CS383C/CAM383C/MATH383E
Fall 2000
Midterm Exam

Note: You may use any notes or book to answer these questions. However, you may not under any circumstances discuss this exam with anyone else.

Question I: Consider the following algorithm for LU factorization with partial pivoting:

$$
\begin{array}{l}
\text{Partition } A = \left(\begin{array}{c||c} A_{TL} & A_{TR} \\ \hline\hline A_{BL} & A_{BR} \end{array} \right) \\
\qquad \text{where } A_{TL} \text{ is } 0 \times 0 \\
\text{do until } A_{BR} \text{ is } 0 \times 0 \\
\\
\quad \text{Determine permutation } P_i \\
\qquad \text{(here } i \text{ is such that } A_{TL} \text{ is } (i-1) \times (i-1)\text{)} \\
\quad \text{Update } \left(\begin{array}{c||c} A_{TL} & A_{TR} \\ \hline\hline A_{BL} & A_{BR} \end{array} \right) \leftarrow P_i^T \left(\begin{array}{c||c} A_{TL} & A_{TR} \\ \hline\hline A_{BL} & A_{BR} \end{array} \right) \\
\quad \text{Partition} \\
\qquad \left(\begin{array}{c||c} A_{TL} & A_{TR} \\ \hline\hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c||c|c} L \backslash U_{00} & a_{01} & A_{02} \\ \hline\hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) \\
\\
\quad a_{10} \leftarrow l_{10} = U_{00}^{-1} a_{10} \\
\quad a_{01} \leftarrow u_{01} = L_{00}^{-1} a_{01} \\
\quad \alpha_{11} \leftarrow \upsilon_{11} = \alpha_{11} - l_{10}^T u_{01} \\
\\
\quad \text{Continue with} \\
\qquad \left(\begin{array}{c||c} A_{TL} & A_{TR} \\ \hline\hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c|c||c} L \backslash U_{00} & \mathbf{u_{01}} & A_{02} \\ \hline \mathbf{l_{10}^T} & \upsilon_{\mathbf{11}} & a_{12}^T \\ \hline\hline A_{20} & a_{21} & A_{22} \end{array} \right) \\
\text{enddo}
\end{array}
$$

Here P_i is chosen so that when P_i^T is applied from the left to the matrix the top row of $\left(\begin{array}{c||c} A_{BL} & A_{BR} \end{array} \right)$ is swapped with the appropriage row of that matrix, as determined from the first column of A_{BR}.

1. **(The purpose of the game is to show that the matrix will be overwritten by the LU factorization of $(P_n^T P_{(n-1)}^T \cdots P_1^T)A$. and that at least in theory the result is the LU factorization without pivoting a matrix that has the rows permuted up front: $R_n^T A = LU$.)**

   Define $R_i^T = P_i^T P_{i-1}^T \cdots P_1^T$. Show that R_i is a permutation matrix.

2. **(So let us suppose that we have proceeded through i steps of LU factorization, and that the current result is equivalent to permuting the matrix with the permutation matrices encountered so far $(P_1, \cdots, P_i)$.)**

Partition
$$R_i^T A = \left(\begin{array}{c||c} A_{TL}^{(i)} & A_{TR}^{(i)} \\ \hline\hline A_{BL}^{(i)} & A_{BR}^{(i)} \end{array}\right)$$
where A is the original matrix. Assume that after i steps the matrix has been overwritten by

(1)
$$A_i = \left(\begin{array}{c||c} L\backslash U_{TL}^{(i)} & U_{TR}^{(i)} \\ \hline\hline L_{BL}^{(i)} & A_{BR}^{(i)} - L_{BL}^{(i)} U_{TR}^{(i)} \end{array}\right)$$

where

- $A_{TL}^{(i)} = L_{TL}^{(i)} U_{TL}^{(i)}$ (the LU factorization of $A_{TL}^{(i)}$),
- $L_{BL}^{(i)} = A_{BL}^{(i)} U_{TL}^{(i)\,-1}$
- $U_{TR}^{(i)} = L_{TL}^{(i)\,-1} A_{TR}^{(i)}$

Show that $A_{BR}^{(i)} - L_{BL}^{(i)} U_{TR}^{(i)} = A_{BR}^{(i)} - A_{BL}^{(i)} A_{TL}^{(i)\,-1} A_{TR}^{(i)}$. This is sometimes called the Shur's complement of the blocked matrix.

(This little result about the Shur's complement is a small digression on the subject: If one were to solve the system of linear equations
$$\left(\begin{array}{c||c} A_{TL}^{(i)} & A_{TR}^{(i)} \\ \hline\hline A_{BL}^{(i)} & A_{BR}^{(i)} \end{array}\right) \left(\begin{array}{c} x_T \\ \hline\hline x_B \end{array}\right) = \left(\begin{array}{c} b_T \\ \hline\hline b_B \end{array}\right)$$
then one could solve this by doing a block equivalent of Gaussian elimination:
$$\left(\begin{array}{c||c} A_{TL}^{(i)} & A_{TR}^{(i)} \\ \hline\hline 0 & A_{BR}^{(i)} - A_{BL}^{(i)} A_{TL}^{(i)\,-1} A_{TR}^{(i)} \end{array}\right) \left(\begin{array}{c} x_T \\ \hline\hline x_B \end{array}\right) = \left(\begin{array}{c} b_T \\ \hline\hline b_B - A_{BL}^{(i)} A_{TL}^{(i)\,-1} b_T \end{array}\right)$$
If one solves this for x_B, one can then substitute that back into the top (blocked) row of the equation and solve for x_T. In this method $A_{BR}^{(i)} - A_{BL}^{(i)} A_{TL}^{(i)\,-1} A_{TR}^{(i)}$ is referred to as the Schur complement.)

3. Show that $P_n^T \cdots P_i^T$ only permute rows of $\left(A_{BL} \,\|\, A_{BR} \right)$ (an informal argument is acceptable).

4. **Now we are ready to examine what changes the computation in the $(i+1)$st step of the factorization will make to the current contents of the matrix.**

Partition *the permuted original matrix*

$$
(2) \qquad R_{i+1}^T A = P_{i+1}^T R_i^T A = \left(\begin{array}{c||c|c} A_{00}^{(i)} & a_{01}^{(i)} & A_{02}^{(i)} \\ \hline\hline a_{10}^{(i)\,T} & \alpha_{11}^{(i)} & a_{12}^{(i)\,T} \\ \hline A_{20}^{(i)} & a_{21}^{(i)} & A_{22}^{(i)} \end{array}\right)
$$

and partition the matrix that holds the current contents of the matrix, $\hat{A}_i$,

$$
(3) \qquad P_{i+1}^T \hat{A}_i = \left(\begin{array}{c||c|c} L\backslash U_{00}^{(i)} & u_{01}^{(i)} & U_{02}^{(i)} \\ \hline\hline l_{10}^{(i)\,T} & \hat{\alpha}_{11}^{(i)} & \hat{a}_{12}^{(i)\,T} \\ \hline L_{20}^{(i)} & \hat{a}_{21}^{(i)} & \hat{A}_{22}^{(i)} \end{array}\right)
$$

Notice that the top-left block of these partitionings are $i \times i$. *Establish the following relation between the current contents in $P_{i+1}^T A_i$ and the original matrix:*

- $\hat{\alpha}_{11}^{(i)} = \alpha_{11}^{(i)} - l_{10}^{(i)\,T} u_{01}^{(i)}$
- $\hat{a}_{12}^{(i)\,T} = a_{12}^{(i)\,T} - l_{10}^{(i)\,T} U_{02}^{(i)}$
- $\hat{a}_{21}^{(i)} = a_{21}^{(i)} - L_{20}^{(i)} u_{01}^{(i)}$
- $\hat{A}_{22}^{(i)} = A_{22}^{(i)} - L_{20}^{(i)} U_{02}^{(i)}$

5. **OK, now let us examine what we would like to see in A_{i+1}. From (4) we see that if we partition**

$$
R_{i+1}^T A = \left(\begin{array}{c||c} A_{TL}^{(i+1)} & A_{TR}^{(i+1)} \\ \hline\hline A_{BL}^{(i+1)} & A_{BR}^{(i+1)} \end{array}\right)
$$

then we like for A_{i+1} to hold

$$
A_{i+1} = \left(\begin{array}{c||c} L\backslash U_{TL}^{(i+1)} & U_{TR}^{(i+1)} \\ \hline\hline L_{BL}^{(i)} & A_{BR}^{(i+1)} - L_{BL}^{(i+1)} U_{TR}^{(i+1)} \end{array}\right)
$$

where

- $A_{TL}^{(i+1)} = L_{TL}^{(i+1)} U_{TL}^{(i+1)}$ **(the LU factorization of $A_{TL}^{(i+1)}$),**
- $L_{BL}^{(i+1)} = A_{BL}^{(i+1)} U_{TL}^{(i+1)\,-1}$
- $U_{TR}^{(i+1)} = L_{TL}^{(i+1)\,-1} A_{TR}^{(i+1)}$

By carefully relating $A_{TL}^{(i+1)}$ etc. to the partitionings in (3) and (??) you should be able to establish the following:

Show

$$
\left(\begin{array}{c||c|c} L\backslash U_{00}^{(i)} & u_{01}^{(i)} & U_{02}^{(i)} \\ \hline\hline l_{10}^{(i)\,T} & v_{11}^{(i)} & u_{12}^{(i)\,T} \\ \hline L_{20}^{(i)} & l_{21}^{(i)} & \hat{A}_{22}^{(i)} - l_{21}^{(i)} u_{12}^{(i)\,T} \end{array}\right) = \left(\begin{array}{c||c} \hat{A}_{TL}^{(i+1)} & \hat{A}_{TR}^{(i+1)} \\ \hline\hline \hat{A}_{BL}^{(i+1)} & \hat{A}_{BR}^{(i+1)} \end{array}\right)
$$

where $v_{11}^{(i)} = \hat{\alpha}_{11}^{(i)}$, $l_{21}^{(i)} = \hat{a}_{21}^{(i)}/\hat{\alpha}_{11}^{(i)}$, $u_{12}^{(i)\,T} = \hat{a}_{12}^{(i)\,T}$, and

$$R_{i+1}^T A = \left(\begin{array}{c||c} A_{TL}^{(i+1)} & A_{TR}^{(i+1)} \\ \hline A_{BL}^{(i+1)} & A_{BR}^{(i+1)} \end{array} \right)$$

Notice that what this says is that the update in the algorithm does exactly what is needed to maintain the contents of the matrix from one iteration to the next.

6. Argue that $A_0 = A$ and A_n holds the LU factorization of $R_n^T A$.

Question II: Question 2.19 in the book.

Question III: Let $A \in R^{n\times n}$ and $x, y \in R^n$.

1. Show that $\mathrm{fl}(A + xy^T) = (A + \delta A) + (x + \delta x)(y + \delta y)^T$ (a rank-1 update). Put bounds on δA, δx, and δy. Mutter something about the stability of this operation.
2. Show that in general one cannot find δx and δy such that $\mathrm{fl}(xy^T) = (x + \delta x)(y + \delta y)^T$. What does this say about the stability of the outer-product?

Question IV Consider an upper-Hessenberg matrix A:

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1(n-2)} & \alpha_{1(n-1)} & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2(n-2)} & \alpha_{2(n-1)} & \alpha_{2n} \\ 0 & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3(n-2)} & \alpha_{3(n-1)} & \alpha_{3n} \\ 0 & 0 & \alpha_{43} & \cdots & \alpha_{4(n-2)} & \alpha_{4(n-1)} & \alpha_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{(n-1)(n-2)} & \alpha_{(n-1)(n-1)} & \alpha_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{n(n-1)} & \alpha_{nn} \end{pmatrix}$$

1. Design an efficient algorithm for computing the LU factorization of this matrix.
2. Compute the operation count (number of floating point operations) for your algorithm.
3. Extend you results for 1. and 2. to the case where the matrix has b subdiagonals:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1(n-2)} & A_{1(n-1)} & A_{1n} \\ U_{21} & A_{22} & A_{23} & \cdots & A_{2(n-2)} & A_{2(n-1)} & A_{2n} \\ 0 & U_{32} & A_{33} & \cdots & A_{3(n-2)} & A_{3(n-1)} & A_{3n} \\ 0 & 0 & U_{43} & \cdots & A_{4(n-2)} & A_{4(n-1)} & A_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{(n-1)(n-2)} & A_{(n-1)(n-1)} & A_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & U_{n(n-1)} & A_{nn} \end{pmatrix}$$

   where A_{ii} is $b \times b$ and $U_{(i+1)i}$ is upper triangular.
4. Mutter a few words about how to formulate your result in 3. as a blocked algorithm (if you haven't done so already).