Chapter 2

Gaussian Elimination and LU Factorization

2.1 Gaussian Elimination

Recall that Gaussian Elimination is the process used to make a set of linear systems and transform it into an upper triangular system, which is easier to solve.

Assume that we are given the set of linear systems

$$
\begin{array}{ccccccccc}
\alpha_{11}\chi_1 & + & \alpha_{12}\chi_2 & + & \cdots & + & \alpha_{1n}\chi_n & = & y_1 \\
\alpha_{21}\chi_1 & + & \alpha_{22}\chi_2 & + & \cdots & + & \alpha_{2n}\chi_n & = & y_2 \\
\alpha_{31}\chi_1 & + & \alpha_{32}\chi_2 & + & \cdots & + & \alpha_{3n}\chi_n & = & y_3 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n1}\chi_1 & + & \alpha_{n2}\chi_2 & + & \cdots & + & \alpha_{nn}\chi_n & = & y_n
\end{array}
$$

We start by computing multiplier $\lambda_{21} = \alpha_{21}/\alpha_{11}$ which multiplies the first row, after which the result is subtracted from the second row:

$$
\begin{array}{c|ccccccccc|}
\cline{2-10}
 & \alpha_{11}\chi_1 & + & \alpha_{12}\chi_2 & + & \cdots & + & \alpha_{1n}\chi_n & = & y_1 \\
\cline{2-10}
\lambda_{21} & \alpha_{21}\chi_1 & + & \alpha_{22}\chi_2 & + & \cdots & + & \alpha_{2n}\chi_n & = & y_2 \\
\cline{2-10}
\multicolumn{1}{c}{} & \alpha_{31}\chi_1 & + & \alpha_{32}\chi_2 & + & \cdots & + & \alpha_{3n}\chi_n & = & \multicolumn{1}{c}{y_3} \\
\multicolumn{1}{c}{} & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} \\
\multicolumn{1}{c}{} & \alpha_{n1}\chi_1 & + & \alpha_{n2}\chi_2 & + & \cdots & + & \alpha_{nn}\chi_n & = & \multicolumn{1}{c}{y_n}
\end{array}
$$

Resulting in

$$
\begin{array}{c|ccccccccc|}
\cline{2-10}
 & \alpha_{11}\chi_1 & + & \alpha_{12}\chi_2 & + & \cdots & + & \alpha_{1n}\chi_n & = & y_1 \\
\cline{2-10}
\lambda_{21} & 0 & + & (\alpha_{22} - \lambda_{21}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{2n} - \lambda_{21}\alpha_{1n})\chi_n & = & y_2 - \lambda_{21}y_1 \\
\cline{2-10}
\multicolumn{1}{c}{} & \alpha_{31}\chi_1 & + & \alpha_{32}\chi_2 & + & \cdots & + & \alpha_{3n}\chi_n & = & \multicolumn{1}{c}{y_3} \\
\multicolumn{1}{c}{} & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} \\
\multicolumn{1}{c}{} & \alpha_{n1}\chi_1 & + & \alpha_{n2}\chi_2 & + & \cdots & + & \alpha_{nn}\chi_n & = & \multicolumn{1}{c}{y_n}
\end{array}
$$

Next, the multiplier to eliminate term $\alpha_{31}\chi_1$ is computed: $\lambda_{31} = \alpha_{31}/\alpha_{11}$:

$$
\begin{array}{c|ccccccccc|}
\cline{2-10}
 & \alpha_{11}\chi_1 & + & \alpha_{12}\chi_2 & + & \cdots & + & \alpha_{1n}\chi_n & = & y_1 \\
\cline{2-10}
\lambda_{21} & 0 & + & (\alpha_{22} - \lambda_{21}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{2n} - \lambda_{21}\alpha_{1n})\chi_n & = & y_2 - \lambda_{21}y_1 \\
\cline{2-10}
\lambda_{31} & \alpha_{31}\chi_1 & + & \alpha_{32}\chi_2 & + & \cdots & + & \alpha_{3n}\chi_n & = & y_3 \\
\cline{2-10}
\multicolumn{1}{c}{} & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} \\
\multicolumn{1}{c}{} & \alpha_{n1}\chi_1 & + & \alpha_{n2}\chi_2 & + & \cdots & + & \alpha_{nn}\chi_n & = & \multicolumn{1}{c}{y_n}
\end{array}
$$

Resulting in

$$
\begin{array}{c|ccccccccc|}
\cline{2-10}
 & \alpha_{11}\chi_1 & + & \alpha_{12}\chi_2 & + & \cdots & + & \alpha_{1n}\chi_n & = & y_1 \\
\cline{2-10}
\lambda_{21} & 0 & + & (\alpha_{22} - \lambda_{21}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{2n} - \lambda_{21}\alpha_{1n})\chi_n & = & y_2 - \lambda_{21}y_1 \\
\cline{2-10}
\lambda_{31} & 0 & + & (\alpha_{32} - \lambda_{31}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{3n} - \lambda_{31}\alpha_{1n})\chi_n & = & y_3 - \lambda_{31}y_1 \\
\cline{2-10}
\multicolumn{1}{c}{} & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} \\
\multicolumn{1}{c}{} & \alpha_{n1}\chi_1 & + & \alpha_{n2}\chi_2 & + & \cdots & + & \alpha_{nn}\chi_n & = & \multicolumn{1}{c}{y_n}
\end{array}
$$

Repeating this for all rows, we get the equation:

$$
\begin{array}{ccc|ccccccc}
 & \alpha_{11}\chi_1 & + & \alpha_{12}\chi_2 & + & \cdots & + & \alpha_{1n}\chi_n & = & y_1 \\
\cline{2-10}
\lambda_{21} & 0 & + & (\alpha_{22} - \lambda_{21}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{2n} - \lambda_{21}\alpha_{1n})\chi_n & = & y_2 - \lambda_{21}y_1 \\
\lambda_{31} & 0 & + & (\alpha_{32} - \lambda_{31}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{3n} - \lambda_{31}\alpha_{1n})\chi_n & = & y_3 - \lambda_{31}y_1 \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\lambda_{n1} & 0 & + & (\alpha_{n2} - \lambda_{n1}\alpha_{12})\chi_2 & + & \cdots & + & (\alpha_{nn} - \lambda_{n1}\alpha_{1n})\chi_n & = & y_n - \lambda_{n1}y_1
\end{array}
$$

After this, the process continues with the parts of the equations in the lower-right quadrant indicated by the lines.

Let us repeat the same, but now use matrix notation, and ignore the right-hand-side:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \cdots & \alpha_{3n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

We start by computing multiplier $\lambda_{21} = \alpha_{21}/\alpha_{11}$ which multiplies the first row, after which the result is subtracted from the second row:

$$\begin{pmatrix} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \lambda_{21} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ & \alpha_{31} & \alpha_{32} & \cdots & \alpha_{3n} \\ & \vdots & \vdots & & \vdots \\ & \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

Resulting in

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \lambda_{21} & (\alpha_{22} - \lambda_{21}\alpha_{12}) & \cdots & (\alpha_{2n} - \lambda_{21}\alpha_{1n}) \\ \hline \alpha_{31} & \alpha_{32} & \cdots & \alpha_{3n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

Here, we overwrite the 0 with the multiplier!

Next, the multiplier to eliminate term α_{31} is computed: $\lambda_{31} = \alpha_{31}/\alpha_{11}$:

$$\begin{pmatrix} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ & \lambda_{21} & (\alpha_{22} - \lambda_{21}\alpha_{12}) & \cdots & (\alpha_{2n} - \lambda_{21}\alpha_{1n}) \\ \lambda_{31} & \alpha_{31} & \alpha_{32} & \cdots & \alpha_{3n} \\ & \vdots & \vdots & & \vdots \\ & \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

Resulting in

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \lambda_{21} & (\alpha_{22} - \lambda_{21}\alpha_{12}) & \cdots & (\alpha_{2n} - \lambda_{21}\alpha_{1n}) \\ \lambda_{31} & (\alpha_{32} - \lambda_{31}\alpha_{12}) & \cdots & (\alpha_{3n} - \lambda_{31}\alpha_{1n}) \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

Repeating this for all rows, we get the result:

$$\left(\begin{array}{c|ccc} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \hline \lambda_{21} & (\alpha_{22} - \lambda_{21}\alpha_{12}) & \cdots & (\alpha_{2n} - \lambda_{21}\alpha_{1n}) \\ \lambda_{31} & (\alpha_{32} - \lambda_{31}\alpha_{12}) & \cdots & (\alpha_{3n} - \lambda_{31}\alpha_{1n}) \\ \vdots & \vdots & & \vdots \\ \lambda_{n1} & (\alpha_{n2} - \lambda_{n1}\alpha_{12}) & \cdots & (\alpha_{nn} - \lambda_{n1}\alpha_{1n}) \end{array}\right)$$

Again, the process continues recursively for the lower-right quadrant.

Finally, let us use blocked notation to perform an LU factorization: Let

$$A = \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right), \quad L = \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right), \text{and} \quad U = \left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right).$$

Then

$$A = \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right) \left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right) = \left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline v_{11}l_{21} & l_{21}u_{12}^T + L_{22}U_{22} \end{array}\right)$$

Thus

$$\begin{array}{rcl} \alpha_{11} & = & v_{11} \\ a_{12}^T & = & u_{12}^T \\ a_{21} & = & v_{11}l_{21} \\ A_{22} & = & l_{21}u_{12}^T + L_{22}U_{22} \end{array}$$

Rewriting, we find that:

$$\begin{array}{rcl} v_{11} & = & \alpha_{11} \\ u_{12}^T & = & a_{12}^T \end{array}$$

Thus, the first row of the matrix is left alone!

Furthermore

$$l_{21} = \begin{pmatrix} \lambda_{21} \\ \lambda_{31} \\ \vdots \\ \lambda_{21} \end{pmatrix} = a_{21}/\alpha_{11} = \begin{pmatrix} \alpha_{21}/\alpha_{11} \\ \alpha_{31}/\alpha_{11} \\ \vdots \\ \alpha_{21}/\alpha_{11} \end{pmatrix}$$

Notice the similarity with the multipliers of Gaussian elimination!

And finally

$$\begin{aligned} A_{22} - l_{21}u_{12}^T &= A_{22} - \begin{pmatrix} \lambda_{21} \\ \lambda_{31} \\ \vdots \\ \lambda_{n1} \end{pmatrix} \left(\alpha_{12} \mid \alpha_{13} \mid \cdots \mid \alpha_{1n} \right) \\ &= \begin{pmatrix} (\alpha_{22} - \lambda_{21}\alpha_{12}) & \cdots & (\alpha_{2n} - \lambda_{21}\alpha_{1n}) \\ (\alpha_{32} - \lambda_{31}\alpha_{12}) & \cdots & (\alpha_{3n} - \lambda_{31}\alpha_{1n}) \\ \vdots & & \vdots \\ (\alpha_{n2} - \lambda_{n1}\alpha_{12}) & \cdots & (\alpha_{nn} - \lambda_{n1}\alpha_{1n}) \end{pmatrix} = L_{22}U_{22} \end{aligned}$$

Notice the similarity with Gaussian elimination!

This last equation shows that the factorization can be completed by computing the LU factorization

$$\begin{pmatrix} (\alpha_{22} - \lambda_{21}\alpha_{12}) & \cdots & (\alpha_{2n} - \lambda_{21}\alpha_{1n}) \\ (\alpha_{32} - \lambda_{31}\alpha_{12}) & \cdots & (\alpha_{3n} - \lambda_{31}\alpha_{1n}) \\ \vdots & & \vdots \\ (\alpha_{n2} - \lambda_{n1}\alpha_{12}) & \cdots & (\alpha_{nn} - \lambda_{n1}\alpha_{1n}) \end{pmatrix} = L_{22}U_{22}$$

Again, just like for Gaussian elimination!!