phi)^{-1} \Phi^T R = B^{-1} b$$

$$B = \Phi^T \Phi - \gamma \Phi^T \underbrace{(P\Phi)}_{\text{from previous slide}}$$

$$b = \Phi^T R$$

- The hard part of which is *B* the kxk matrix:
- Both B and b can be computed incrementally for each (s,a,r,s') sample: (initialize to zero)

$$B_{ij} \leftarrow B_{ij} + \phi_i(s)\phi_j(s) - \gamma \phi_i(s)\phi_j(s')$$

$$b_i \leftarrow b_i + r \cdot \phi_i(s)$$

LSTD Algorithm

- Collect data by executing trajectories of current policy
- For each (s,a,r,s') sample:

$$B_{ij} \leftarrow B_{ij} + \phi_i(s)\phi_j(s) - \gamma\phi_i(s)\phi_j(s')$$

$$b_i \leftarrow b_i + r \cdot \phi_i(s,a)$$

$$w \leftarrow B^{-1}b$$

LSTD Summary

- Does $O(k^2)$ work per datum
  - Linear in amount of data.
- Approaches model-based answer in limit
- Finding fixed point requires inverting matrix
- Fixed point almost always exists
- Stable; efficient

Approximate Policy Iteration with LSTD

Policy Iteration: iterates between policy improvement and policy evaluation

Idea: use LSTD for approximate policy evaluation in PI

Start with random weights **w** (i.e. value function)

Repeat Until Convergence

$\pi(s) = \text{greedy}(\hat{V}(s, \mathbf{w}))$ // policy improvement

Evaluate π using LSTD

- Generate sample trajectories of π
- Use LSTD to produce new weights **w**
  (**w** gives an approx. value function of π)

What Breaks?

- No way to execute greedy policy without a model
- Approximation is biased by current policy
  - We only approximate values of states we see when executing the current policy
  - LSTD is a *weighted* approximation toward those states
- Can result in Learn-forget cycle of policy iteration
  - Drive off the road; learn that it's bad
  - New policy never does this; forgets that it's bad
- Not truly a batch method
  - Data must be collected from current policy for LSTD

LSPI

- LSPI is similar to previous loop but replaces LSTD with a new algorithm LSTDQ
- LSTD: produces a value function
  - Requires sample from policy under consideration
- LSTDQ: produces a Q-function
  - Can learn Q-function for policy from any (reasonable) set of samples---sometimes called an off-policy method
  - No need to collect samples from current policy
- Disconnects policy evaluation from data collection
  - Permits reuse of data across iterations!
  - Truly a batch method.

Implementing LSTDQ

- Both LSTD and LSTDQ compute: $B = \Phi^T \Phi - \lambda \Phi^T (P\Phi)$
- But LSTDQ basis functions are indexed by actions

$$\hat{Q}_w(s,a) = \sum_k w_k \cdot \phi_k(s,a)$$

defines greedy policy: $\pi_w(s) = \arg\max_a \hat{Q}_w(s,a)$

- For each (s,a,r,s') sample:

$$B_{ij} \leftarrow B_{ij} + \phi_i(s,a)\phi_j(s,a) - \lambda \phi_i(s,a)\phi_j(s',\pi_w(s'))$$

$$b_i \leftarrow b_i + r \cdot \phi_i(s,a)$$

$$w \leftarrow B^{-1} b$$

$$\arg\max_a \hat{Q}_w(s',a)$$

Running LSPI

- There is a Matlab implementation available!

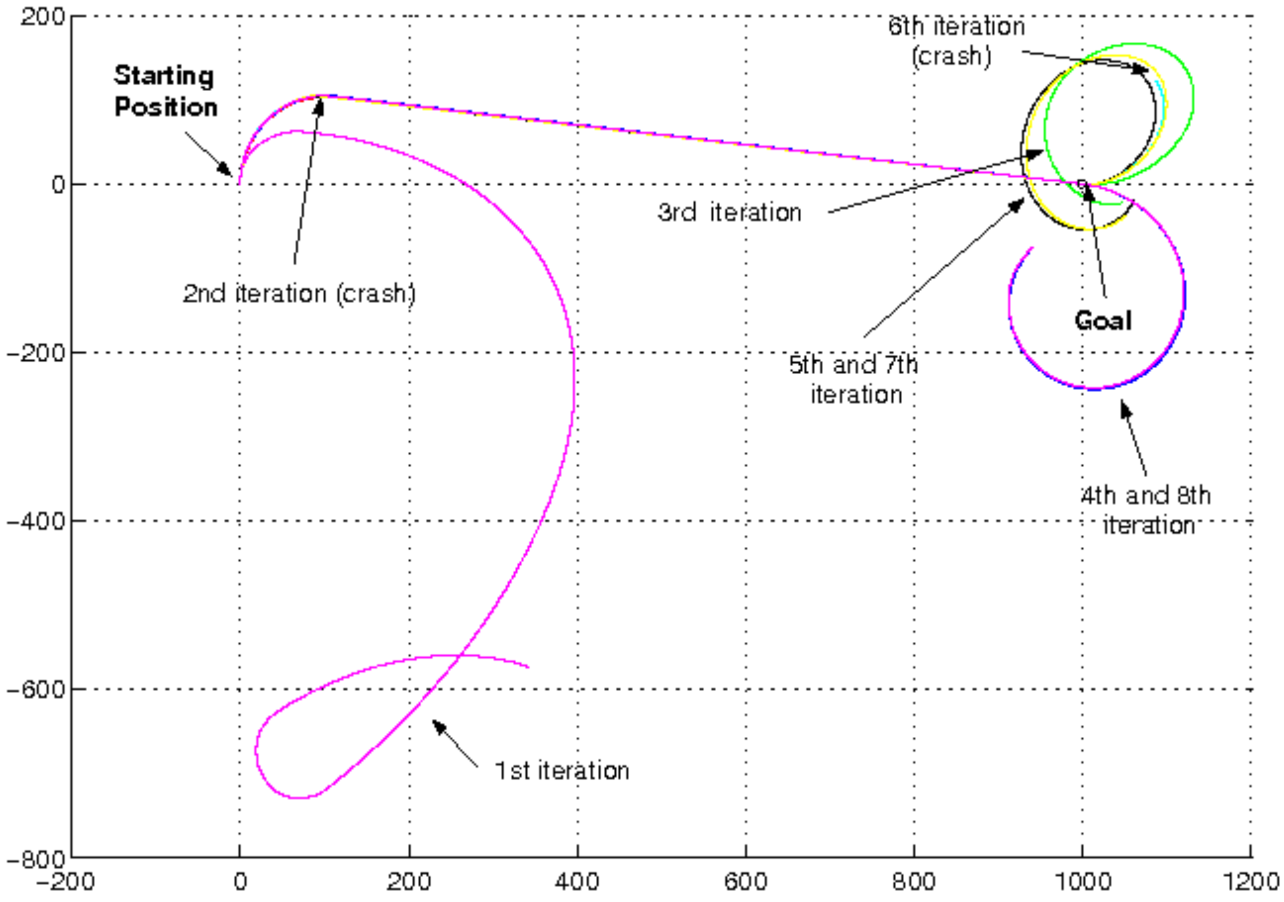
1. Collect a database of (s,a,r,s') experiences (this is the magic step)
2. Start w/random weights (= random policy)
3. Repeat
   - Evaluate current policy against database
     - Run LSTDQ to generate new set of weights
     - New weights imply new Q-function and hence new policy
   - Replace current weights with new weights

- Until convergence

Results: Bicycle Riding

- Watch random controller operate bike
- Collect ~40,000 (s,a,r,s') samples
- Pick 20 simple basis functions (×5 actions)
- Make 5-10 passes over data (PI steps)
- Reward was based on distance to goal + goal achievement
- Result:
  Controller that balances and rides to goal

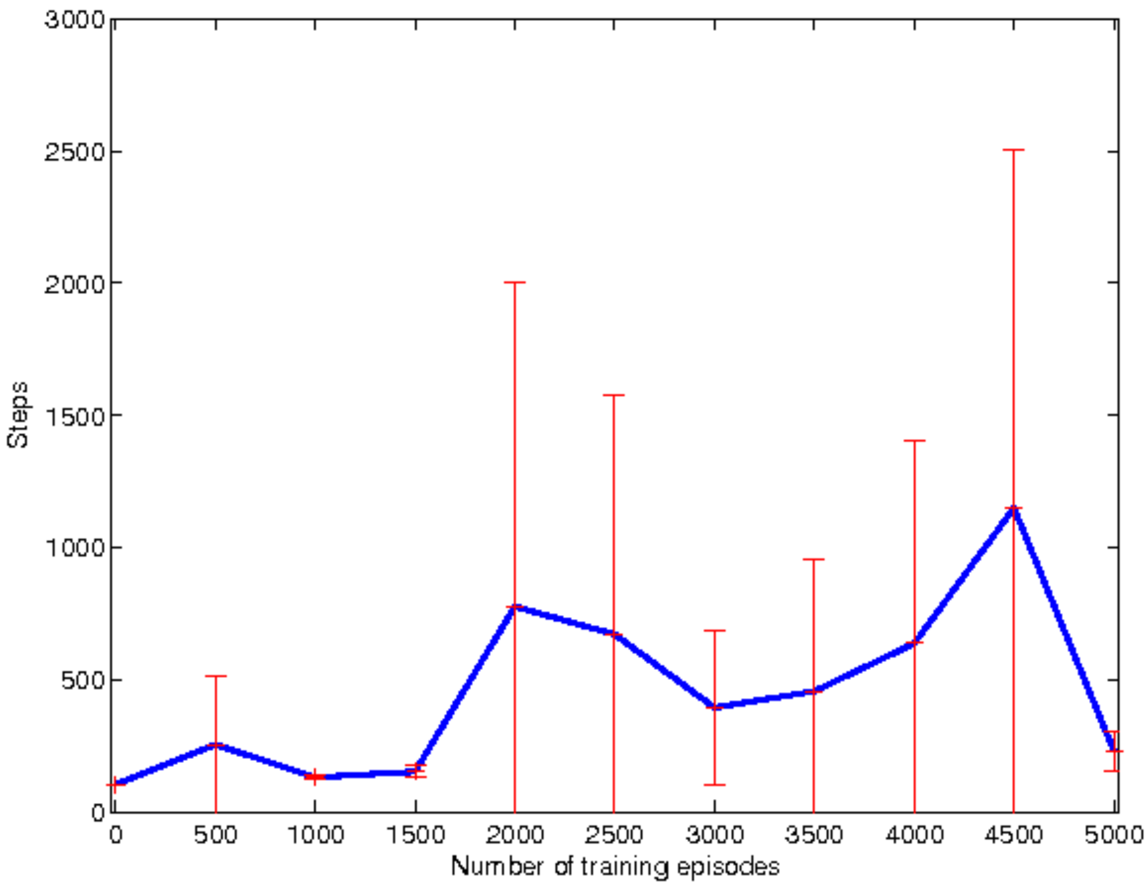
Bicycle Trajectories



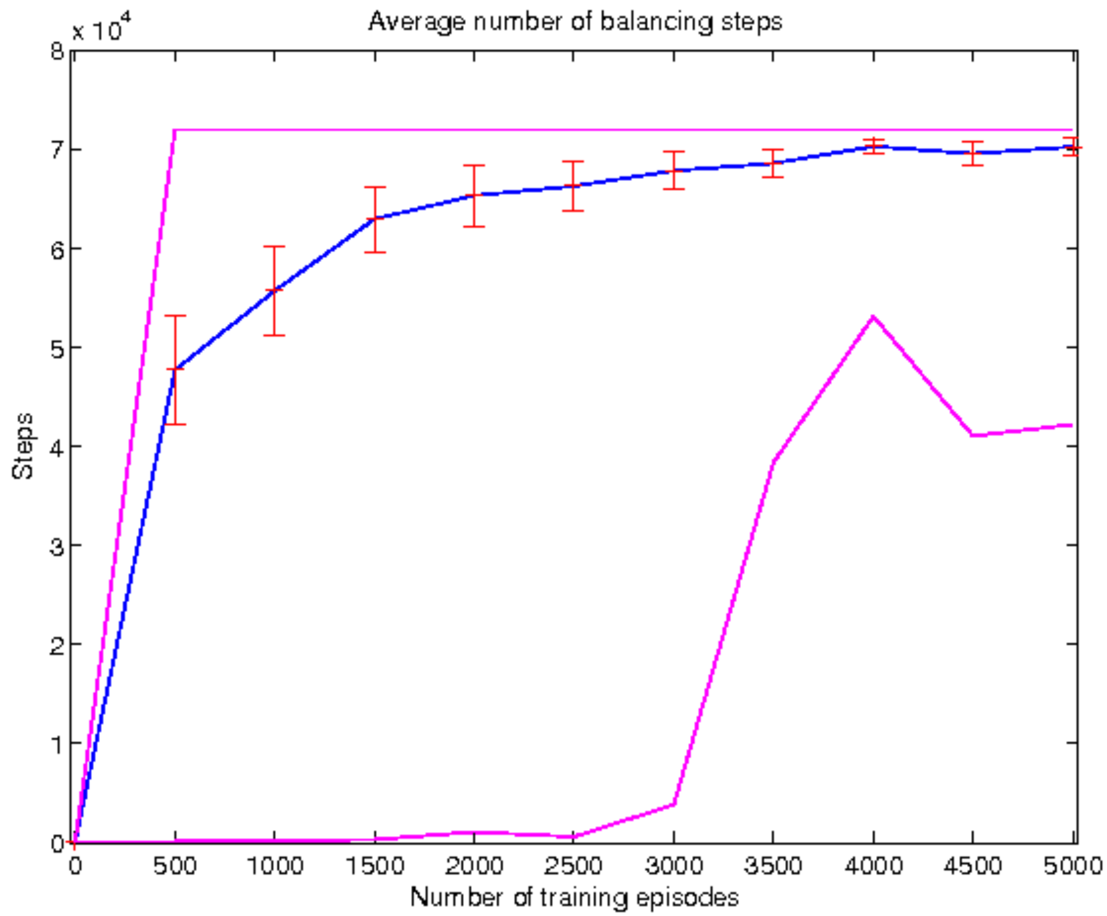
What about Q-learning?

- Ran Q-learning with same features
- Used experience replay for data efficiency

Q-learning Results



LSPI Robustness



Some key points

- LSPI is a batch RL algorithm
  - Can generate trajectory data anyway you want
  - Induces a policy based on global optimization over full dataset
- Very stable with no parameters that need tweaking

So, what's the bad news?

- LSPI does not address the exploration problem
  - It decouples data collection from policy optimization
  - This is often not a major issue, but can be in some cases
- k^2 can sometimes be big
  - Lots of storage
  - Matrix inversion can be expensive
- Bicycle needed "shaping" rewards
- Still haven't solved
  - Feature selection (issue for all machine learning, but RL seems even more sensitive)