\[ w \leftarrow w - \frac{1}{2} \alpha \nabla \left[ V_{\pi}(s_k) - \hat{V}(s_k, w) \right] \]

\[ w \leftarrow w + \alpha \left[ U_k - \hat{V}(s_k, w) \right] \cdot \nabla \hat{V}(s_k, w) \]

**MC Estimate:**
\[ U_k = \sum_{t \geq t_k} y^{i-t-1} R_i \]
- Unbiased
- Fixed when \( w \) changes
- Converges near global opt

**Bootstrap Estimate:**
\[ U_k = R_k + \delta \hat{V}(s_k, w) \]
- Biased
- Function of \( w \) changes with \( w \)
- Gradient calc treated \( U_k \) as a constant!
- Converges to TD fixed point (local opt)

**For TD fixed point** \( w_{TD} \):
\[ \overline{VE}(w_{TD}) \leq \frac{1}{1-\gamma} \min_w \left[ \overline{VE}(w) \right] \]
The deadly triad

Divergence is possible when all 3 parts of the deadly triad are present:

• Function approximation
• Bootstrapping
• Off-Policy training
Off-policy semigradient methods

Stability of semigradient methods depends on on-policy distribution of updates. Why?

Imagine only updating one state $S$ over and over again (i.e. off-policy):

- In tabular case, updating one state's value leaves all others changed
- With function approx + MC, multiple state values are updated, but $V(S)$ is estimated independently of them via rewards only
- With function approx + TD (semigradient), multiple values are updated, which are then used to help estimate $V(S)$ via bootstrapping, which are then updated again, which are then used to help estimates $V(S)$…

On-policy distribution forces state values to be “grounded” to something real
Proof of Convergence of Linear TD(0)

What properties assure convergence of the linear TD(0) algorithm (9.9)? Some insight can be gained by rewriting (9.10) as

$$\mathbb{E}[\mathbf{w}_{t+1} | \mathbf{w}_t] = (\mathbf{I} - \alpha \mathbf{A}) \mathbf{w}_t + \alpha \mathbf{b}. \quad (9.13)$$

Note that the matrix $\mathbf{A}$ multiplies the weight vector $\mathbf{w}_t$ and not $\mathbf{b}$; only $\mathbf{A}$ is important to convergence. To develop intuition, consider the special case in which $\mathbf{A}$ is a diagonal matrix. If any of the diagonal elements are negative, then the corresponding diagonal element of $\mathbf{I} - \alpha \mathbf{A}$ will be greater than one, and the corresponding component of $\mathbf{w}_t$ will be amplified, which will lead to divergence if continued. On the other hand, if the diagonal elements of $\mathbf{A}$ are all positive, then $\alpha$ can be chosen smaller than one over the largest of them, such that $\mathbf{I} - \alpha \mathbf{A}$ is diagonal with all diagonal elements between 0 and 1. In this case the first term of the update tends to shrink $\mathbf{w}_t$, and stability is assured. In general, $\mathbf{w}_t$ will be reduced toward zero whenever $\mathbf{A}$ is positive definite, meaning $\mathbf{y}^T \mathbf{A} \mathbf{y} > 0$ for any real vector $\mathbf{y} \neq 0$. Positive definiteness also ensures that the inverse $\mathbf{A}^{-1}$ exists.

For linear TD(0), in the continuing case with $\gamma < 1$, the $\mathbf{A}$ matrix (9.11) can be written

$$\mathbf{A} = \sum_s \mu(s) \sum_a \pi(a|s) \sum_{r,s'} p(r,s'|s,a) \mathbf{x}(s) (\mathbf{x}(s) - \gamma \mathbf{x}(s'))^T$$

$$= \sum_s \mu(s) \sum_{s'} p(s'|s) \mathbf{x}(s) (\mathbf{x}(s) - \gamma \mathbf{x}(s'))^T$$

$$= \sum_s \mu(s) \mathbf{x}(s) \left( \mathbf{x}(s) - \gamma \sum_{s'} p(s'|s) \mathbf{x}(s') \right)^T$$

$$= \mathbf{X}^T \mathbf{D} (\mathbf{I} - \gamma \mathbf{P}) \mathbf{X},$$

where $\mu(s)$ is the stationary distribution under $\pi$, $p(s'|s)$ is the probability of
Init: $w_0 = 10$  
Thus: $\hat{V}(A) = 10$, $\hat{V}(B) = 20$

Assume $\alpha = 0.5$, $\beta = 0.9$
Init: \( \omega_0 = 10 \)  

Thus: \( \hat{V}(A) = 10 \), \( \hat{V}(B) = 20 \)

Assume \( \alpha = 0.5 \), \( \beta = 0.9 \)

Observe transition from \( A \) to \( B \)

\[
\omega_{t+1} = \omega_t + \alpha \, \beta \left[ R_t + \beta \hat{V}(B) - \hat{V}(A) \right] \nabla \hat{V}(A)
\]

\[
= 10 + (0.5)(0.9)(20 - 10) \cdot 1
\]

\[
= 10 + 4
\]

\[
= 14
\]

Thus: \( \hat{V}(A) = 14 \), \( \hat{V}(B) = 28 \)
Init: \( W_0 = 10 \)  
Thus: \( \hat{V}(A) = 10 \), \( \hat{V}(B) = 20 \)

Assume \( \alpha = 0.5 \), \( \beta = 0.9 \)

Observe transition from A to B

\[
W_{t+1} = W_t + \alpha \beta \left[ R_t + \beta \hat{V}(B) - \hat{V}(A) \right] \nabla \hat{V}(A)
\]

\[
= 10 + (0.5)(1)\left[ 0 + 0.9(20) - 10 \right] \cdot 1
\]

\[
= 10 + 4
\]

\[
= 14
\]

Thus: \( \hat{V}(A) = 14 \), \( \hat{V}(B) = 28 \)
\[ r = 0 \text{ everywhere} \]

\[
\begin{align*}
2w_1 + w_5 & \\
2w_2 + w_5 & \\
2w_3 + w_5 & \\
w_4 + 4w_5 & \\
\end{align*}
\]

\[
\Rightarrow \quad \hat{\mathbf{v}} = \begin{bmatrix} 3 & 3 & 3 & 14 \end{bmatrix}^T
\]

if \( w_0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \end{bmatrix}^T \) then,

\[
x = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}
\]
\[ x = \begin{bmatrix} 20001 \\ 02001 \\ 00201 \\ 00014 \end{bmatrix} \]

if \( w_0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \end{bmatrix}^T \) then,

\[ \hat{V} = \begin{bmatrix} 3 & 3 & 3 & 14 \end{bmatrix}^T \]

DP update:

\[ w_{t+1} = w_t + \frac{\alpha}{151} \sum_s \left( E_{t+1} [R_{t+1} + \gamma \hat{V}(s_{t+1})] - \hat{V}(s) \right) \triangledown V(s) \]
\[ x: \begin{bmatrix} 20001 \\ 02001 \\ 00201 \\ 000014 \end{bmatrix} \]

If \( w_0 = \begin{bmatrix} 111101 \end{bmatrix}^T \) then,
\[ \hat{V} = \begin{bmatrix} 33314 \end{bmatrix}^T \]

**DP update:**
\[
W_{t+1} = W_t + \frac{\alpha}{151} \sum_s \left( E_{\pi}[R_{t+1} + \gamma \hat{V}(S_{t+1})] - \hat{V}(S) \right) \nabla V(S)
\]

\[
= W_t + \frac{1}{4} \left[ (14-3) \begin{bmatrix} 20001 \end{bmatrix}^T + \\
(14-3) \begin{bmatrix} 02001 \end{bmatrix}^T + \\
(14-3) \begin{bmatrix} 00201 \end{bmatrix}^T + \\
(14-14) \begin{bmatrix} 000014 \end{bmatrix}^T \right]
\]
\[
\begin{align*}
2w_1 + w_5 \\
2w_2 + w_5 \\
2w_3 + w_6 \\
w_4 + 4w_5
\end{align*}
\]

\[r = 0 \text{ everywhere} \]

\[
\Rightarrow
\]

\[
x: \begin{bmatrix}
20 & 00 & 01 \\
02 & 00 & 01 \\
00 & 20 & 01 \\
00 & 00 & 14
\end{bmatrix}
\]

if \( w_0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \end{bmatrix}^\top \) then,

\[
\hat{V} = \begin{bmatrix} 3 & 3 & 3 & 14 \end{bmatrix}^\top
\]

**DP update:**

\[
\begin{aligned}
\omega_{k+1} &= \omega_k + \frac{\alpha}{151} \sum_s \left( E_\pi [R_{k+1} + 8 \hat{V}(s_{k+1})] - \hat{V}(s) \right) \nabla V(s) \\
&= \omega_k + \frac{1}{4} \left[ (14 - 3) \begin{bmatrix} 20 & 00 & 01 \end{bmatrix}^\top + \\
(14 - 3) \begin{bmatrix} 02 & 00 & 01 \end{bmatrix}^\top + \\
(14 - 3) \begin{bmatrix} 00 & 20 & 01 \end{bmatrix}^\top + \\
(14 - 14) \begin{bmatrix} 00 & 00 & 14 \end{bmatrix}^\top \right] \\
&= \begin{bmatrix} 1 & 1 & 1 & 10 & 1 \end{bmatrix}^\top + \frac{1}{4} \cdot \begin{bmatrix} 22 & 22 & 22 & 0 & 33 \end{bmatrix}^\top \\
&= \begin{bmatrix} 6.5 & 6.5 & 6.5 & 10 & 9.25 \end{bmatrix}^\top
\end{aligned}
\]
\( r = 0 \) everywhere

\[ 2w_1 + w_5 \]
\[ 2w_2 + w_5 \]
\[ 2w_3 + w_5 \]
\[ w_4 + 4w_5 \]

\[
\begin{align*}
x & \in \begin{bmatrix} 00001 \\ 02001 \\ 00201 \\ 00014 \\ \end{bmatrix} \\
\text{if } w_0 &= \begin{bmatrix} 1 & 1 & 1 & 10 & 1 \end{bmatrix}^T \text{ then, } \\
\hat{V} &= \begin{bmatrix} 3 & 3 & 3 & 14 \end{bmatrix}^T
\end{align*}
\]

**DP update:**

\[
\begin{align*}
\omega_{k+1} &= \omega_k + \frac{\alpha}{151} \sum_s \left( E_i [R_{k+1} + \gamma \hat{V}(s_{k+1})] - \hat{V}(s) \right) \nabla V(s)
\end{align*}
\]

\[
= \omega_k + \frac{1}{4} \left[ (14 - 3) \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \end{bmatrix}^T + \\
(14 - 3) \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \end{bmatrix}^T + \\
(14 - 3) \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \end{bmatrix}^T + \\
(14 - 14) \begin{bmatrix} 0 & 0 & 0 & 1 & 4 \end{bmatrix}^T \right]
\]

\[
= \begin{bmatrix} 1 & 1 & 1 & 10 & 1 \end{bmatrix}^T + \frac{1}{4} \cdot \begin{bmatrix} 22 & 22 & 22 & 0 & 33 \end{bmatrix}^T
\]

\[
= \begin{bmatrix} 6.5 & 6.5 & 6.5 & 10 & 9.25 \end{bmatrix}^T \quad \Rightarrow \quad \hat{V} = \begin{bmatrix} 22.25 & 22.25 & 22.25 & 4.7 \end{bmatrix}
\]
DP update:

\[
\begin{align*}
    w_{t+1} &= w_t + \frac{\alpha}{151} \sum_s \left( E_t \left[ R_{t+1} + \gamma \hat{V}(s_{t+1}) \right] - \hat{V}(s) \right) \nabla \hat{V}(s) \\
    &= w_t + \frac{1}{4} \left( (14-3) \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \end{bmatrix}^T + \\
    &\quad (14-3) \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \end{bmatrix}^T + \\
    &\quad (14-3) \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \end{bmatrix}^T + \\
    &\quad (14-14) \begin{bmatrix} 0 & 0 & 0 & 1 & 4 \end{bmatrix}^T \right) \\
    &= \begin{bmatrix} 1 & 1 & 1 & 10 & 1 \end{bmatrix}^T + \frac{1}{4} \cdot \begin{bmatrix} 22 & 22 & 22 & 0 & 33 \end{bmatrix}^T \\
    &= \begin{bmatrix} 6.5 & 6.5 & 6.5 & 10 & 9.25 \end{bmatrix}^T \\
\end{align*}
\]

- DP updates with off-policy state dist (uniform), but policy \( \pi \) always follows solid lines.
- Updates top states more often than it should.
- Bottom state looks better and raises value of top state.
- ... which raises value of bottom state, which...

\[
x: \begin{bmatrix} 20001 \\ 02001 \\ 00201 \\ 00014 \end{bmatrix} \Rightarrow \hat{v}: \begin{bmatrix} 22.25 & 22.25 & 22.25 & 47 \end{bmatrix}^T
\]
Orthogonal Projection

Projection to a unit vector: $P_u = uu^T$
So: $P_u x = uu^T x$
Orthogonal Projection

Projection to a unit vector: \( P_u = uu^T \)
So: \( Pu \cdot x = uu^T \cdot x \)

Why?
\( u^T x = \|x\| \cos \theta \)
\( uu^T \cdot x \) is a vector of magnitude \( \|x\| \cos \theta \) in the direction of \( u \)
Orthogonal Projection

Projection to a unit vector: $P_u = uu^T$
So: $P_u x = uu^T x$

Why?
$u^T x = \|x\| \cos \theta$
$uu^T x$ is a vector of magnitude $\|x\| \cos \theta$ in the direction of $u$

More Generally:
If $A = [u_1, ..., u_k]$ is an orthonormal basis of the subspace $U$, then:

$$P_A = AA^T$$
Orthogonal Projection

Projection to a unit vector: \( P_u = uu^T \)
So: \( P_u x = uu^T x \)

Why?
\( u^T x = \|x\| \cos \theta \)
\( uu^T x \) is a vector of magnitude \( \|x\| \cos \theta \)
in the direction of \( u \)

What if \( u, \ldots u_k \) not orthonormal?
\( P_A = A (A^T A)^{-1} A^T \)

More Generally:
if \( A = [u_1 \ldots u_k] \) is an orthonormal basis of the subspace \( U \), then:
\( P_A = AA^T \)
Orthogonal Projection

Projection to a unit vector: \( P_u = uu^T \)
So: \( P_u x = uu^T x \)

Why?
\( u^T x = \|x\| \cos \theta \)
\( uu^T x \) is a vector of magnitude \( \|x\| \cos \theta \)
in the direction of \( u \)

What if \( u, \ldots, u_k \) not orthonormal?
\( P_A = A (A^T A)^{-1} A^T \)

normalizing factor

More Generally:
if \( A = [u, \ldots, u_k] \) is an orthonormal basis of the subspace \( U \), then:
\( P_A = AA^T \)

What about other inner products?
\( P_A = A (A^T DA)^{-1} A^T D \)

\[ \langle x, y \rangle = \sqrt{D} x \]
Orthogonal Projection

Projection to a unit vector: \( P_u = uu^T \)
So: \( P_u x = uu^T x \)

Why?
\( u^T x = \|x\| \cos \theta \)
\( uu^T x \) is a vector of magnitude \( \|x\| \cos \theta \)
in the direction of \( u \)

What if \( u_1, \ldots, u_k \) not orthonormal?
\[ P_A = A (A^T A)^{-1} A^T \]
(normalizing factor)

Linear regression:
\[ \hat{y} = X (X^T X)^{-1} X^T y \]

What about other inner products?
\[ P_A = A (A^T D A)^{-1} A^T D \]

More Generally:
if \( A = [u_1, \ldots, u_k] \) is an orthonormal basis of the subspace \( U \), then:
\[ P_A = AA^T \]
Basis functions vs. vectors

\[ \begin{bmatrix} 3.5 & 4 & 1.5 \end{bmatrix} \]
Basis functions vs. vectors

If increased to an infinite number of points, an infinite-dimensional vector can represent a function exactly!

Thus, basis vectors can represent basis functions... and be projected onto!
Basis functions vs. vectors

If increased to an infinite number of points, an infinite-dimensional vector can represent a function exactly!

Thus, basis vectors can represent basis functions ... and be projected onto!

\[ x_1: \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array} \quad x_2: \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array} \quad y: \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array} \]
Basis functions vs. vectors

If increased to an infinite number of points, an infinite-dimensional vector can represent a function exactly!

Thus, basis vectors can represent basis functions... and be projected onto!

Project $y$ onto $X_1, X_2$
Basis functions vs. vectors

If increased to an infinite number of points, an infinite-dimensional vector can represent a function exactly!

Thus, basis vectors can represent basis functions ... and be projected onto!

\[ P_x = X(X^TDX)^{-1}X^TD \]

where \( X \) is \( |S| \times |\text{features}| \) and \( D \) is:

\[
\begin{bmatrix}
\mu(s_1) \\
\vdots \\
\mu(s_n)
\end{bmatrix}
\]

Project \( y \) onto \( X_1, X_2 \)
The 3D space of all value functions over 3 states

The subspace of all value functions representable as $v_w$

Bellman error vector (BE)

Value error (VE)

$\min \text{TDE}$

$\Pi v_\pi$ ($\min \text{VE} = \|\text{VE}\|_\mu^2$)

$\Pi B_\pi v_w$

$\frac{w_{TD}}{PBE} = \delta$

$\min \text{BE} = \|\text{BE}\|_\mu^2$