# FORWARD KINEMATICS AND JACOBIAN

RBT350 – Gateway to Robotics

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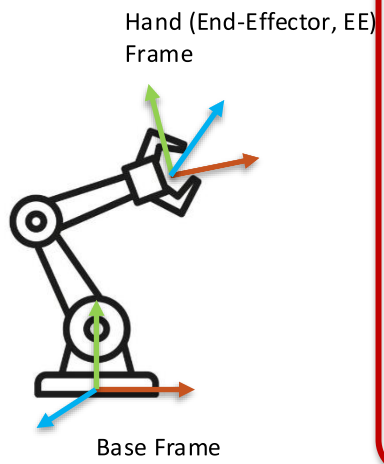
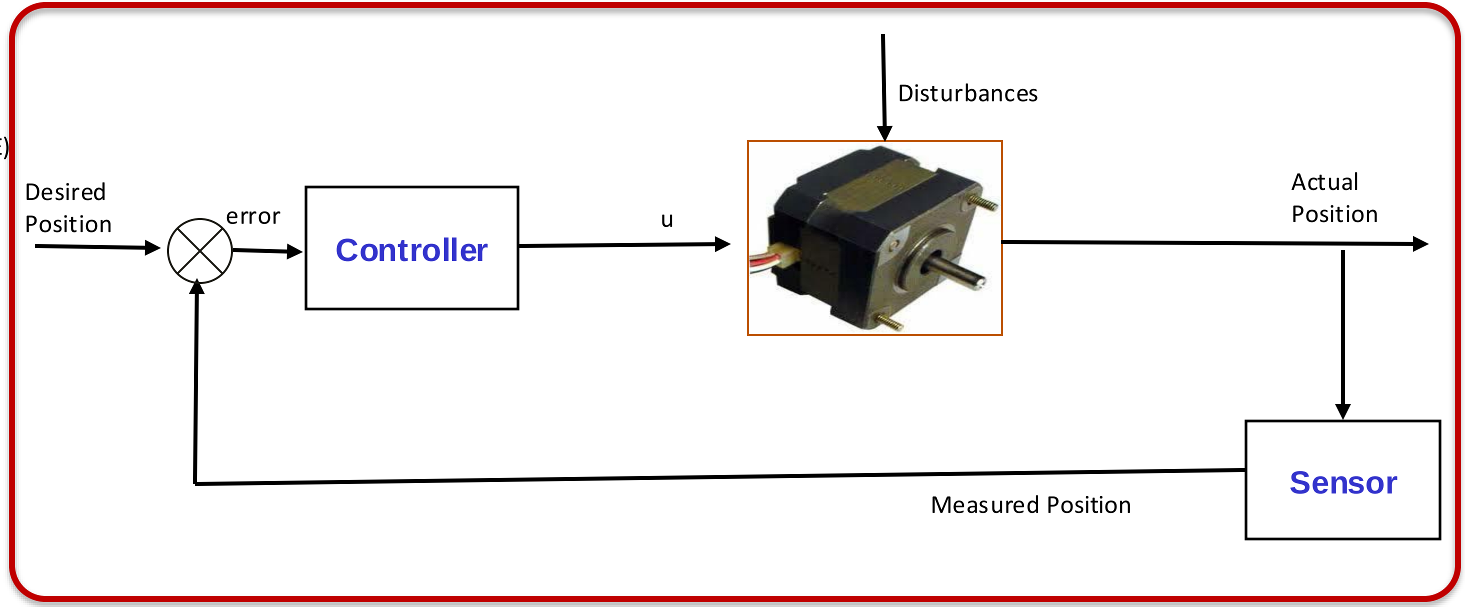
# We know how to control joints, but what is then the pose of the hand (end-effector)?




Given the joint values/angles for each joint, what is the pose of the hand? → Forward Kinematics
[Next: Given a desired pose of the hand, what are the necessary joint values/angles? → Inverse Kinematics

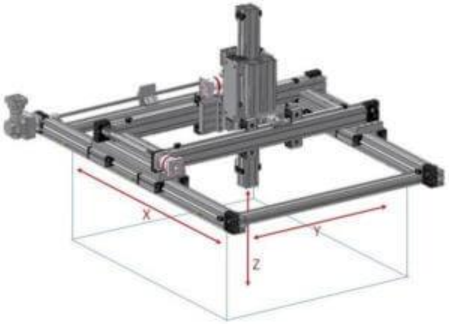
# What will you learn today?

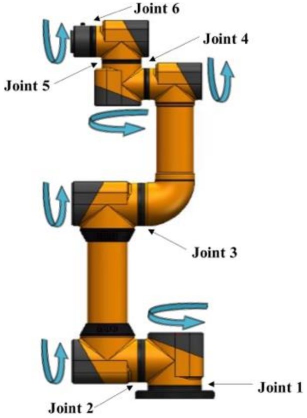
- Robot kinematics of motion
  - Configuration Space and Workspace
  - Forward Kinematics
- Jacobian
  - Definition and Meaning
  - Computation
  - Inversion
  - Singularities

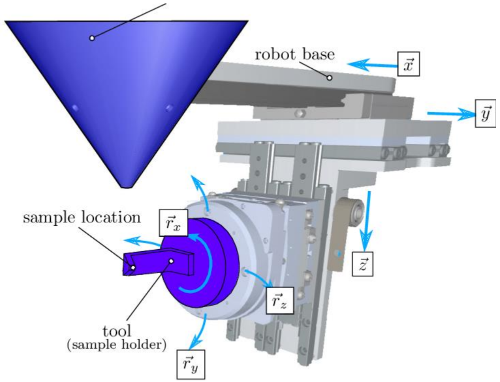
# A Note on Notation

- R → Revolute joint
- P → Prismatic joint

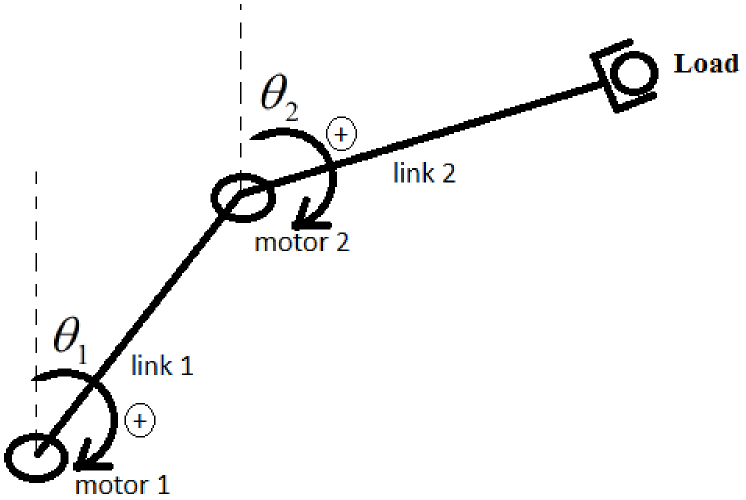
- Examples:
  - 3R
  - 3P (gantry)
  - 3P3R (CNC machine)
  - 6R



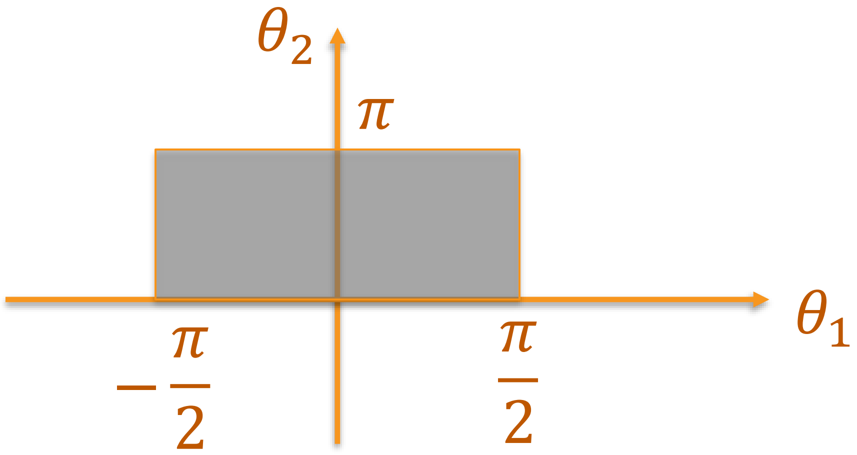



# Configuration Space (C-Space) of a Robot

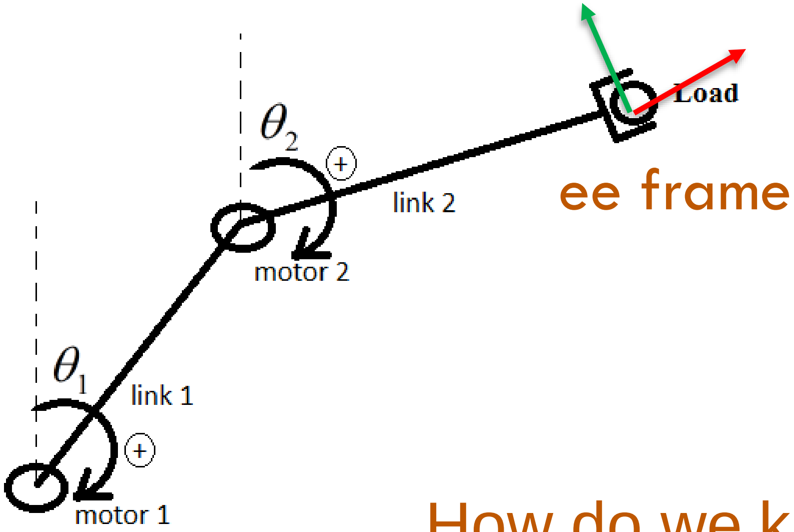



- Space of all possible joint values
- The dimensions of this space depends on the number of joints
- The “shape” of this space depends on the limits of the joints




[Lozano-Perez ’79]

# Workspace of a Robot




- We care about a special frame: the task frame
  - aka end-effector frame, gripper frame, tool frame…
- The space of all possible poses of the ee frame given the possible joint configurations of the robot is called the workspace of the robot
- The dimensions of the workspace are the DoF of a pose (6)

How do we know the pose of the ee frame from the joint configuration?

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# Exercise

- Given two coordinate frames {b}, and {s} and the homogeneous transformation between them $T_{sb} = \begin{pmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - What is the position of the origin of coordinates of {b} in the coordinate frame {s}?
  - If a point p has coordinates $p_b=(0,1,0)^T$ in frame {b}, what are its coordinates wrt. frame {s}?

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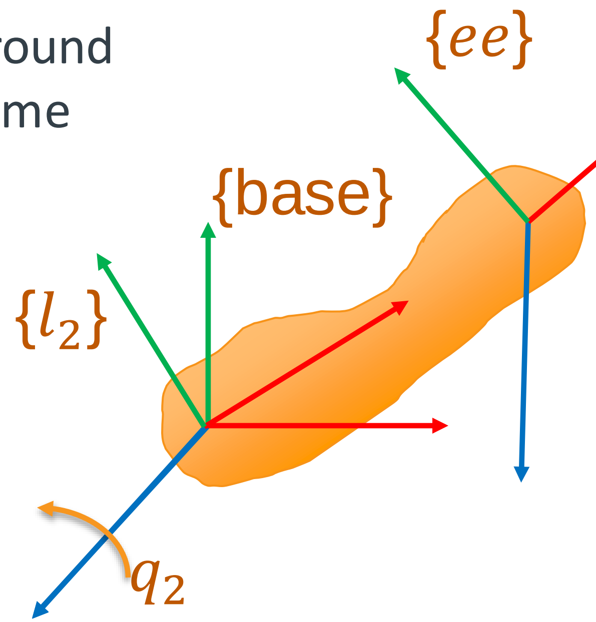
# Exercise

- A rigid body $l_2$ rotates an angle $q_2$ around the z axis with respect to another frame {base}. $l_2$ has a second frame {ee} attached rigidly to it

1) What is the homogeneous transformation matrix representing the pose of $\{l_2\}$ wrt. the $\{base\}$ frame?

2) What is the homogeneous transformation representing the pose of the frame {ee} wrt. the {base} frame?



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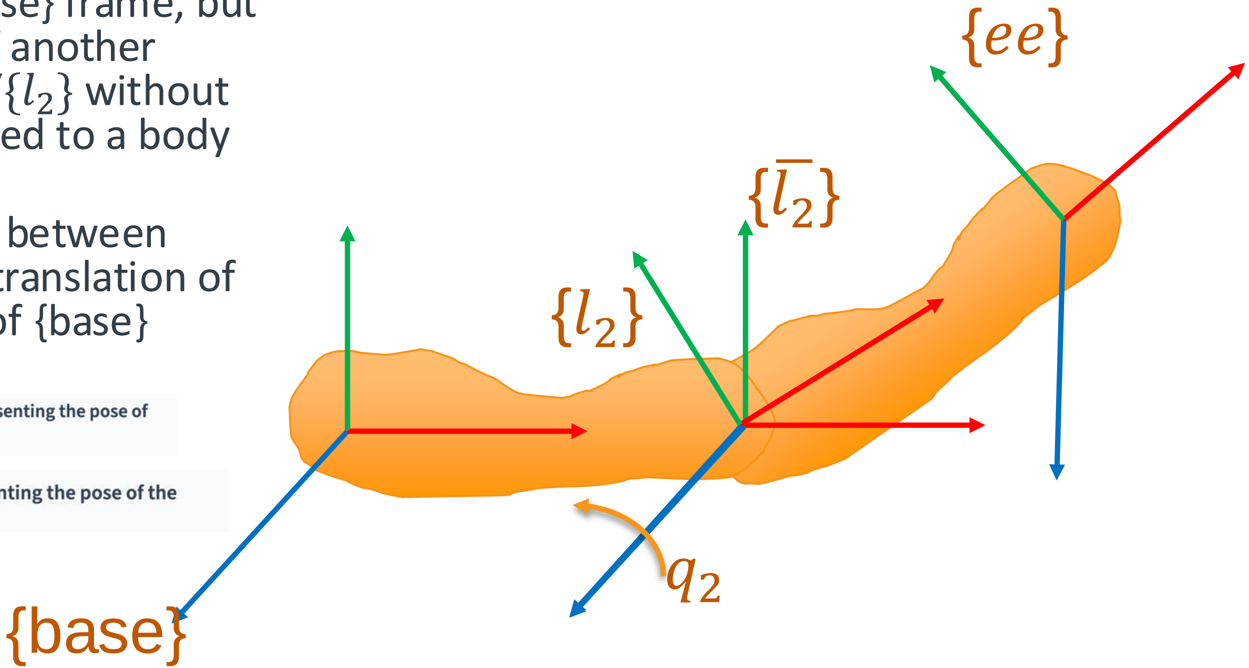
# Exercise

- Imagine the rotation is not around the z axis of the {base} frame, but around the z axis of another frame we call $\{\overline{l_2}\}$ ("$\{l_2\}$ without rotation") , connected to a body $l_1$
- The transformation between {base} and $\{\overline{l_2}\}$ is a translation of L1 along the x axis of {base}




3) What is the homogeneous transformation matrix representing the pose of $l_2$ wrt. the base frame?

4) What is the homogeneous transformation representing the pose of the frame {ee} wrt. the {base} frame?

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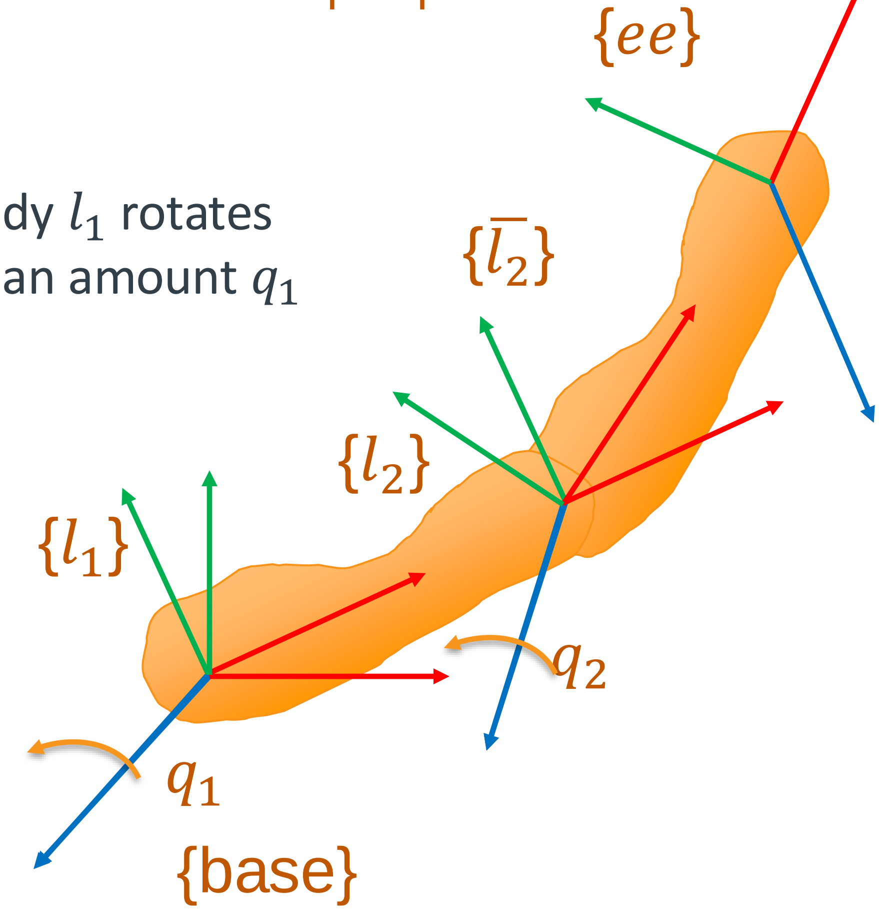
# Exercise

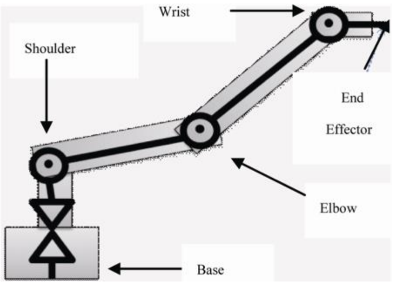
- Now imagine that the body $l_1$ rotates around the z axis as well an amount $q_1$

5) What is the homogeneous transformation representing the pose of the frame {ee} wrt. the {base} frame?



# Forward Kinematics

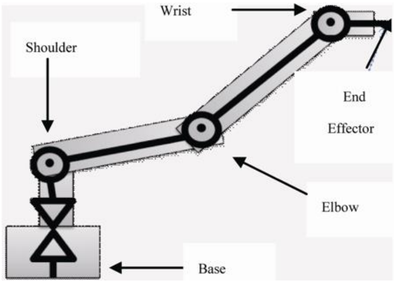
- Given the kinematic model of the robot
  - Links (distance between joints)
  - Joints (type, axis of rotation/translation)
- Given the joint configuration of the robot q = (q1, q2, …)




Forward Kinematics

pose of the end-effector frame

$$^{base}_{ee}T(q)$$



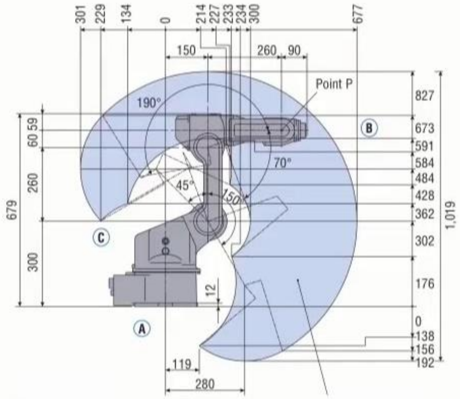
# Configuration Space <> Workspace

- Forward kinematics transforms from configuration space (joint values) to workspace (Cartesian poses)
- Next class: Inverse kinematics: from Cartesian poses to configuration space

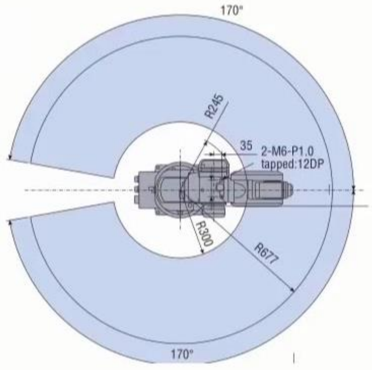
https://www.cs.unc.edu/~jeffi/c-space/robot.xhtml

# Reachable workspace, Dexterous workspace

- The set of all reachable end point positions

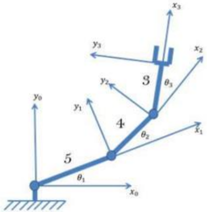


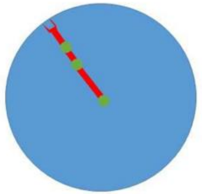

Yaskawa America, Inc., Motoman Robotics Division




Yaskawa America, Inc., Motoman Robotics Division

**Yaskawa Motoman SV3X**





(b)

(d)

- Reachable workspace: all 3D points around the robot I can reach with “some” orientation
- Dexterous workspace: all 3D points around the robot I can reach with “any” orientation

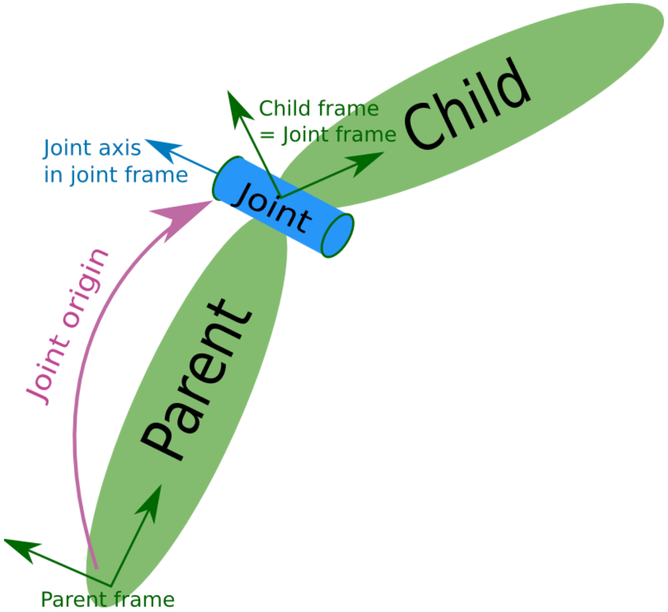
# URDF – Universal Robot Description Language

```xml
<robot name="My 2R robot">
  <link name="base_link">
    ...
  </link>

  <link name="link_1">
    ...
  </link>

  <link name="link_2">
    ...
  </link>

  <joint name="joint_1" type="revolute">
   ...
    <parent link="base_link"/>
    <child link="link_1"/>
  </joint>

  <joint name="joint_2" type="revolute">
    ...
    <parent link="link_1"/>
    <child link="link_2"/>
  </joint>
</robot>
```

```xml
<link name="link_1">
  <visual>
    <origin xyz="0 0 0" rpy="0 1.5708 0" />
    <geometry>
      <cylinder length="1" radius="0.1"/>
    </geometry>
    <material name="grey">
      <color rgba="0.6 0.6 .6 1"/>
    </material>
  </visual>
  <collision>
    <origin xyz="0 0 0" rpy="0 1.5708 0" />
    <geometry>
      <cylinder length="1.0" radius="0.1"/>
    </geometry>
  </collision>
  <inertial>
    <mass value="2.0"/>
    <inertia ixx="0.2" ixy="0.0" ixz="0.0" iyy="0.6" iyz="0.0" izz="0.6"/>
  </inertial>
</link>
```

# URDF – Universal Robot Description Language

```
<joint name="joint_2" type="revolute">
  <origin xyz="1 0 0" rpy="0 0 0"/>
  <axis xyz="0 1 0"/>
  <parent link="link_1" />
  <child link="link_2" />
</joint>
```

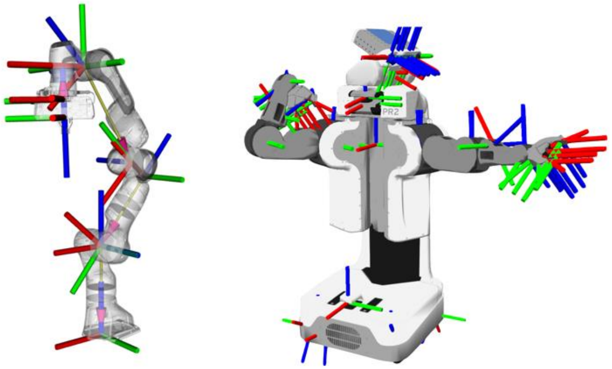


# ROS Robot State Publisher → Forward Kinematics

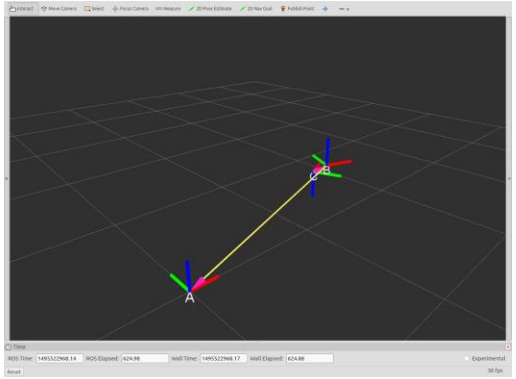
URDF description

+

joint configuration (q)







# Jacobian

- So far, we have defined the pose of the end-effector frame as a function of the joint configuration
- What happens if we have infinitesimally small changes to the joint configuration?

$$x_{ee} = f(q)$$

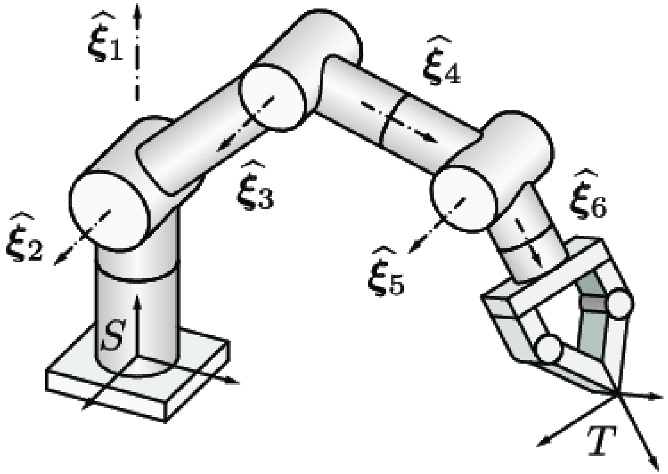
# VIP: (End-Effector) Jacobian

- The Jacobian relates infinitesimal changes in joint space and in the pose of the end-effector frame
- It also relates velocities in joint space and of the end-effector frame in Cartesian space

$$\dot{x}_{ee} = J(q)\dot{q}$$

$$\dot{x}_{ee} = \frac{\delta \bar{x}_{ee}}{\delta q}\dot{q}$$

# Understanding the Jacobian




$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}_{6\times 1} = \begin{bmatrix} J_{11} & J_{12} & . & . & . & J_{1n} \\ J_{21} & J_{22} & . & . & . & J_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & \cdots & . & . & . & . \\ J_{61} & \cdots & . & . & . & J_{6n} \end{bmatrix}_{6\times n} * \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

96.PNG

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial z}{\partial q_n} \end{bmatrix}_{3\times n}$$

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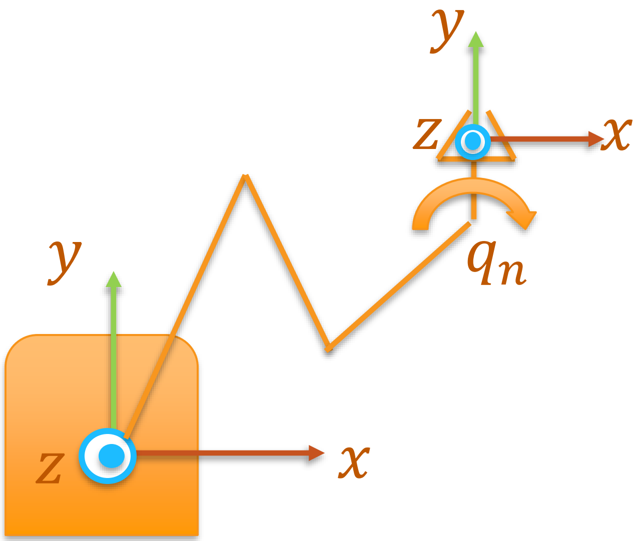
# Exercise

- Assuming a simple robot in the given configuration




a) What element of the Jacobian will be positive?

b) What is the value of the entry in the third row, n column?

# Uses of the Jacobian

- Translate between joint space velocities and Cartesian space velocity of the end-effector
  - Velocity control
- Translate between joint space torques and Cartesian space wrenches

$$\tau = J^T F$$

# Derivation of $\tau = J^T F$

- Energy dissipated by a system should be the same, no matter the way we represent the state of the system (joint or Cartesian space)

- Derivation
  1. Work is the application of force over a distance: $W = \int F^T v \, dt$
  2. Power is the rate at which work is performed: $P = \frac{W}{t}$
  3. Substituting in the equation for work into the equation for power gives:
  $$P = \frac{W}{t} = \frac{F^T d}{t} = F^T \frac{d}{t} = F^T v$$
  4. Because of energy equivalence, work is performed at the same rate regardless of the characterization of the system.
     - Rewriting this in terms of end-effector space gives:
       $P = F^T \dot{x}$ , where $F$ is the force applied by/to the hand, and $\dot{x}$ is the velocity of the hand
     - Rewriting the above in terms of joint space gives:
       $P = \tau_q^T \dot{q}$ , where $\tau_q$ is the torque applied by/to the joints, and $\dot{q}$ is the angular velocity of the joints
  5. Setting these two equations equal to each other and substituting in our equation for the Jacobian gives:
  $$\tau^T \dot{q} = F^T \dot{x}$$
  $$\tau^T \dot{q} = F^T J(q) \dot{q}$$
  $$\tau^T = F^T J(q) \qquad (A \cdot b)^T = b^T A^T$$
  $$\tau = J^T(q) F$$

# Inverting the Jacobian

- Problem! Often, the Jacobian is not square
  - 7 DoF robot arm
- We will use the pseudo-inverse $J^+$ obtained from the Moore-Penrose inversion:

$$\boxed{J^+ = J^T(J \cdot J^T)^{-1}}$$

- $J \in R^{m \times n}$ then $J^+ \in R^{n \times m}$

# Kinematic Singularities

- Configuration in which the robot loses the ability to move in one of the Cartesian dimensions
- What does that mean for the Jacobian?

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial z}{\partial q_n} \end{bmatrix}_{3\times n}$$

Same!

Linearly dependent columns

Matrix is not full rank

# Reminder

- Given a matrix $A$, the rank of $A$ is
  - The rank of a matrix is equal to the number of linearly independent rows (or columns) in it.
    - Rows or columns? Whatever the smallest!
  - Order (~dimensions) of the highest ordered non-zero minor
    - A minor is the determinant of some smaller square matrix generated from A by removing one or more of its rows and columns
- For a matrix $A \in \mathbb{R}^{nx,}$
  - If $A$ is square (n=m):
    - Check if $|A| = 0$
      - Yes? Find the highest ordered non-zero minor
      - No? → Rank of $A$ is n
  - If $A$ is NOT square (n!=m):
    - Find the highest ordered non-zero minor

# Kinematic Singularities

- Configuration in which the robot loses the ability to move in one of the Cartesian dimensions
- What does that mean for the Jacobian?

96.PNG

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} & .. & .. & .. & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} & .. & .. & .. & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} & .. & .. & .. & \frac{\partial z}{\partial q_n} \end{bmatrix}_{3\times n}$$

Same!

Linearly dependent columns

Determinant =0

Inverse or pseudo-inverse will get close to infinite!

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj } A$$

# How does a singularity look like?

- Trying to follow a smooth trajectory in Cartesian space with the end-effector is not possible:
  - Either robot “stops”

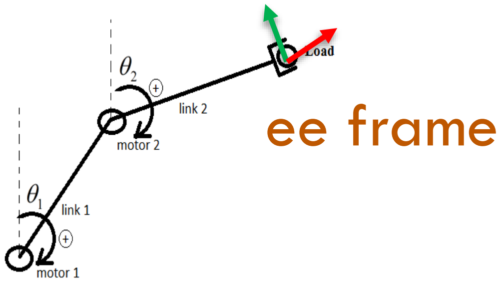
  or

  - Robot moves one/several joint(s) at very fast speed

$$\dot{q} = J^{(-1\ or\ +)}\dot{x}_{ee}$$

# Summary

- Surface barely scratched
- Starts with how to represent the location of rigid bodies (EE) in space as a function of the joint values
- This enables Forward Kinematics
- Finding the influence coefficients (Jacobian) informs us how joint motions impact EE motion as a function of the current joint angles
  - This is a linear (or linearize) system




- Next class
  - How do we go from EE pose to joint values? Inverse Kinematics!