Some Definitions about Sets

Definition: Two sets are **equal** if they contain the same elements. I.e., sets A and B are equal if
$\forall x[x \in A \leftrightarrow x \in B]$.
Notation: $A = B$.

Recall: Sets are unordered and we do not distinguish between repeated elements. So:
$\{1, 1, 1\} = \{1\}$, and $\{a, b, c\} = \{b, a, c\}$.

Definition: A set A is a **subset** of set B, denoted $A \subseteq B$, if every element x of A is also an element of B. That is, $A \subseteq B$ if $\forall x(x \in A \rightarrow x \in B)$.

Example: $\mathbb{Z} \subseteq \mathbb{R}$.
$\{1, 2\} \subseteq \{1, 2, 3, 4\}$

Notation: If set A is not a subset of B, we write $A \not\subseteq B$.
Example: $\{1, 2\} \not\subseteq \{1, 3\}$

More on Subsets

Definition: If $A \subseteq B$ and $A \neq B$, we say that A is a **proper subset** of B. Notation: $A \subset B$.

Example: $\{1, 2\} \subset \{3, 2, 1\}$

Note: One way of proving that $A = B$ is by proving that $A \subseteq B$ and $B \subseteq A$.

Definition: The set that contains no elements is the **empty set**, and is denoted by $\emptyset$ or less commonly, $\{\}$.

Theorem: Prove that for every set S, $\emptyset \subseteq S$.

Creating New Sets - Set Operations

Binary Operations

The **union** of two sets A and B is denoted $A \cup B$ and is defined as $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

The **intersection** of two sets A and B is denoted $A \cap B$ and is defined as $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Two sets are **disjoint** if they have no elements in common, that is, A and B are disjoint if $A \cap B = \emptyset$.

Some Proofs about Sets

Theorem: For any sets A and B, $A \cap B \subseteq A$.

Theorem: For any sets A and B, $A \subseteq A \cup B$.

Venn Diagrams and More Set Operations

Sets and relationships between sets are represented visually using **Venn Diagrams**, which were introduced by mathematician **John Venn**.

Venn Diagrams

- The **universal set** U, which contains all objects under consideration, is represented by a rectangle.
- Circles and other shapes are used inside the rectangle to represent sets (which are subsets of U).
- Elements of U (or other sets) are represented by dots.

Example: Venn diagrams that represent $A \subseteq B$, $A \cup B$, and $A \cap B$.

Definition: For sets A and B, the **difference** or **set difference** of A and B, denoted $A - B$, is given by
$A - B = \{x | x \in A \text{ and } x \notin B\}$.

Example: Venn diagram for $A - B$.

Example: $A = \{a, b, d, f\}$, $B = \{b, f, h, i, j\}$. What is $A - B$? What is $B - A$?

More Set Operations

Definition: Let U be the universal set. The **complement** of a set A, denoted $\overline{A}$ or A^c, is $U - A$, or equivalently,
$A^c = \{x | x \notin A\}$.

Example: Venn diagram for A^c.

Example: Let the universe $U = \mathbb{Z}^+$, and let $A = \{x | x \geq 10\}$. What is A^c?

Set Identities

Identity	Name
$A \cup \emptyset = A$ $A \cap U = A$	identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	domination laws
$A \cup A = A$ $A \cap A = A$	idempotence
$(A^c)^c = A$	double complement law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	absorption laws
$A \cup A^c = U$ $A \cap A^c = \emptyset$	complement laws

Proving Set Equality

Set equality proofs are usually a special type of linear equivalence proof that uses definitions about set operations and logical identities.

Example: Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Let $x \in \overline{A \cap B}$
$\equiv x \notin A \cap B$ {Definition of set complement}
$\equiv \neg(x \in A \cap B)$ {Meaning of $\notin$ abbreviation}
$\equiv \neg(x \in A \wedge x \in B)$ {Definition of set intersection}
$\equiv x \notin A \vee x \notin B$ {De Morgan, $\notin$ abbreviation}
$\equiv x \in \overline{A} \vee x \in \overline{B}$ {Definition of set complement (twice)}
$\equiv x \in \overline{A} \cup \overline{B}$ {Definition of set union}
Therefore $\forall x(x \in \overline{A \cap B} \leftrightarrow x \in \overline{A} \cup \overline{B})$ {Universal Generalization}
In other words: $\overline{A \cap B} = \overline{A} \cup \overline{B}$ {Definition of set equality}

Note: Some set equality proofs are more challenging, and require two separate subset proofs instead. However, your general approach to a set equality proof should be to do a linear equivalence proof. If, along the way, you find that some necessary step can only be accomplished with an inference (not an equivalence), then your set equality proof becomes a subset proof, and you will also need to do another subset proof in the opposite direction.

More Proofs of Identities

Theorem: For any sets A, B and C, $\overline{A \cup (B \cap C)} = (\overline{B} \cup \overline{C}) \cap \overline{A}$.

Proof:

Exercise: Prove the complement law.

More on the Cardinality of a Set

Definition: Let A be a set. If A contains exactly n distinct elements and $n \in \mathbb{Z}^{\geq 0}$, then A is a **finite set** and n is the **cardinality** of A. Notation: $|A| = n$.

Example: $A = \{a, b, c, ..., z\}$. What is $|A|$?

Example: What is $|\emptyset|$?

What is the cardinality of the following sets:

1. $\{\emptyset\}$
2. $\{\emptyset, \{\emptyset\}\}$
3. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
4. $\{\{a\}\}$
5. $\{a, \{a\}\}$

Definition: If a set is not finite, then it is **infinite**.

The Power Set

Definition: For any set S, the **power set** of S, denoted $P(S)$ or 2^S, is the set of all subsets of S.

Example: $A = \{a, b, c\}$. What is 2^A?

Example: What is $2^\emptyset$?

Cartesian Products

Definition: The **ordered n-tuple** $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_i as its ith element for all integers i such that $1 \leq i \leq n$.

Two ordered n-tuples are equal if each pair of corresponding entries are equal. That is, $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ if and only if $a_i = b_i$ for all $i \in \{1, 2, ..., n\}$.

We call ordered 2-tuples **ordered pairs**.

Definition: For sets A and B, the **Cartesian product** of A and B, denoted $A\text{x}B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. That is, $A\text{x}B = \{(a, b) | a \in A \wedge b \in B\}$.

Example: $A = \{0, 1\}$, $B = \{a, b, c\}$. What is $A\text{x}B$?

Example: $A = \{1, 2\}$. What is $A\text{x}\emptyset$?

More on Cartesian Products

Example: Disprove: For all sets A and B, $A\text{x}B = B\text{x}A$.

Definition: The **Cartesian product** of sets $A_1, A_2, ..., A_n$, denoted $A_1\text{x}A_2\text{x}...\text{x}A_n$, is the set of ordered n-tuples $(x_1, x_2, ..., x_n)$ where $x_i \in A_i$ for all $i \in \{1, 2, ..., n\}$. That is, $A_1\text{x}A_2\text{x}...\text{x}A_n = \{(x_1, x_2, ..., x_n)|x_i \in A_i \text{ for } i = 1, 2, ..., n\}$.

Example: $A = \{0, 1\}$, $B = \{a, b, c\}$, $C = \{cat, dog\}$.
What is $A\text{x}B\text{x}C$?