Identity Relation

Definition: The identity relation on set $A$ is defined as $i_A = \{(x, y) \in A \times A | x = y\}$.

Example: If $A = \{1, 2, 3\}$, then $i_A = \{(1, 1), (2, 2), (3, 3)\}$

What properties does this relation have? Is it reflexive, irreflexive, symmetric, anti-symmetric, and/or transitive?
Equivalence Relations

Example: Different Forms for Rational Numbers
\[ \frac{1}{2} = \frac{2}{4} = \frac{16}{32} \ldots \]
Even though these fractions have different numerators and denominators, and they look different, they are all equal.

Equivalence relations group together objects that may appear different but that are equal in some sense.

Definition: Let \( A \) be a set and \( R \) be a relation on \( A \). \( R \) is an equivalence relation if \( R \) is reflexive, symmetric and transitive.

Example: Let \( A \) be the set of all people and let \( R \) be the relation on \( A \) defined by:
\[ xRy \leftrightarrow x \text{ and } y \text{ have the same birthday.} \]
Show that \( R \) is an equivalence relation on \( A \).
More on Equivalence Relations

**Example:** Some compilers ignore all symbols in an identifier name after some fixed number of characters. If this limit is 8, for example, then the identifiers `NumberOfSquarePegs` and `NumberOfRoundHoles` would be the same.

Let $S$ be the set of all strings that are legal identifiers in some programming language. We can define a relation $R$ on $S$ as follows:

$x Ry \iff$ the first eight characters of $x$ are the same as the first 8 characters of $y$

**Exercise:** Prove that $R$ is an equivalence relation.
Examples

Definition: A string over a set A is a finite sequence of elements from A. The length of a string x, denoted $|x|$, is the number of elements in the sequence. Notation: String $a = a_1a_2...a_n$, where $a_i \in A$ for all $i \in \{0, 1, ..., n\}$.

Example: Some bit strings (strings over the set $\{0, 1\}$): $0, 110, 00000$. $|110| = 3$.

Example: Let A be the set of all bit strings and define relation R on A as follows:
$xRy \iff |x| = |y|$.
Is R an equivalence relation?
Equivalence Classes

**Definition:** Let $A$ be a set and let $R$ be an equivalence relation on $A$. For each element $x \in A$, the **equivalence class of $x$**, denoted $[x]$, is $[x] = \{y \in A | xRy\}$.

**Notation:** If it’s not clear which equivalence relation is being considered, we write $[a]_R$ for the equivalence class of $a$ under relation $R$.

**Example:** Let $A = \{1, 2, 3, 4, 5\}$ and define relation $R$ on $A$ by:

$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (3, 2), (3, 3), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4), (5, 1), (5, 5)\}$.

Verify that $R$ is an equivalence relation and find the equivalence class for each element of $A$. 
More on Equivalence Relations

**Definition:** For integers $a$, $b$ and $n$, we say that $a$ is congruent to $b \mod n$ if $n$ divides $a - b$.
Notation: $a \equiv_n b$ or $a \equiv b(\mod n)$.

**Example:** Congruence mod 3
$7 \equiv_3 1$, $8 \equiv_3 2$, $9 \equiv_3 0$.

**Example:** Assume $a$, $b$ and $n$ are integers and $n > 1$. Define the relation $R$ on $\mathbb{Z}$ as follows:
$R = \{(a, b) | a \equiv b(\mod n)\}$.
Is $R$ an equivalence relation? Prove or disprove.
More on Equivalence Classes

**Recall:** Let $R$ be an equivalence relation on set $A$. For $x \in A$, the equivalence class of $x$ is $[x]_R = \{ y \in A | (x, y) \in R \}$.

**Definition:** Let $R$ be an equivalence relation on set $A$, and let $x \in A$. For any $t \in [x]_R$, $t$ is a **representative** of this equivalence class.

**Example:** Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$. Define relation $R$ on $A$ such that $(a, b) \in R \leftrightarrow |a| = |b|$. Verify that $R$ is an equivalence relation, and list the equivalence classes.
Equivalence Classes for congruence mod 3

**Example:** Define the equivalence classes of 0, 1 and 2 for congruence mod 3. Do the equivalence classes of 0 and 1 have any common elements? How many (distinct) equivalence classes are there under congruence mod 3?

**Example:** Draw the digraph for the congruent mod 3 equivalence relation on the set $A = \{0, 1, 2, 3, 4, 5, 6\}$. 
Equivalence Classes and Partitions

**Note:** Let \( R \) be an equivalence relation on set \( A \).
1. Then every element \( x \in A \) is in an equivalence class - so the union of the equivalence classes is the set \( A \).
2. The equivalence classes for any \( x, y \in A \) are either equal or disjoint. So \( [x] = [y] \) or \( [x] \cap [y] = \emptyset \).

**Definition:** Let \( A \) be a set and let \( A_1, A_2, \ldots, A_n \) be subsets of \( A \). Then \( \{A_1, A_2, \ldots, A_n\} \) is a **partition** of \( A \) if:

- \( A_i \neq \emptyset \) for all \( i \in \{1, 2, \ldots, n\} \).
- \( A_i \cap A_j = \emptyset \) whenever \( i \neq j \), all \( i, j \in \{1, 2, \ldots, n\} \).
- \( \bigcup_{i=1}^{m} A_i = A \).
Partition Examples

**Examples:** Let \( A = \{1, 2, 3, 4, 5\} \). Then the following are partitions of \( A \).

1. \( \{\{1\}, \{2, 3, 4\}, \{5\}\} \)
2. \( \{\{1, 2, 3\}, \{4, 5\}\} \)

What about \( \{\{1, 5\}, \{2, 3\}, \{4, 5\}\} \)? Is it a partition of \( A \)? What about \( \{\{1\}, \{2, 4, 5\}\} \)?

**Example:** Define two different partitions of the set of lowercase English letters.
Counting

Because the members of a partition are pair-wise disjoint, if we have partitioned a finite set, the sum of the cardinalities (sizes) of each member of the partition will equal the cardinality (size) of the partitioned set.

Formally: If \( \{A_1, \ldots, A_n\} \) is a partition of \( A \), then \( \sum_{i=0}^{n} |A_i| = |A| \)

This fact is actually a special case of a counting rule known as the Inclusion/Exclusion Principle. The fully general version of this rule is too complicated for this class, but consider a situation where \( A, B, \) and \( C \) are all subsets of \( X \), but the sets \( A, B, \) and \( C \) are NOT necessarily pair-wise disjoint (which means they do not represent a partition).

**Theorem: Inclusion/Exclusion Principle (on three sets):** Let \( A, B, C \) be subsets of \( X \). Then

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]

**Example:** Let \( A = \{1, 2, 3\}, B = \{2, 3, 4, 5, 6\}, C = \{2, 5, 6, 7, 8\} \). Then

\[
|A \cup B \cup C| = |\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8, \text{ and}
\]

\[
|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]

\[
= |\{1, 2, 3\}| + |\{2, 3, 4, 5, 6\}| + |\{2, 5, 6, 7, 8\}| - |\{2, 3\}| - |\{2\}| - |\{2, 5, 6\}| + |\{2\}|
\]

\[
= 3 + 5 + 5 - 2 - 1 - 3 + 1
\]

\[
= 13 - 6 + 1 = 8
\]

**Note:** Remember that this rule applies for ANY three subsets of \( X \). They need not be a partition. However, in the special case where \( \{A, B, C\} \) is a partition of \( X \), it is clear that the pair-wise disjointness of \( A, B, C \) will make the above expression simplify to \( |A \cup B \cup C| = |A| + |B| + |C| \).
More on Inclusion/Exclusion

**Theorem: Inclusion/Exclusion Principle (on two sets):** Let $A, B$ be subsets of $X$. Then
\[ |A \cup B| = |A| + |B| - |A \cap B| \]

**Exercise:** How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?

**Exercise:** A class of 50 CS students has 30 who know C++, 18 who know Haskell, and 26 who know Python. Additionally, 9 know both C++ and Haskell, 16 know both C++ and Python, and 8 know both Haskell and Python. 47 know at least one of the three languages. How many students know all three languages?
Equivalence Classes and Partitions

**Example:** Let $A = \{1, 2, 3, 4, 5\}$. Let $R$ be the equivalence relation on $A$ defined by:
$aRb \iff a - b$ is even.

1. Draw the digraph of $R$.
2. Write down the equivalence classes of $R$.
3. Is the set of equivalence classes a partition of $A$?
**Theorem:** Assume $R$ is an equivalence relation on a set $A$. Let $x$ and $y$ be arbitrary elements of $A$. Then $[x] = [y] \lor [x] \cap [y] = \emptyset$.

**Proof:** Proof by cases.
1. Let $x, y \in A$ (arbitrary)
2. Case 1: $xRy$ (Prove $[x] = [y]$)
3. Let $z \in [x]$ (Prove $[x] \subseteq [y]$)
4. $xRz$ {Definition of equivalence class}
5. $zRx$ {$R$ is an equivalence relation, and therefore symmetric: 4}
6. $zRy$ {$R$ is an equivalence relation, and therefore transitive: 2,5}
7. $z \in [y]$ {Definition of equivalence class}
8. Therefore $[x] \subseteq [y]$ {Definition of subset: 3-7}
9. Let $z \in [y]$ (Prove $[y] \subseteq [x]$)
10. $yRz$ {Definition of equivalence class}
11. $zRy$ {$R$ is an equivalence relation, and therefore symmetric: 10}
12. $yRx$ {$R$ is an equivalence relation, and therefore symmetric: 2}
13. $zRx$ {$R$ is an equivalence relation, and therefore transitive: 11,12}
14. $z \in [x]$ {Definition of equivalence class}
15. Therefore $[y] \subseteq [x]$ {Definition of subset: 9-14}
16. $[x] = [y]$ {Definition of set equality: 8,15}
17. Case 2: $xRy$ (Prove $[x] \cap [y] = \emptyset$)
18. Assume $[x] \cap [y] \neq \emptyset$ BWoC
19. $b \in [x] \cap [y]$ for some $b$ {Definition of $\emptyset$, and $\exists$ instantiation}
20. $b \in [x] \land b \in [y]$ {Definition of set intersection}
21. $xRb \land yRb$ {Definition of equivalence class}
22. $xRb \land bRy$ {$R$ is an equivalence relation, and therefore symmetric}
23. $xRy$ {$R$ is an equivalence relation, and therefore transitive}
24. Line 23 contradicts line 17, therefore the assumption on line 18 is wrong.
25. Therefore $[x] \cap [y] = \emptyset$
Equivalence Classes and Partitions

**Corollary:** The set of equivalence classes of an equivalence relation \( R \) on set \( A \) is a partition of \( A \).

**Note:** It is also the case that a partition of a set \( A \) corresponds to an equivalence relation on \( A \): Suppose \( \{A_1, A_2, \ldots, A_n\} \) is a partition of \( A \). Define relation \( R \) on \( A \) as follows:
\[ xRy \leftrightarrow x, y \in A_i \text{ for some } i \in \{1, 2, \ldots, n\}. \]
Exercise: Verify that \( R \) is an equivalence relation.

**Example:** Let \( A = \{1, 2, 3, 4, 5, 6\} \).
The set \( \{\{1\}, \{2, 6\}, \{3\}, \{4, 5\}\} \) is a partition of \( A \). Define the equivalence relation determined by this partition.
Equivalence Relations and Set Operations

**Question:** Let R and S be equivalence relations on a set A. Prove your answers to the following questions.

1. Is \( R \cup S \) an equivalence relation?
2. Is \( R \cap S \) an equivalence relation?