

# Topic 23

## Red Black Trees

"People in every direction  
No words exchanged  
No time to exchange  
And all the little ants are marching  
**Red** and **Black** antennas waving"  
-*Ants Marching*, Dave Matthews's Band

"Welcome to L.A.'s Automated Traffic Surveillance and Control Operations Center. See, they use video feeds from intersections and specifically designed algorithms to predict traffic conditions, and thereby control traffic lights. So all I did was come up with my own... **kick ass algorithm** to sneak in, and now we own the place."

-Lyle, the Napster, (Seth Green), *The Italian Job*

# Clicker 1

▶ 2000 elements are inserted one at a time into an initially empty binary search tree using the simple algorithm. What is the maximum possible height of the resulting tree?

A. 1

B. 11

C. 21

D. 500

E. 1999

# Binary Search Trees

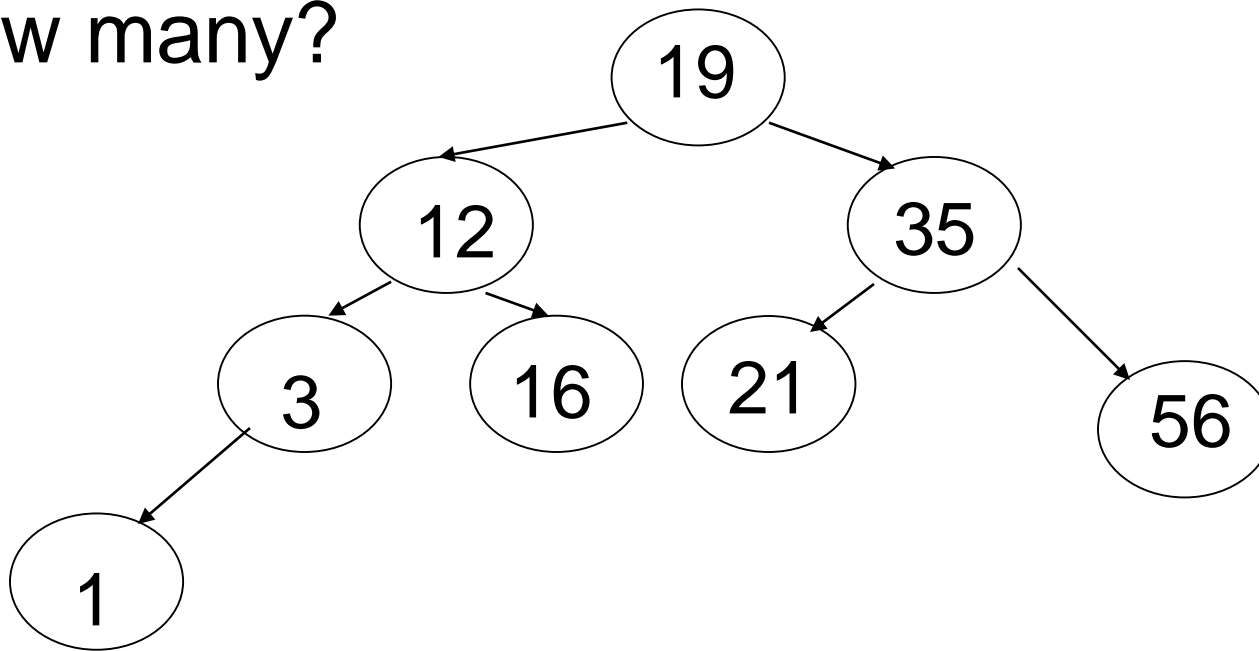
- ▶ Average case and worst case Big O for
  - insertion
  - deletion
  - access
- ▶ Balance is important. Unbalanced trees give worse than  $\log N$  times for the basic tree operations
- ▶ Can balance be guaranteed?

# Red Black Trees

- ▶ A BST with more complex algorithms to ensure balance
- ▶ Each node is labeled as **Red** or Black.
- ▶ Path: A unique series of links (edges) traverses from the root to each node.
  - The number of edges (links) that must be followed is the path length
- ▶ In **Red** Black trees paths from the root to elements with 0 or 1 child are of particular interest

# Paths to Single or Zero Child Nodes

► How many?

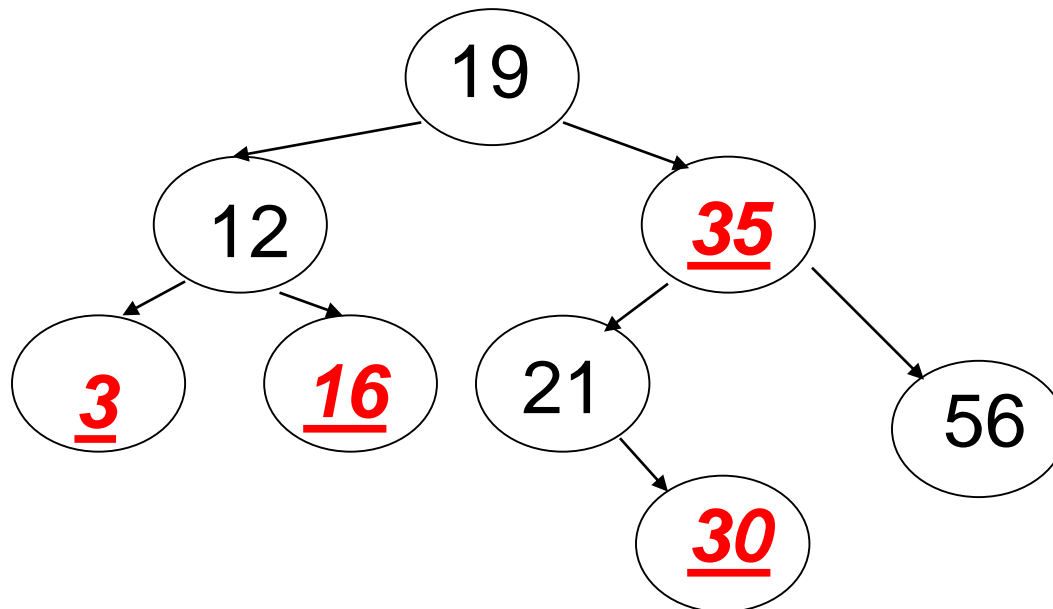


# Red Black Tree Rules

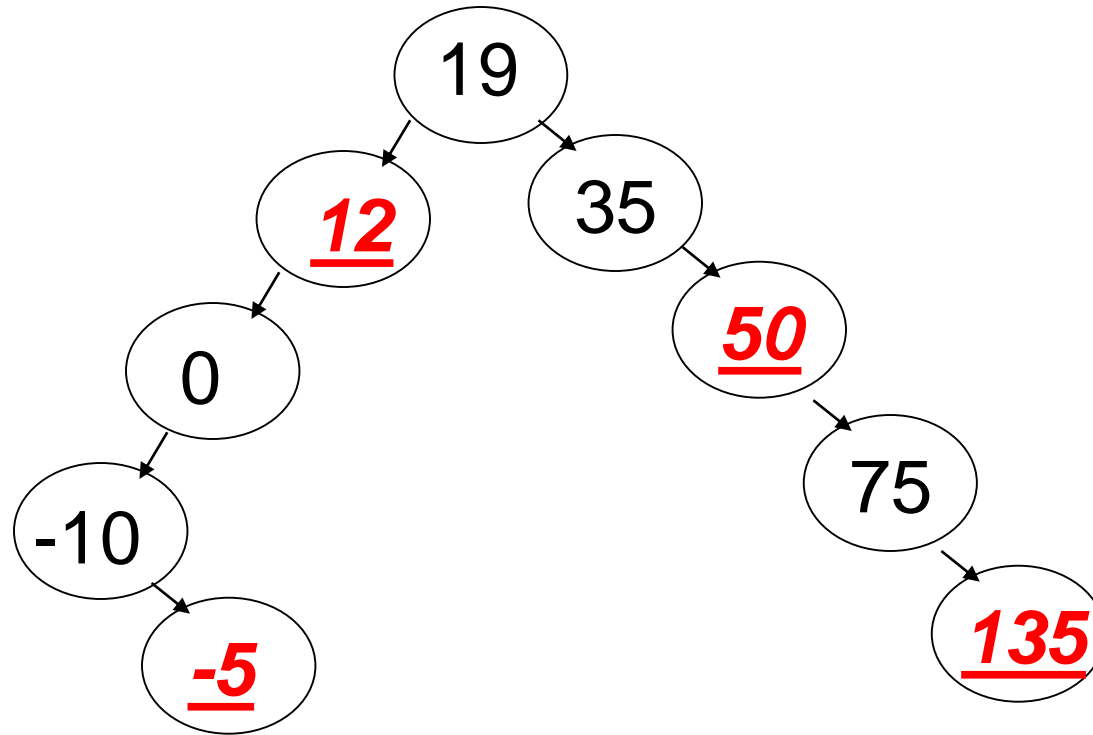
1. Every node is colored either **red** or black
2. The root is black
3. If a node is **red** its children must be black. (a.k.a. the **red** rule)
4. Every path from a node to a null link must contain the same number of black nodes (a.k.a. the path rule)

# Example of a **Red** Black Tree

- ▶ The root of a **Red** Black tree is black
- ▶ Every other node in the tree follows these rules:
  - Rule 3: If a node is **Red**, all of its children are Black
  - Rule 4: The number of Black nodes must be the same in all paths from the root node to null nodes



# Red Black Tree?





# Clicker 2

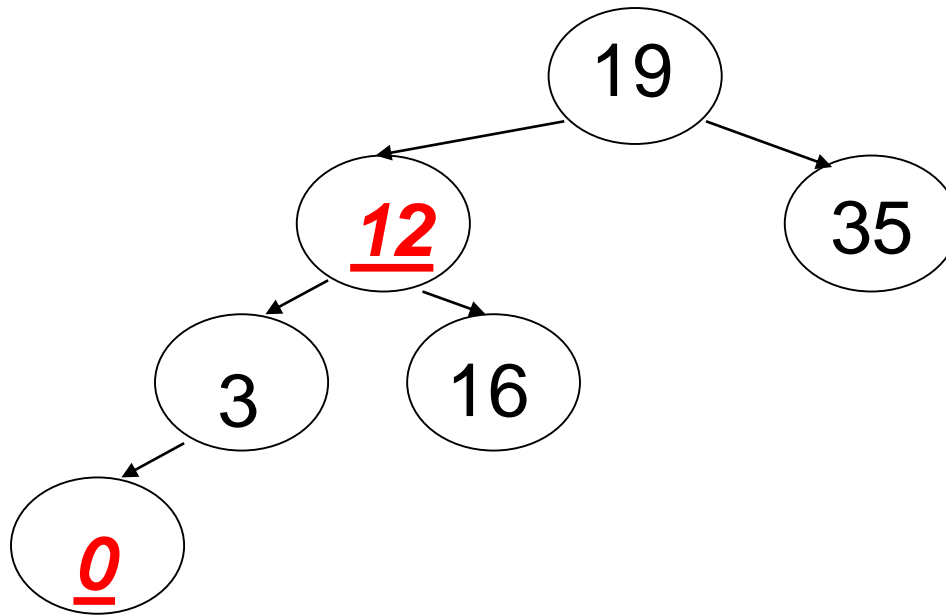
- ▶ Is the tree on the previous slide a binary search tree? Is it a red black tree?

BST?

Red-Black?

- |    |     |     |
|----|-----|-----|
| A. | No  | No  |
| B. | No  | Yes |
| C. | Yes | No  |
| D. | Yes | Yes |

# Red Black Tree?



Perfect?

Full?

Complete?

# Clicker 3

- ▶ Is the tree on the previous slide a binary search tree? Is it a red black tree?

BST?

Red-Black?

- |    |     |     |
|----|-----|-----|
| A. | No  | No  |
| B. | No  | Yes |
| C. | Yes | No  |
| D. | Yes | Yes |

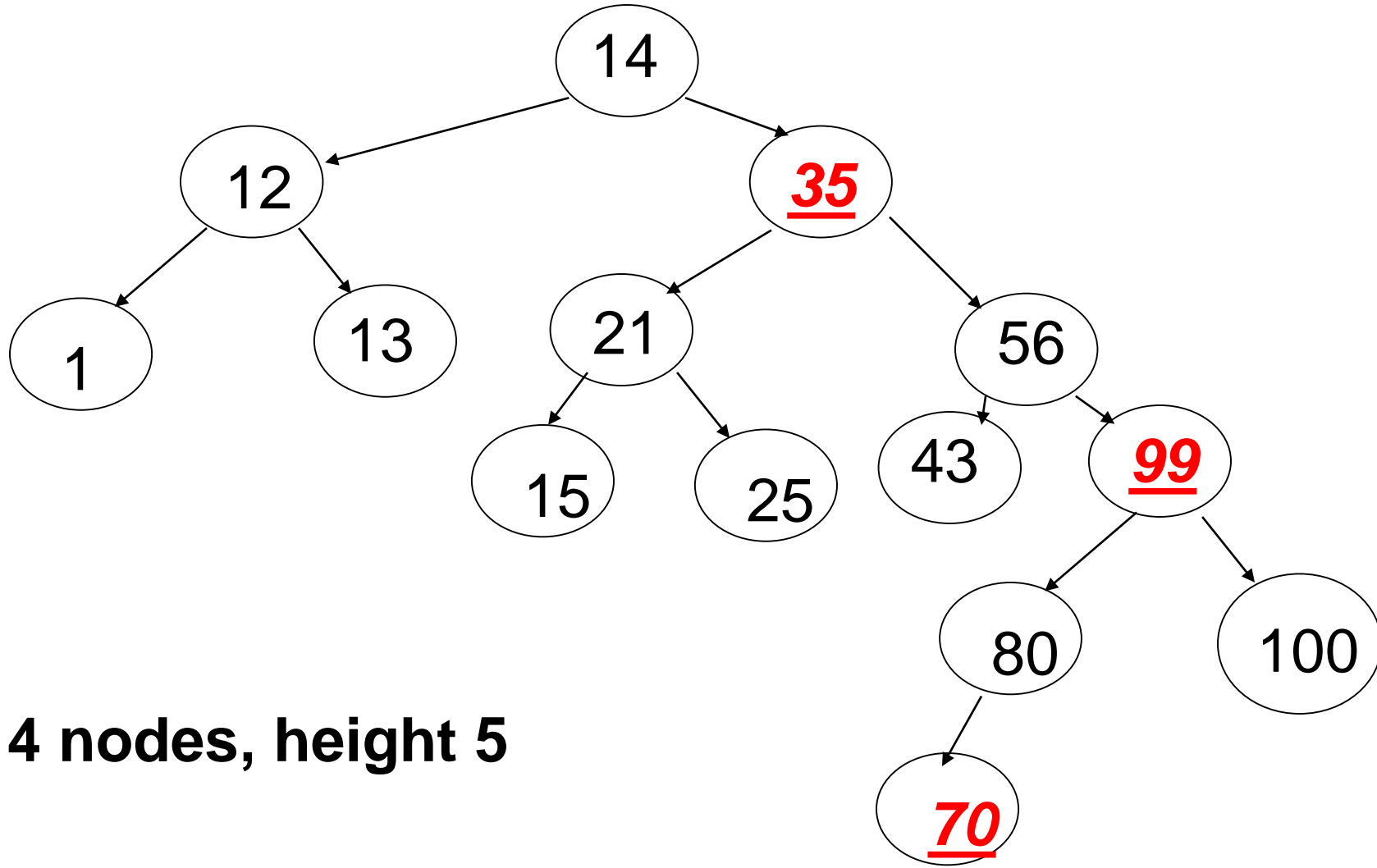
# Implications of the Rules

- ▶ If a **Red** node has any children, it must have two children and they must be Black. (Why?)
- ▶ If a Black node has only one child that child must be a **Red** leaf. (Why?)
- ▶ Due to the rules there are limits on how unbalanced a **Red** Black tree may become.
  - on the previous example may we hang a new node off of the leaf node that contains **0**?

# Properties of Red Black Trees

- ▶ If a Red Black Tree is complete, with all Black nodes except for Red leaves at the lowest level the height will be minimal,  $\sim \log N$
- ▶ To get the max height for N elements there should be as many Red nodes as possible down one path and all other nodes are Black
  - This means the max height would be approximately  $2 * \log N$  (don't use this as a formula)
  - typically less than this
  - see example on next slide
  - interesting exercise, draw max height tree with N nodes

# Max Height **Red** Black Tree



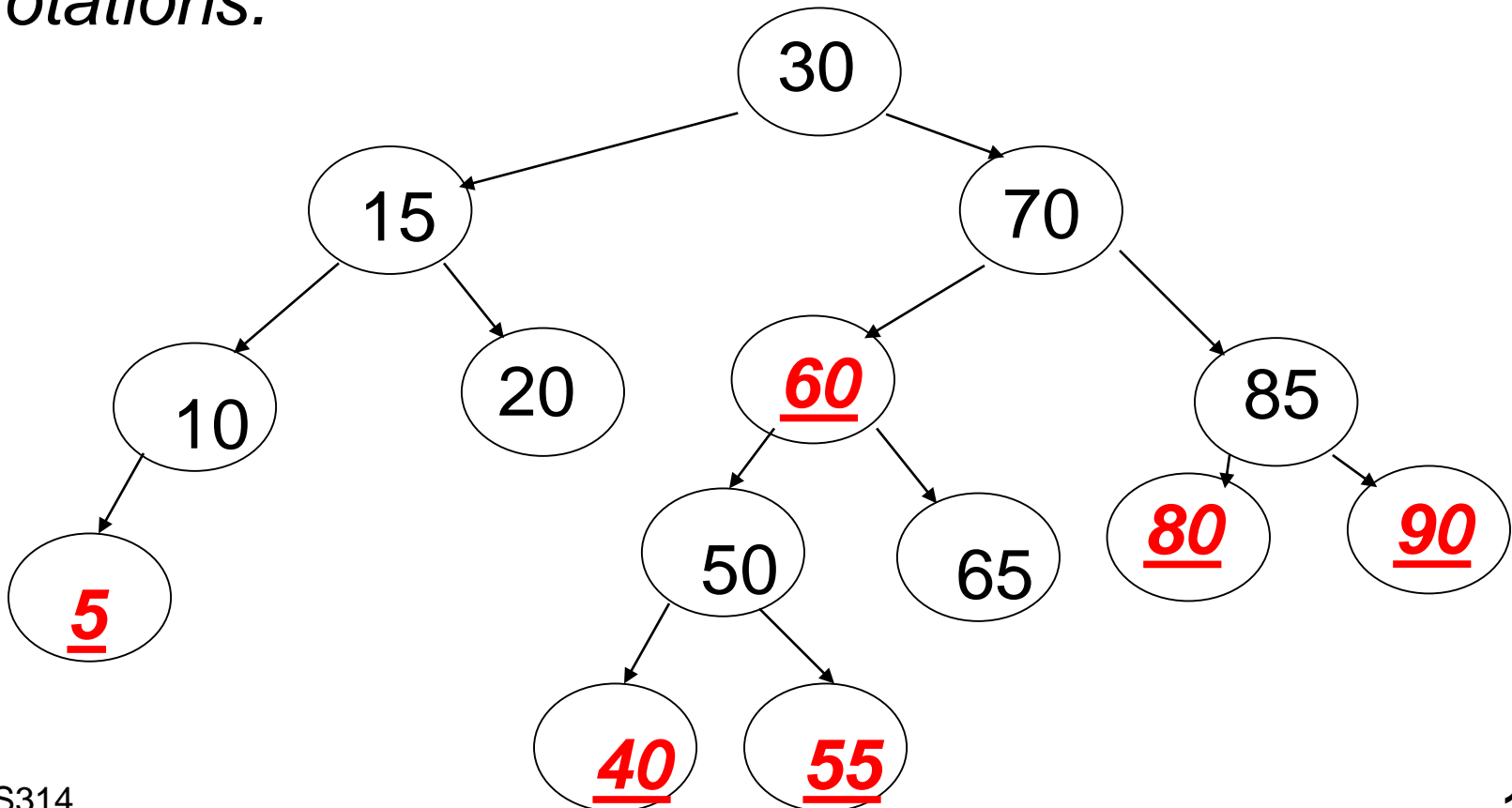
**14 nodes, height 5**

# Maintaining the Red Black Properties in a Tree

- ▶ Insertions
- ▶ Must maintain rules of Red Black Tree.
- ▶ New Node always a leaf
  - can't be black or we will violate rule 4
  - therefore the new leaf must be red
  - If parent is black, done (trivial case)
  - if parent red, things get interesting because a red leaf with a red parent violates rule 3

# Insertions with **Red** Parent - Child

Must modify tree when insertion would result in **Red** Parent - Child pair using color changes and *rotations*.

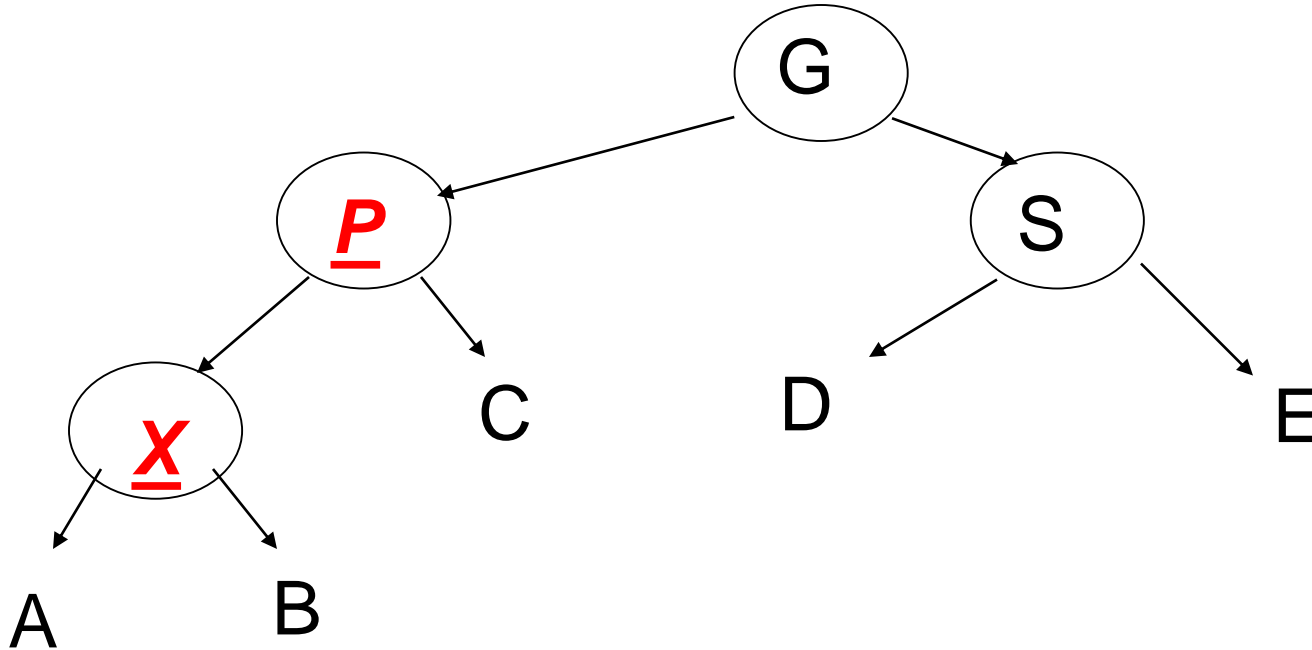




# Case 1

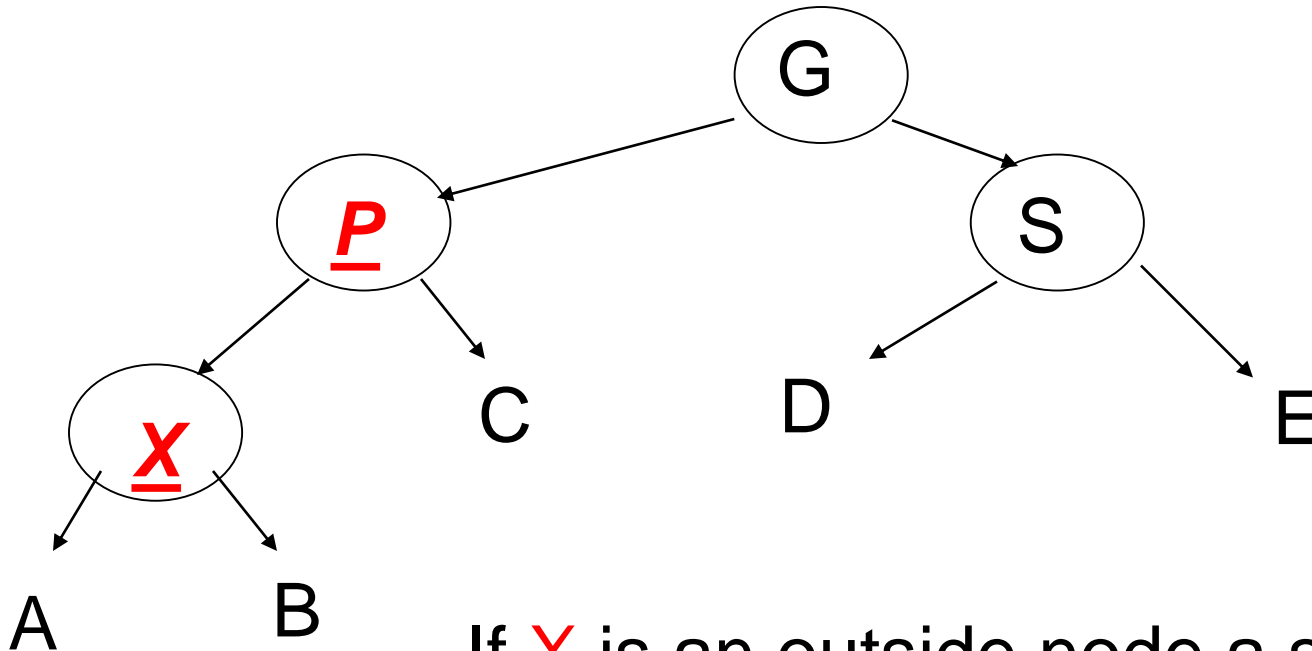
- ▶ Suppose sibling of parent is Black.
  - by convention null nodes are black
- ▶ In the previous tree, true if we are inserting a 3 or an 8.
  - What about inserting a 99? Same case?
- ▶ Let  $X$  be the new leaf Node,  $P$  be its Red Parent,  $S$  the Black sibling and  $G$ ,  $P$ 's and  $S$ 's parent and  $X$ 's grandparent
  - What color is  $G$ ?

# Case 1 - The Picture



Relative to G, **X** could be an *inside* or *outside* node.  
Outside -> left left or right right moves  
Inside -> left right or right left moves

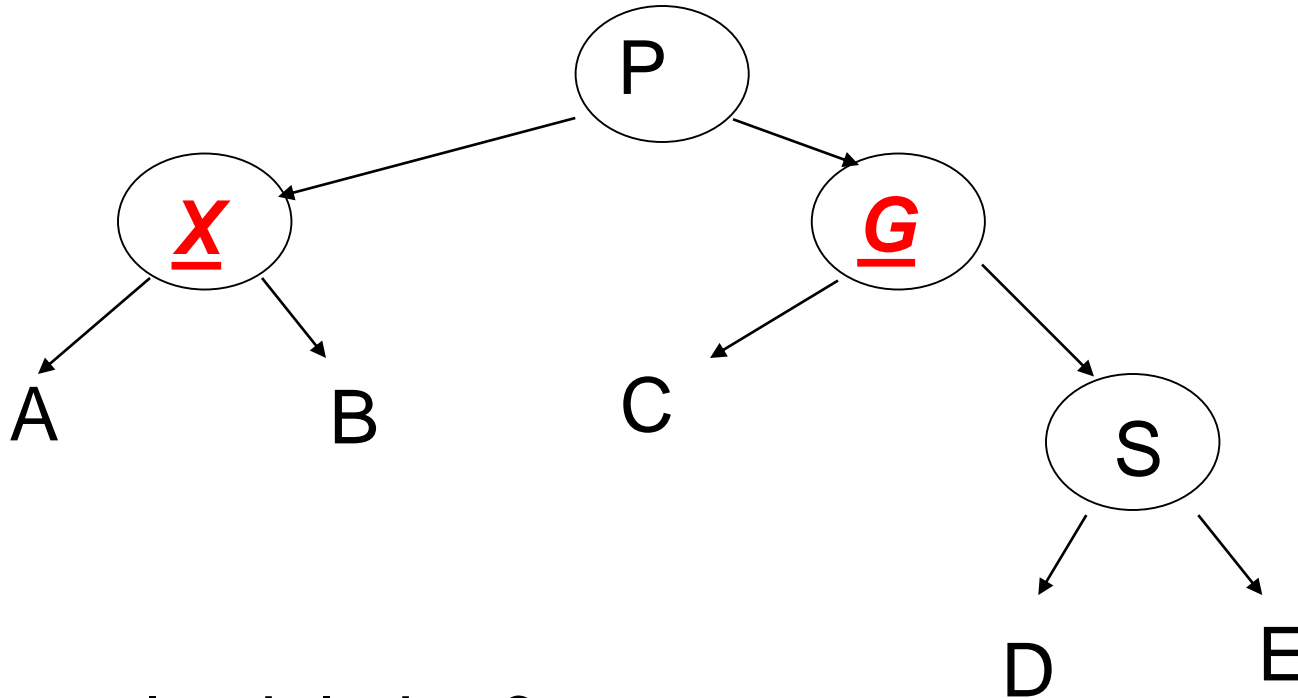
# Fixing the Problem



If **X** is an outside node a single *rotation* between **P** and G fixes the problem.

A rotation is an exchange of roles between a parent and child node. So P becomes G's parent. Also must recolor **P** and G.

# Single Rotation



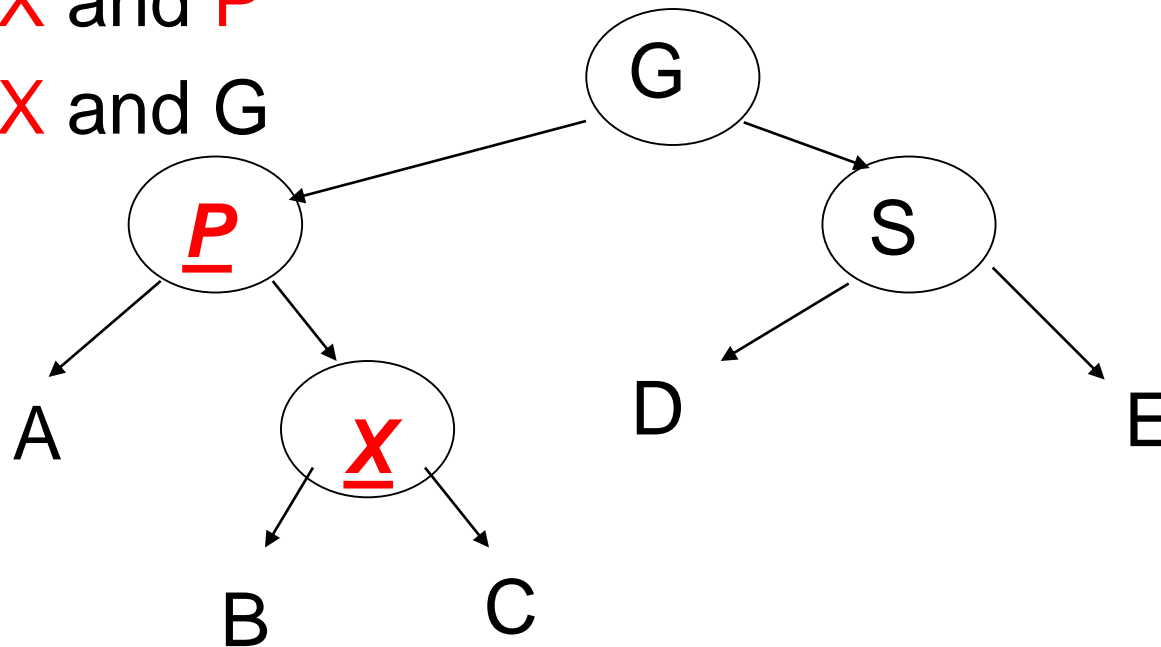
Apparent rule violation?

Recall, S is null if X is a leaf, so no problem

If this occurs higher in the tree (why?) subtrees A, B, and C will have one more black node than D and E.

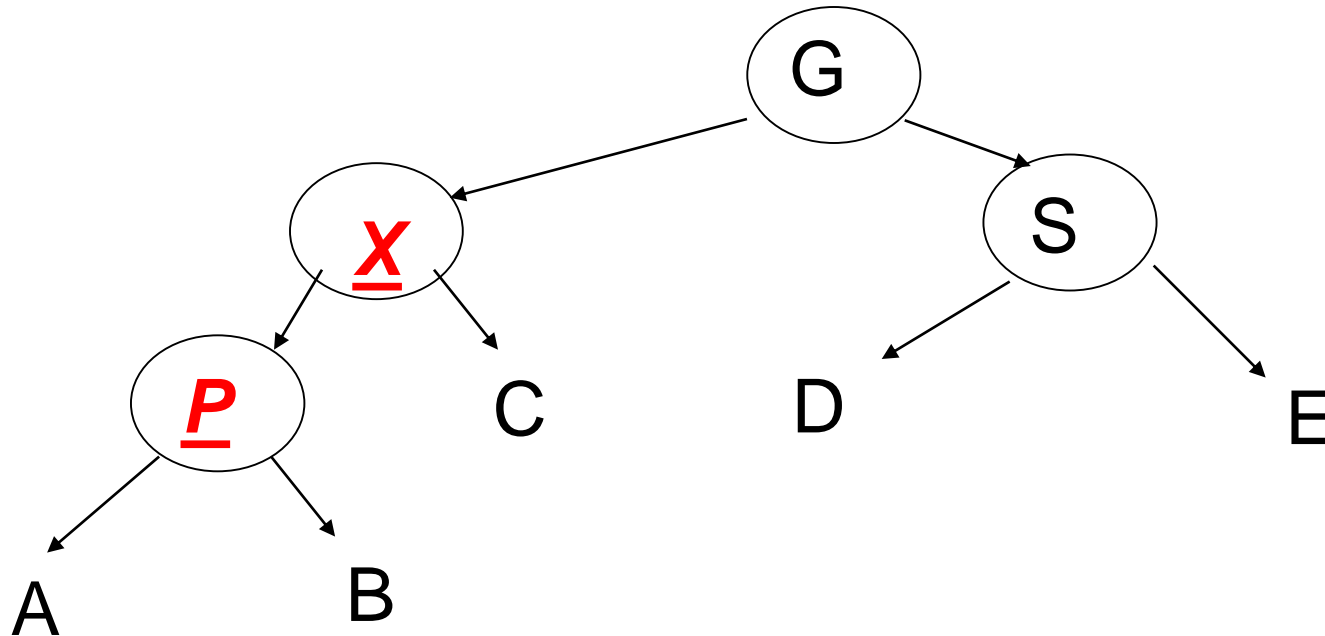
# Case 2

- ▶ What if **X** is an inside node relative to G?
  - a single rotation will not work
- ▶ Must perform a double rotation
  - rotate **X** and **P**
  - rotate **X** and G



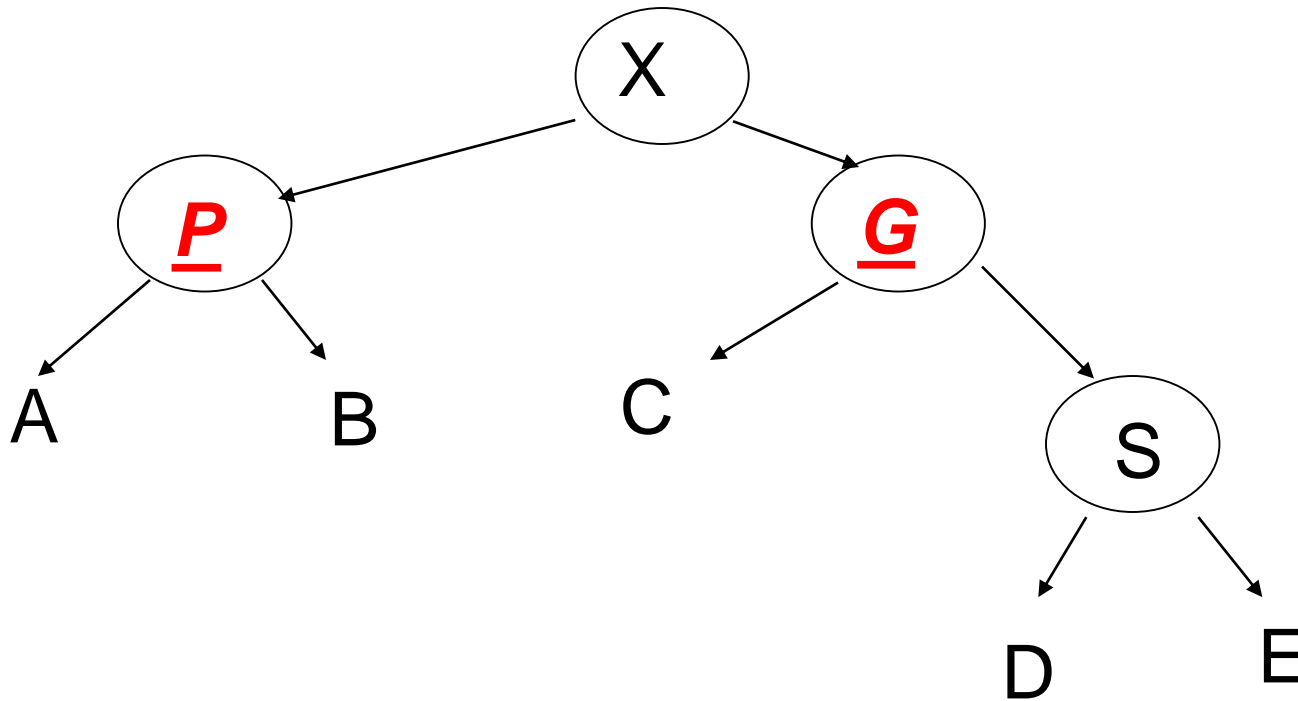
# First Rotation

- ▶ Rotate **P** and **X**, no color change



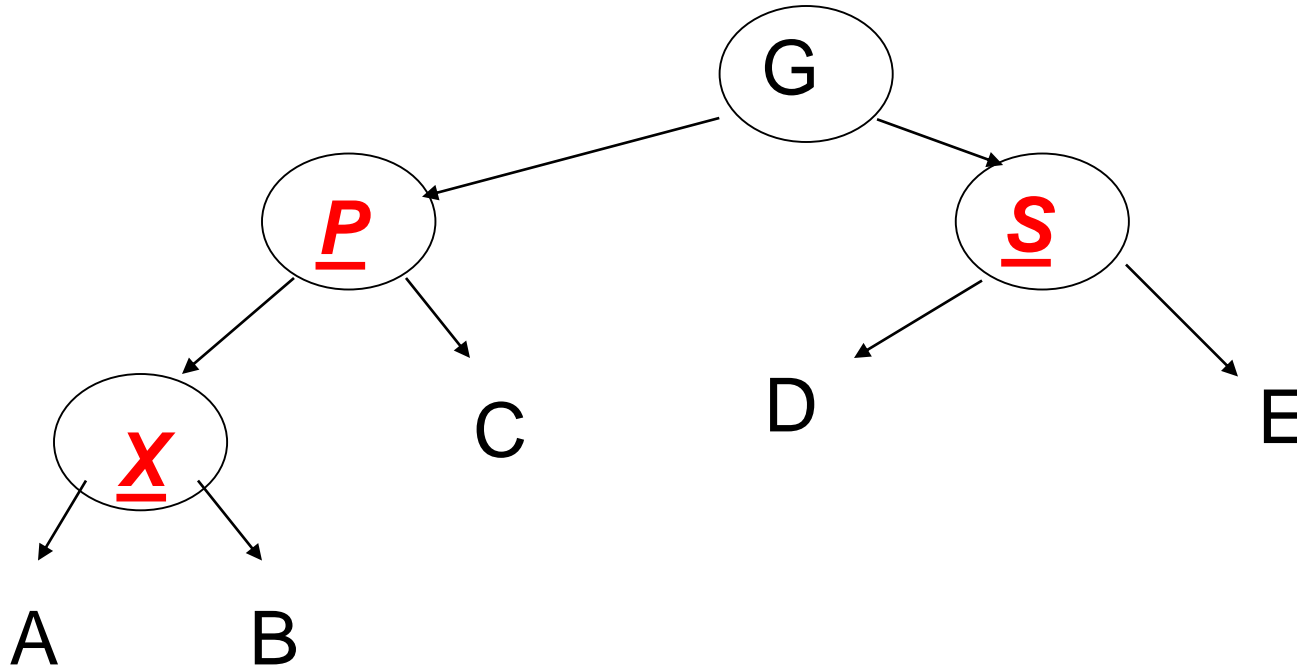
- ▶ What does this actually do?

# After Double Rotation



# Case 3

## Sibling is **Red**, not Black

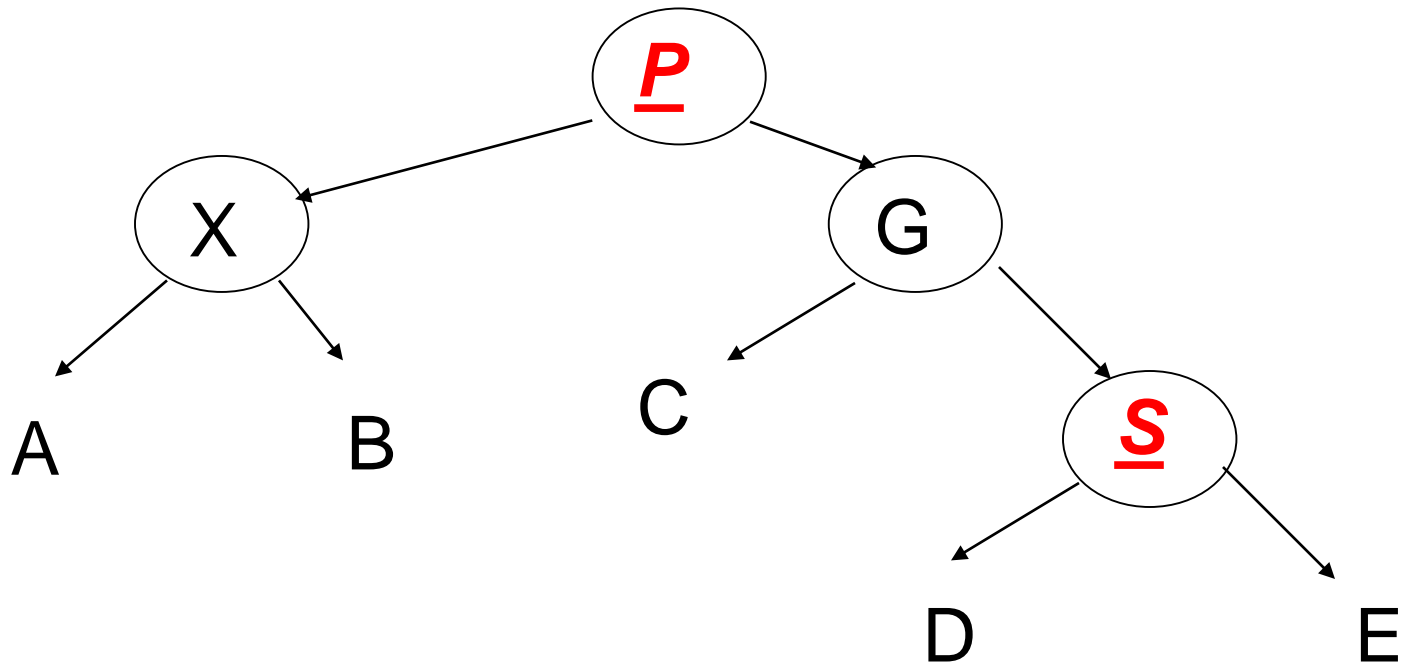


Any problems?



# Fixing Tree when S is **Red**

- ▶ Must perform single rotation between parent, P and grandparent, G, and then make appropriate color changes

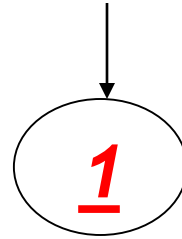


# More on Insert

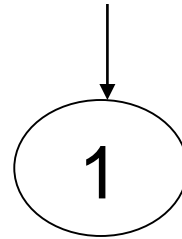
- ▶ Problem: What if on the previous example G's parent (GG!) had been **red**?
- ▶ Easier to never let Case 3 ever occur!
- ▶ On the way down the tree, if we see a node X that has 2 **Red** children, we make X **Red** and its two children black.
  - if recolor the root, recolor it to black
  - the number of black nodes on paths below X remains unchanged
  - If X's parent was **Red** then we have introduced 2 consecutive **Red** nodes.(violation of rule)
  - to fix, apply rotations to the tree, same as inserting node

# Example of Inserting Sorted Numbers

▶ 1 2 3 4 5 6 7 8 9 10

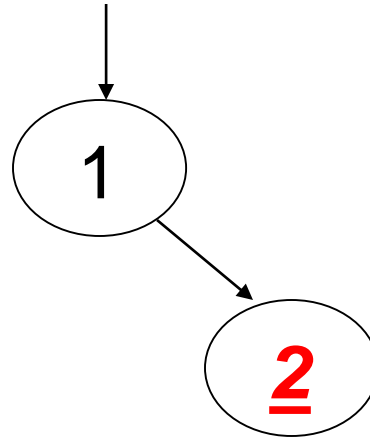


Insert 1. A leaf so red. Realize it is root so recolor to black.



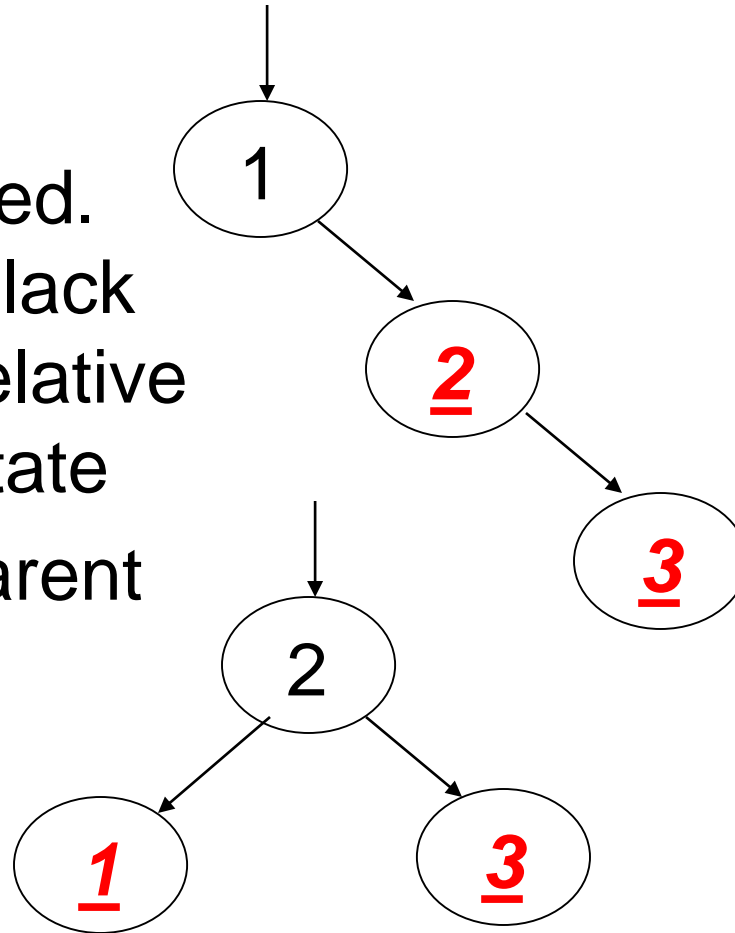
# Insert 2

make 2 red. Parent  
is black so done.



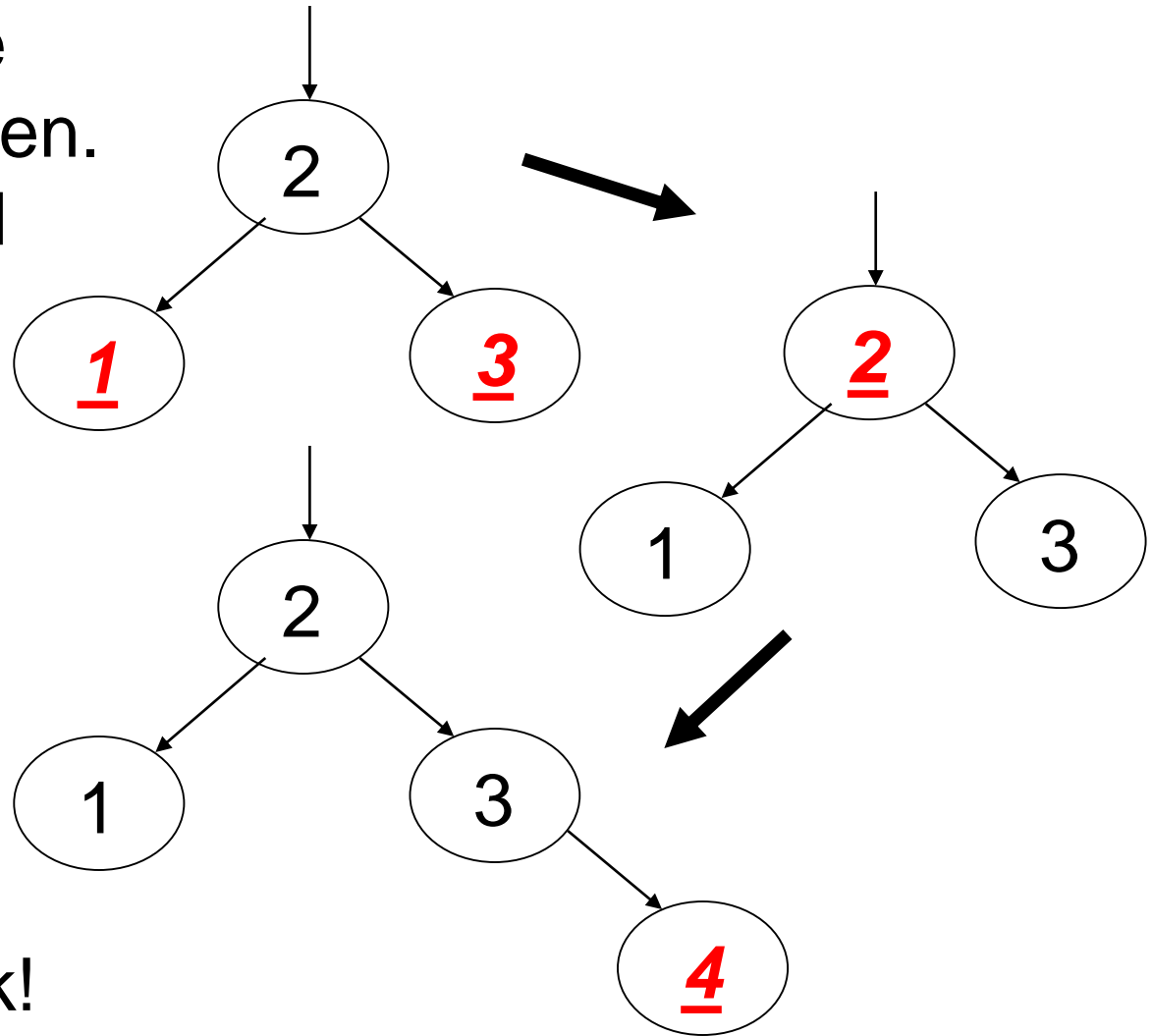
# Insert 3

Insert 3. Parent is red.  
Parent's sibling is black  
(null) 3 is outside relative  
to grandparent. Rotate  
parent and grandparent



# Insert 4

On way down see  
2 with 2 red children.  
Recolor 2 red and  
children black.

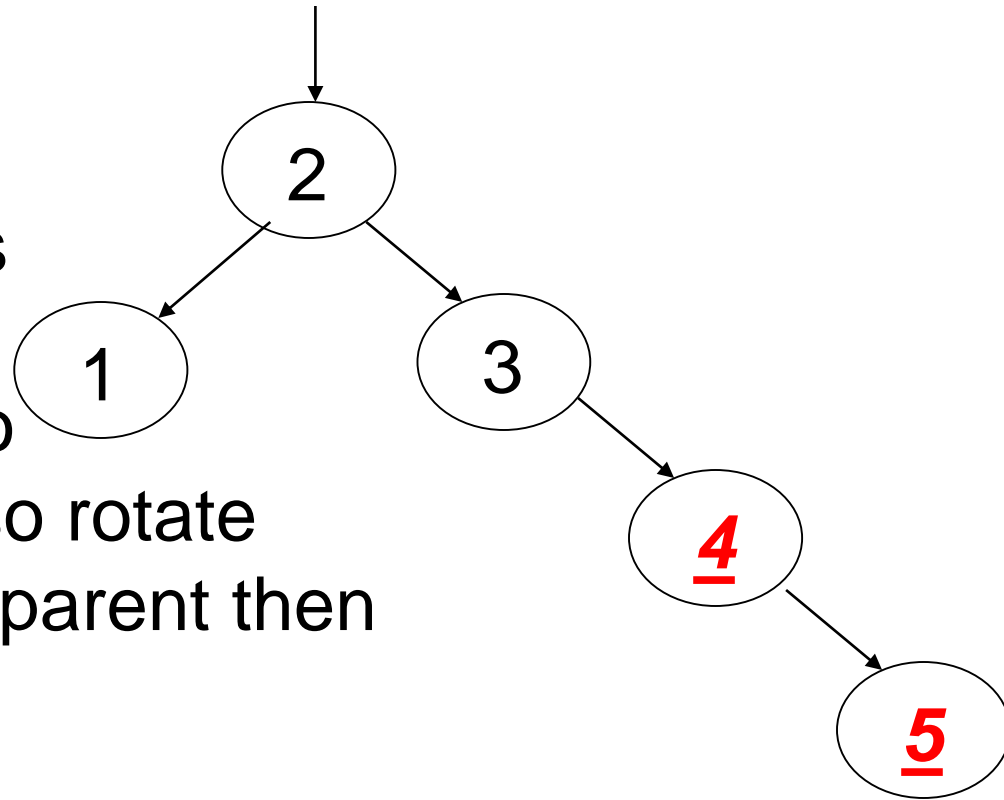


When adding 4  
parent is black  
so done.

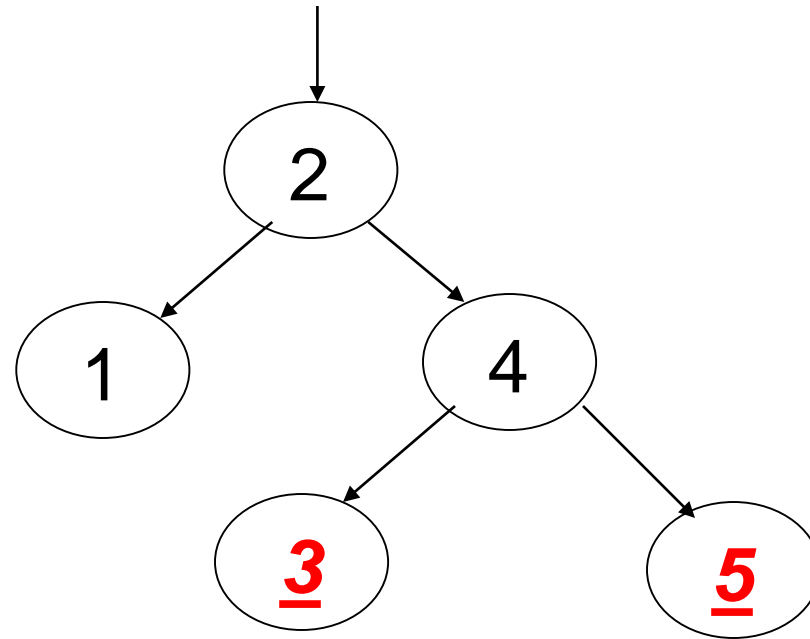
Set root to black!

# Insert 5

5's parent is red.  
Parent's sibling is  
black (null). 5 is  
outside relative to  
grandparent (3) so rotate  
parent and grandparent then  
recolor



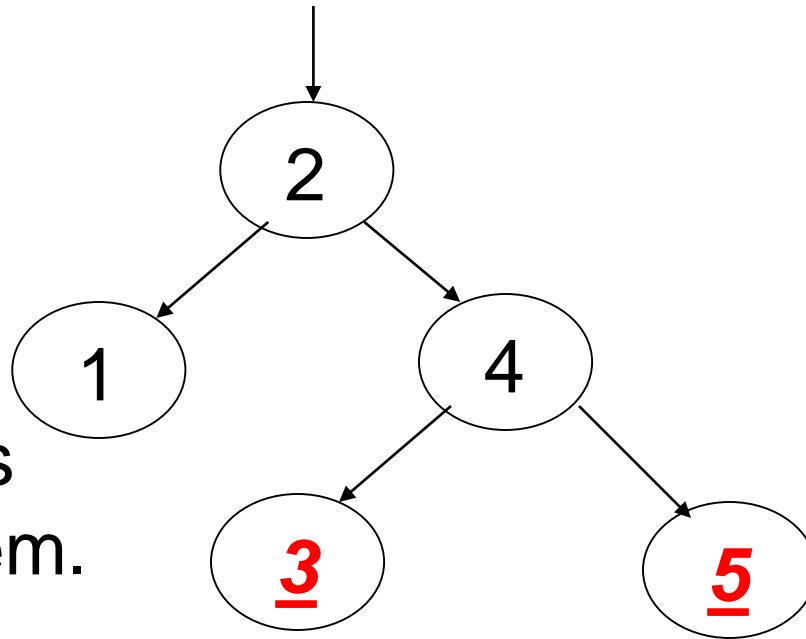
# Finish insert of 5





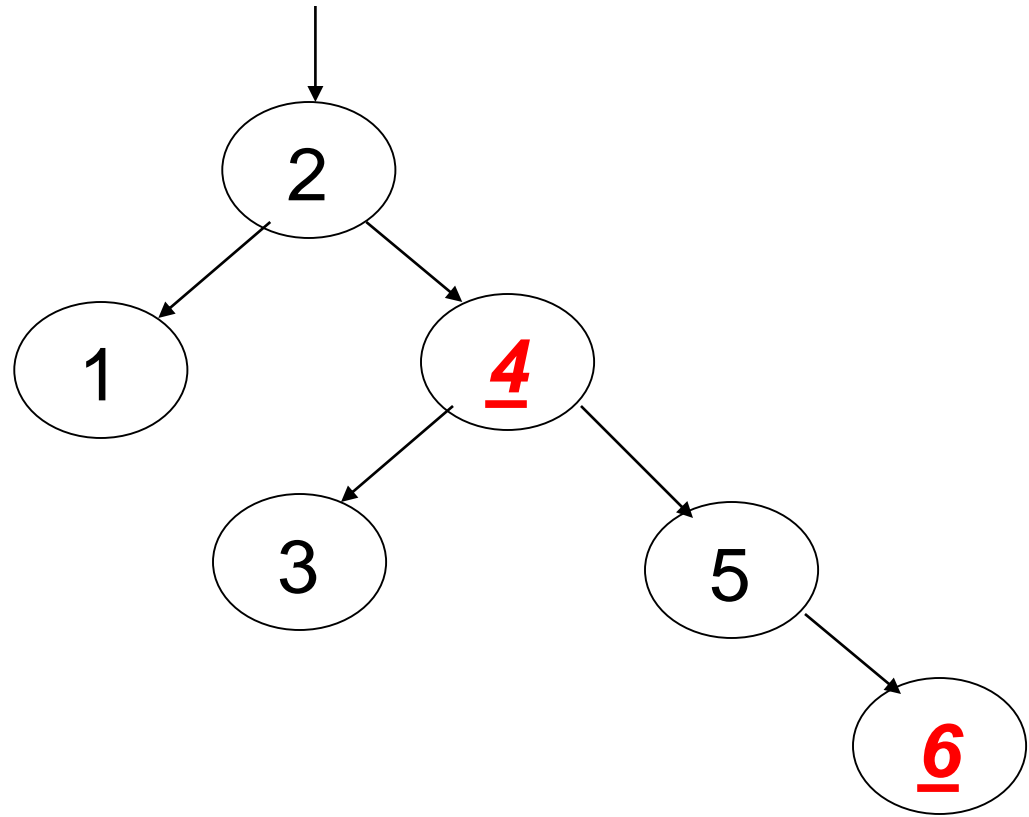
# Insert 6

On way down see  
4 with 2 red  
children. Make  
4 red and children  
black. 4's parent is  
black so no problem.



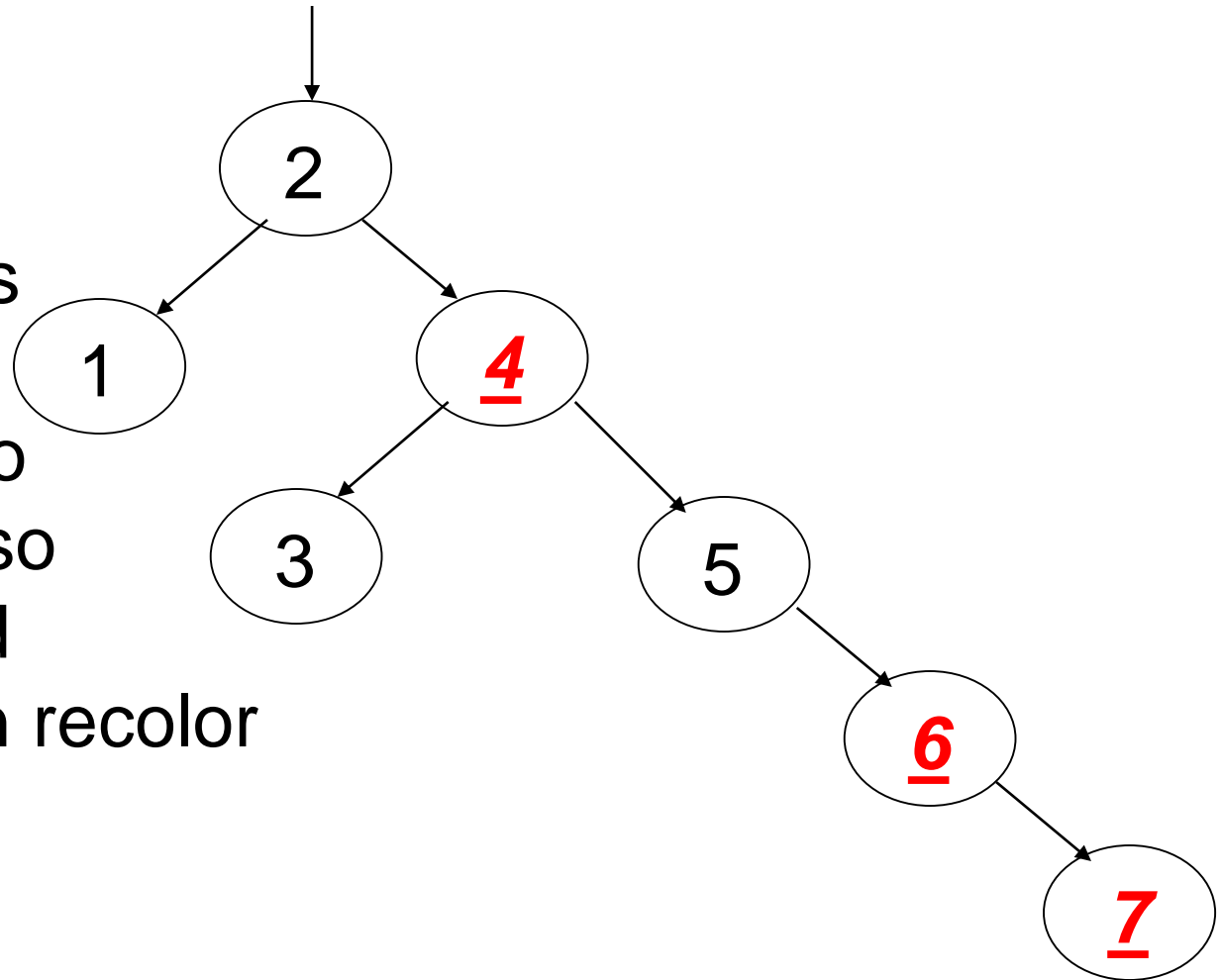
# Finishing insert of 6

6's parent is black  
so done.

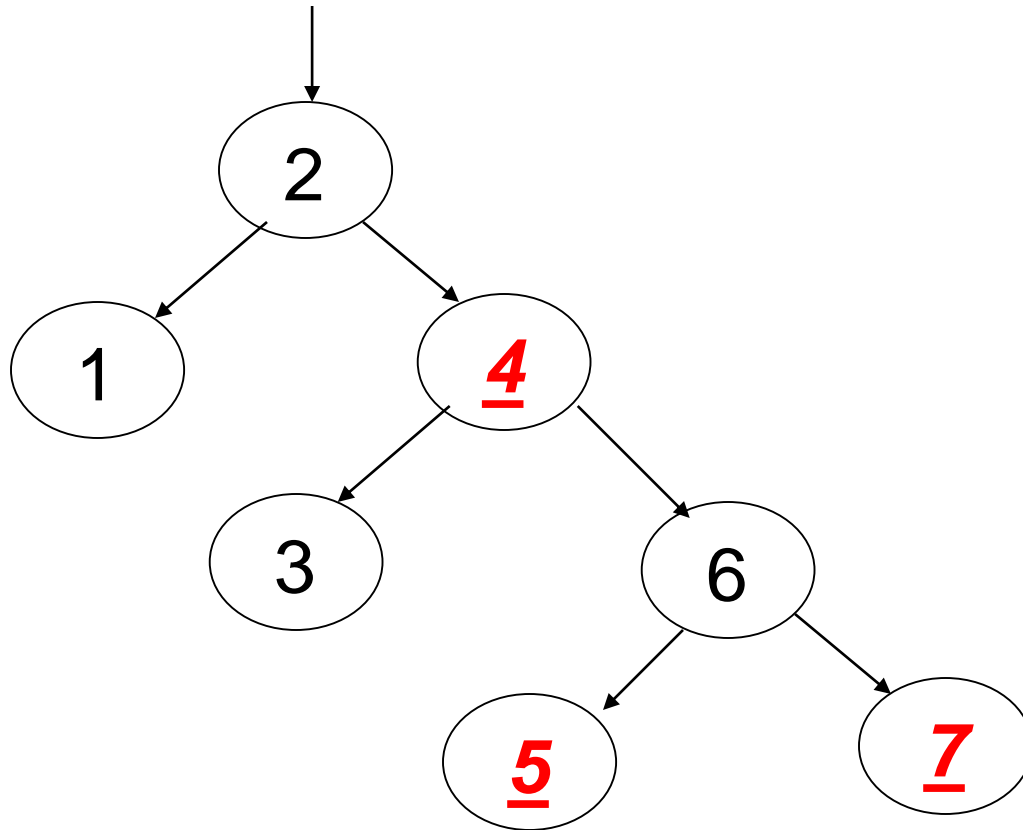


# Insert 7

7's parent is red.  
Parent's sibling is  
black (null). 7 is  
outside relative to  
grandparent (5) so  
rotate parent and  
grandparent then recolor

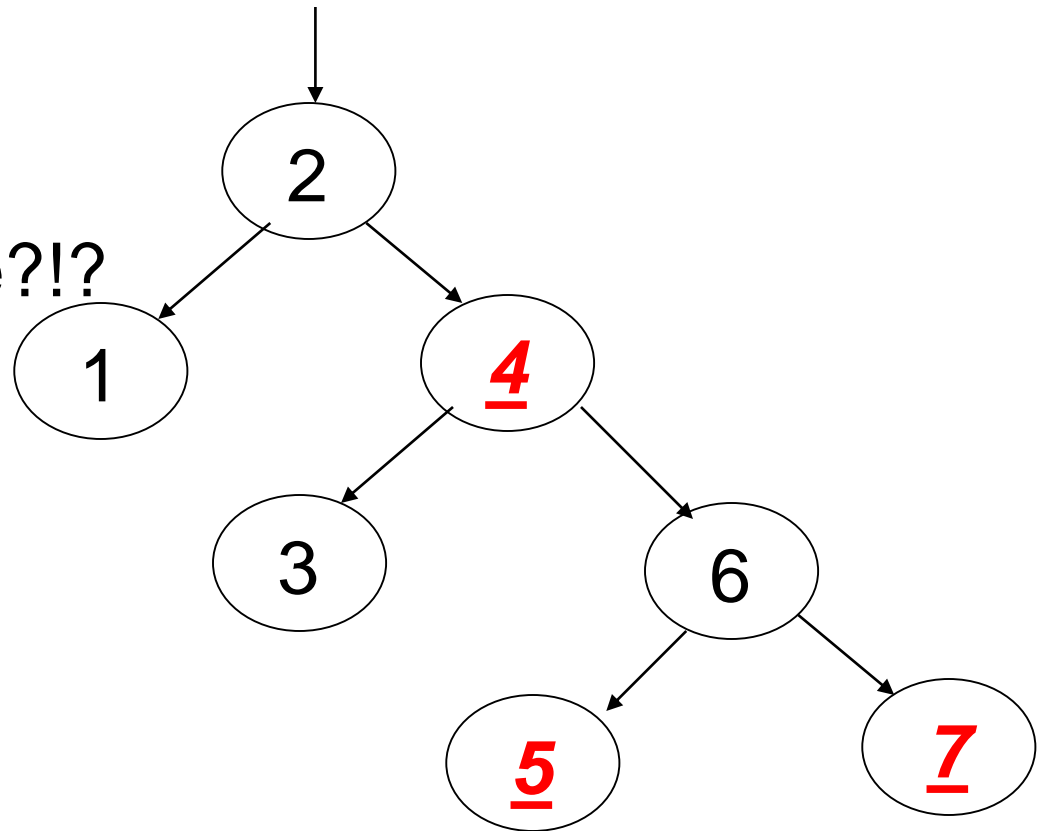


# Finish insert of 7



# Insert 8

The caveat!!!  
Getting unbalanced  
on that right subtree?!?

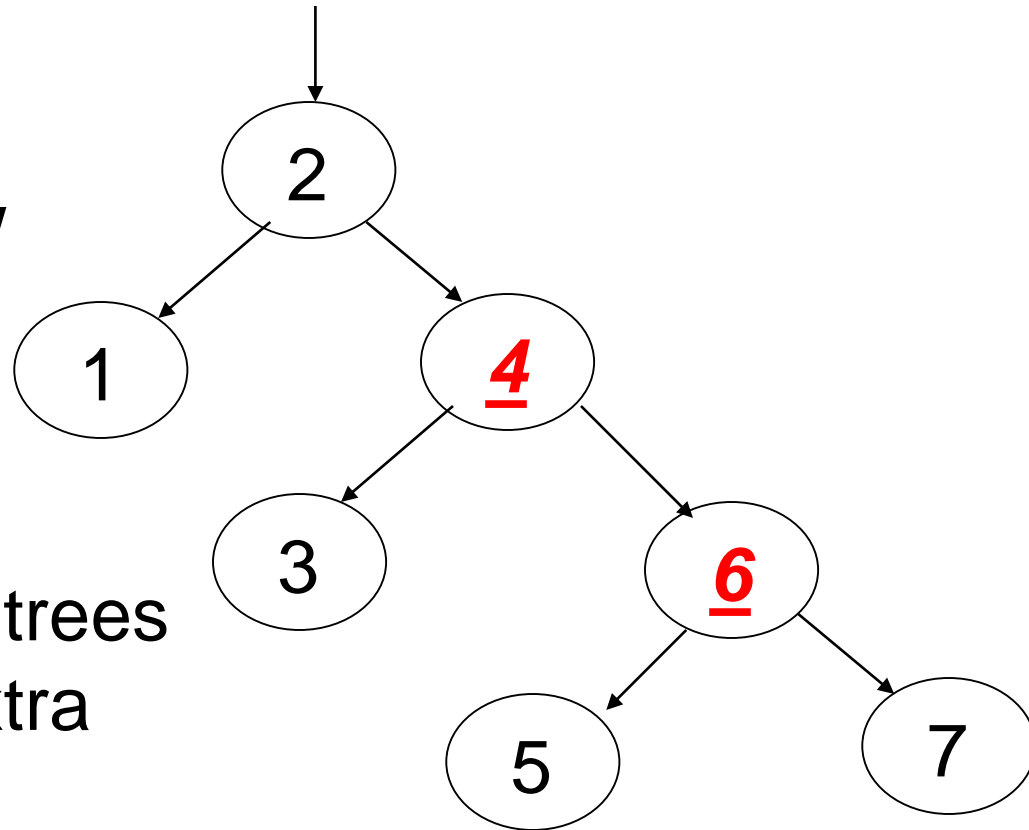


On way down see 6  
with 2 red children.  
Make 6 red and  
children black. This  
creates a problem  
because 6's parent, 4, is  
also red. Must perform  
rotation.

# Still Inserting 8

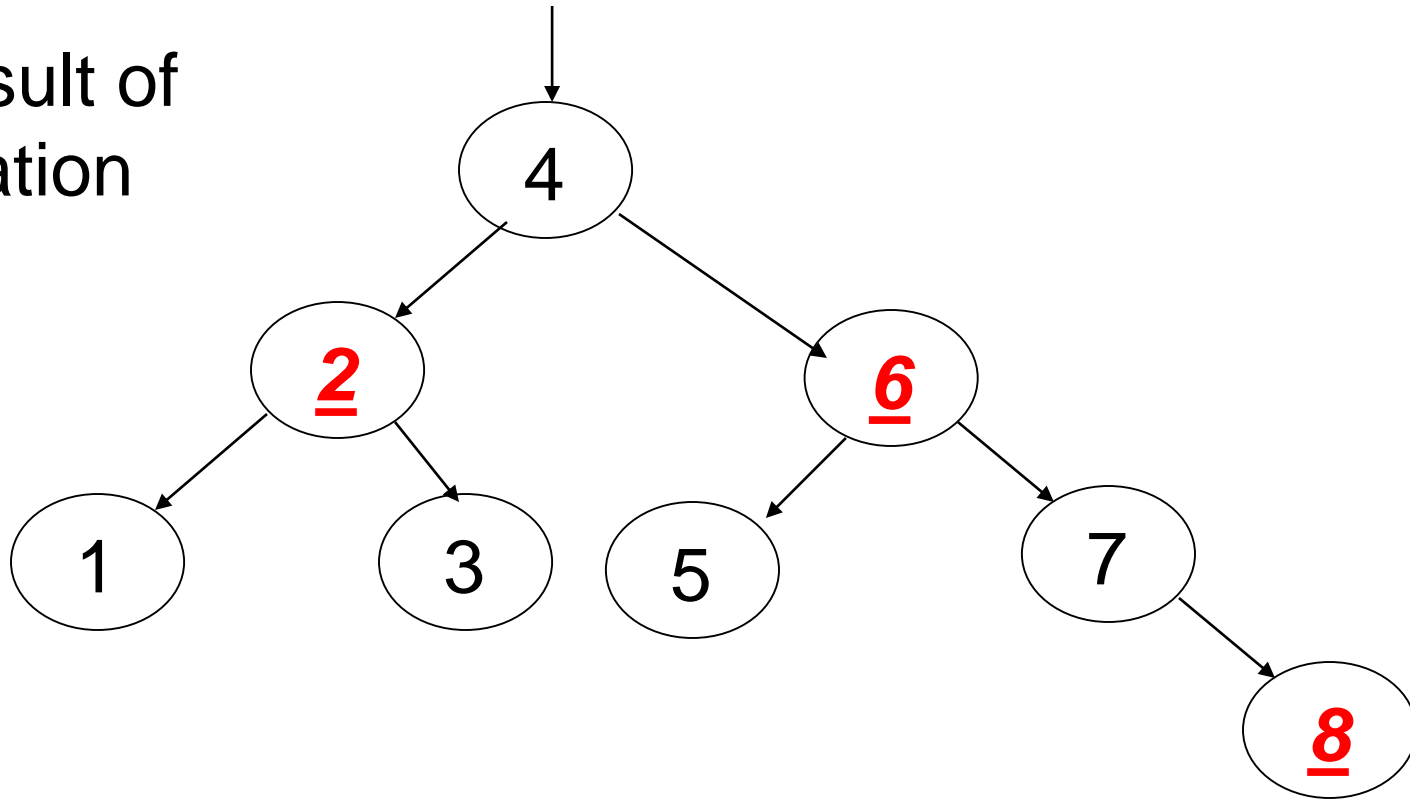
Recolored now  
need to  
rotate.

Recall, the subtrees  
and the one extra  
black node.

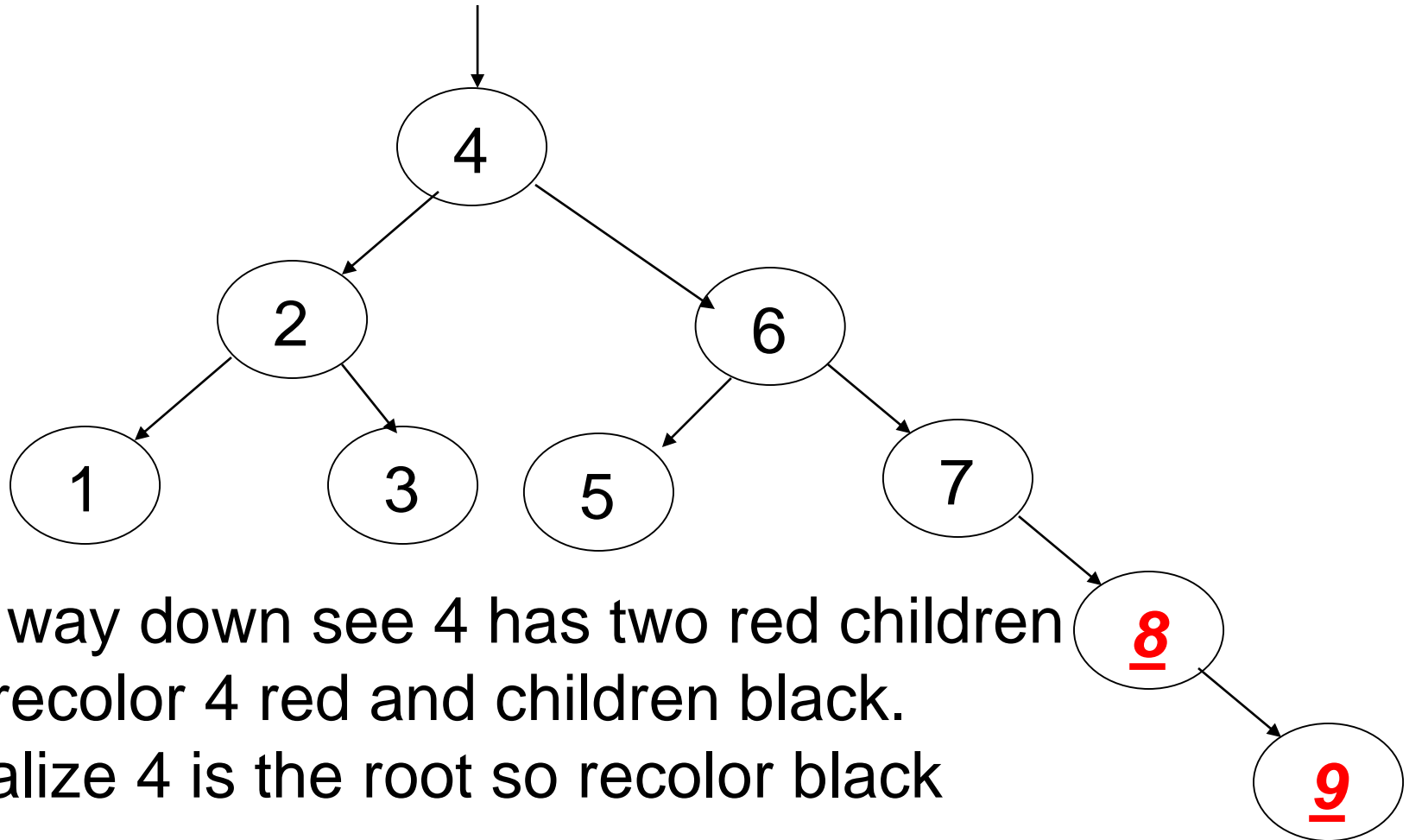


# Finish inserting 8

Result of rotation

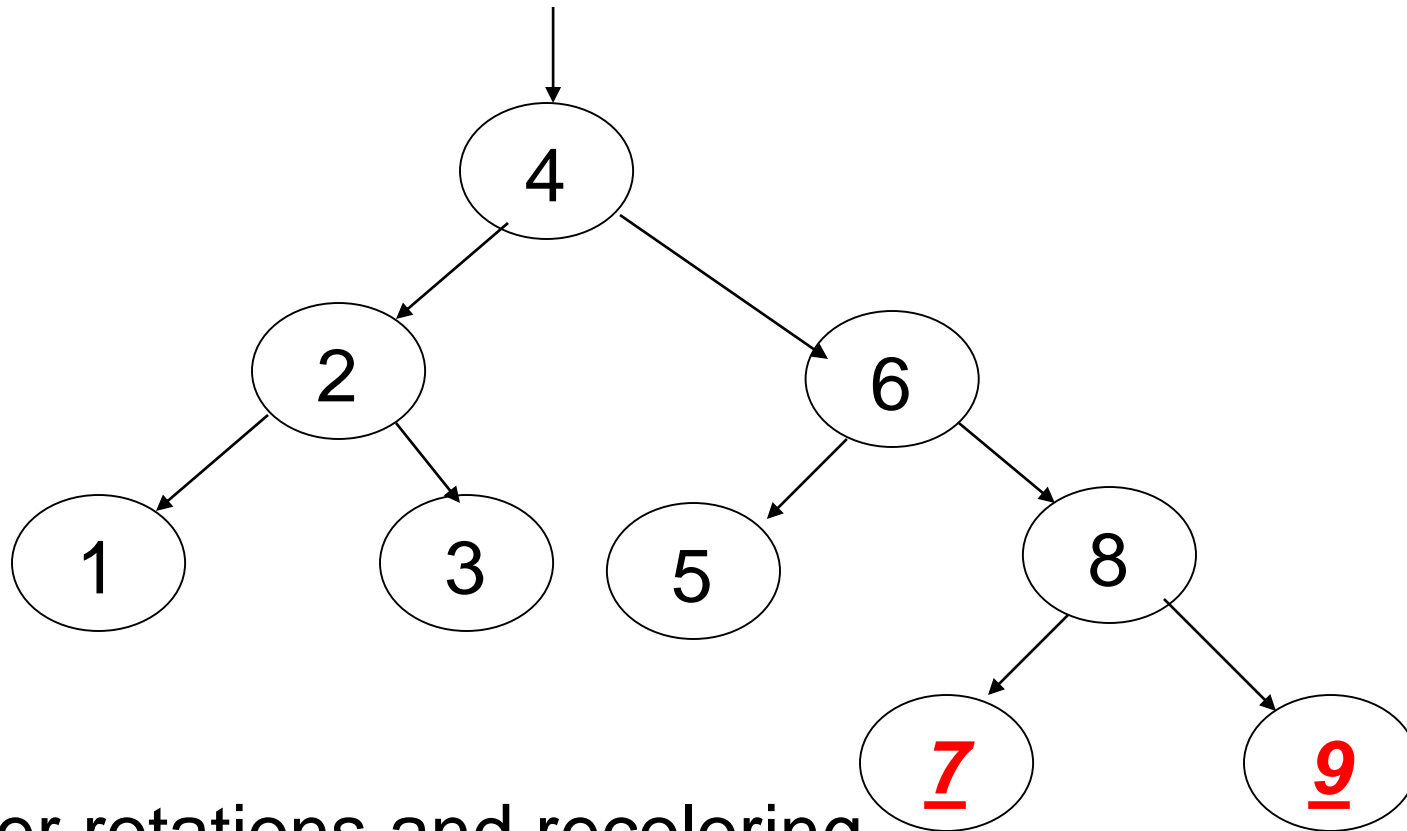


# Insert 9



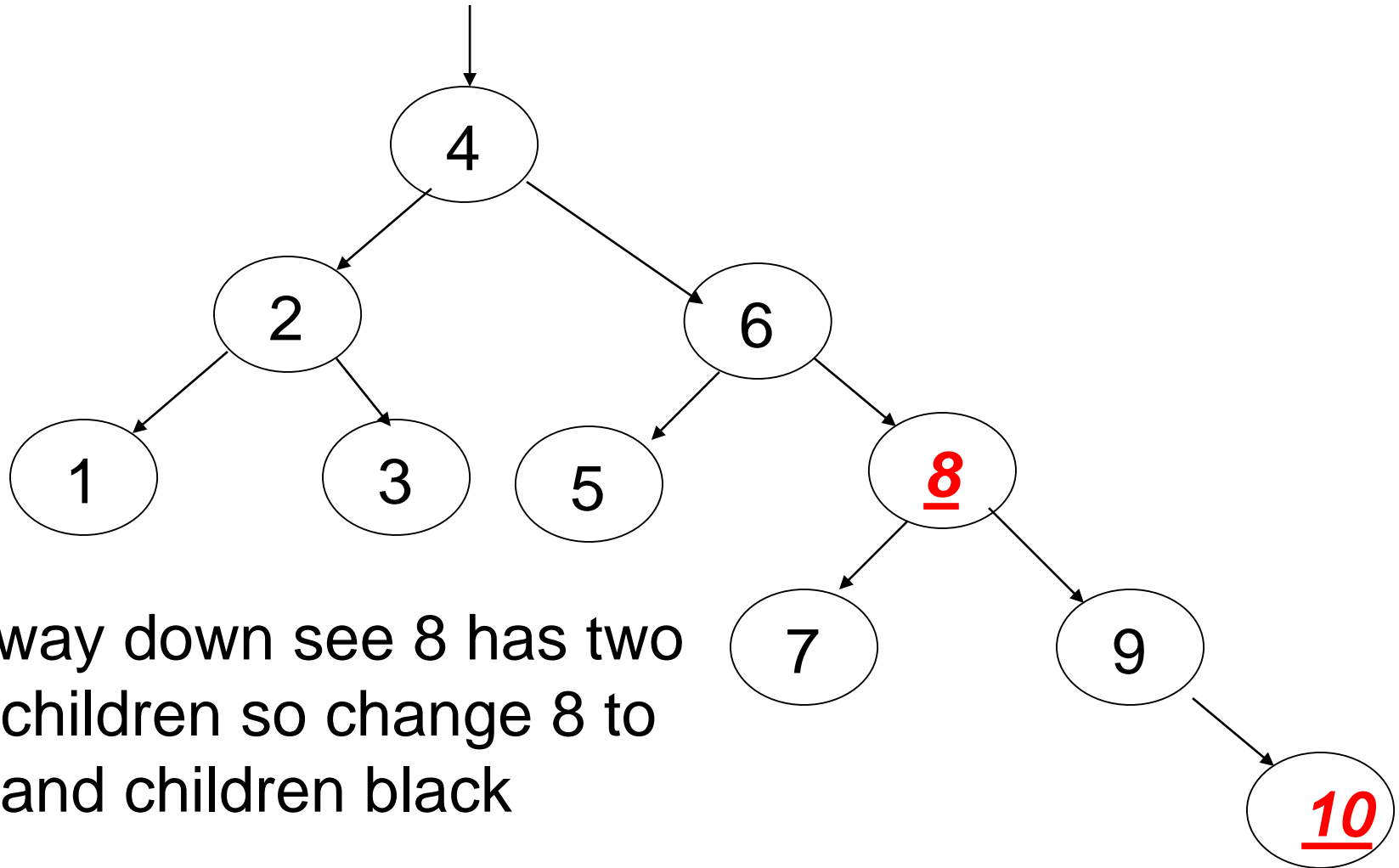


# Finish Inserting 9



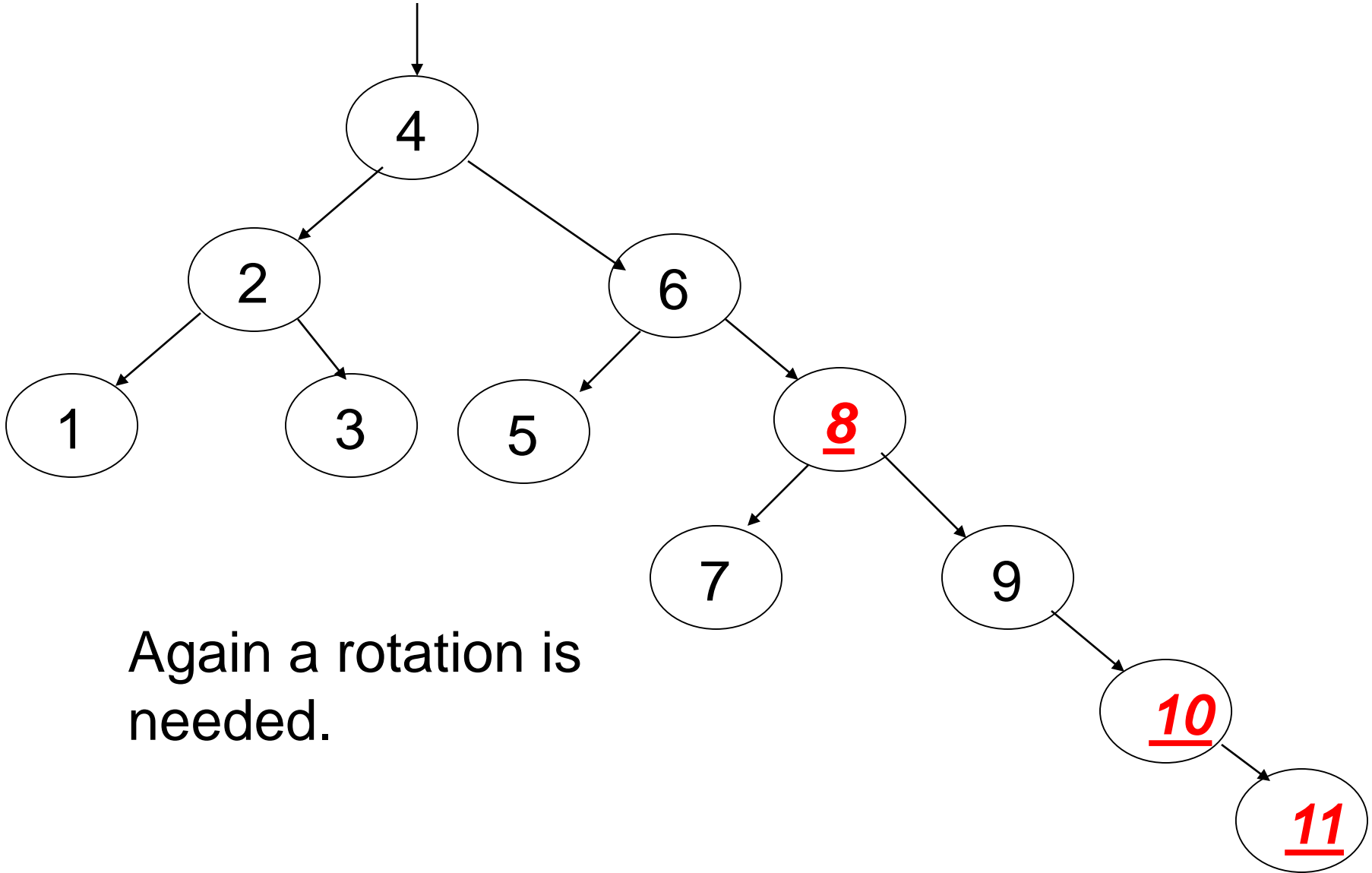
After rotations and recoloring

# Insert 10



On way down see 8 has two red children so change 8 to red and children black

# Insert 11



Again a rotation is needed.

# Finish inserting 11

