Vector and Affine Math II
Linear Transformations

Given vector space $V$ and $W$, function $f: V \rightarrow W$ is a linear map (linear transformation) if

$$f(a_1v_1 + \ldots + a_m v_m) = a_1 f(v_1) + \ldots + a_m f(v_m)$$
Transformations

A 2D transformation matrix: \[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

Applied to a 2D vector: \[ v' = Mv \]

\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

In which case: \[ x' = ax + by \]
\[ y' = cx + dy \]
Scaling

Suppose \( b = c = 0 \), but \( a \) and \( d \) can take on any positive value…

Scaling matrix:

\[
\begin{bmatrix}
  a & 0 \\
 0 & d
\end{bmatrix}
\]

What happens if \( a \) and \( d \) are not equal?
Reflection

Suppose $b = c = 0$, but either $a$ or $d$ goes negative

Reflection matrices:

\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Across which axes will each of these matrices reflect?
Shear

Suppose $a = d = 1$, but $b$ or $c$ changes value

Shear matrix:

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad x' = x + by$$

$$y' = y$$

Skews in one dimension in 2D

What does a shear do in 3D?
In-class Exercises

1. Create a 2D box with Euclidean coordinates. Now separately:
   1. Apply a uniform and non-uniform scaling to its vertices
   2. Apply reflection to its vertices
   3. Apply a shear to its vertices
2. Draw all of these transformations

**For all activities, show matrices**
Rotation

Rotation about the origin:

\[ M_R = R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[
\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ -\sin(\theta) \end{bmatrix}
\]
Linear Transformation Limitations

No notion of an origin

What important graphics operation does this leave out?
Affine Transformations

- Augment linear space \( u, w \) with an origin, \( t \)
- \( u \) and \( w \) are basis vectors
- \( t \) is a point
- A change of frame looks like:

\[
p' = x \cdot u + y \cdot w + t
\]

- How do you represent linear transformations within affine frames?
Homogeneous Coordinates

Loft problem into next dimension:

\[ p' = Mp \]
\[
\begin{bmatrix}
  a & b & t_x \\
  c & d & t_y \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix}
=\begin{bmatrix}
  u & w & t
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix}
= x \cdot u + y \cdot w + 1 \cdot t
\]

Note that \([a c 0]^T \text{ and } [b d 0]^T\) represent vectors and \([t_x t_y 1]^T, [x y 1]^T\) and \([x' y' 1]^T\) represent points.
In-class Exercises

1. Create a 2D box with Euclidean coordinates. Now separately:
   1. Apply a uniform and non-uniform scaling to its vertices
   2. Apply reflection to its vertices
   3. Apply a shear to its vertices
   4. Apply a translation then a rotation
   5. Apply a rotation then a translation

2. Draw all of these transformations

**For all activities, show matrices**
Rotation Around Arbitrary Points

1. Translate q to origin
2. Rotate
3. Translate back

Note that transformation order matters!
Additional Concepts

- Parametric Line Segments
- Plane Equation
- Barycentric Coordinates

All core concepts for working with raytracing! (Assignment 1)
Parametric Line Segment

Linear interpolation along a line, ray or line segment:

\[ p(t) = p_0 + t(p_1 - p_0) = (1 - t)p_0 + tp_1 \]

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix}
  x_0 \\
  y_0 \\
  z_0
\end{bmatrix} + t \begin{bmatrix}
  x_1 - x_0 \\
  y_1 - y_0 \\
  z_1 - z_0
\end{bmatrix} = \begin{bmatrix}
  (1-t)x_0 + tx_1 \\
  (1-t)y_0 + ty_1 \\
  (1-t)z_0 + tz_1
\end{bmatrix}
\]

Line segment: \( 0 \leq t \leq 1 \)
Ray: \( 0 \leq t \leq \infty \)
Line: \( -\infty \leq t \leq \infty \)
Plane Equation

Given normal vector \( N \) orthogonal to the plane and any point \( p' \) in the plane, \( p \) is in plane if:

\[
(p - p') \cdot N = 0
\]

This can be rewritten:

\[
N \cdot p + d = 0
\]

Where

\[
d = - (N_x p'_x + N_y p'_y + N_z p'_z)
\]
Plane Equation

\[ N \cdot p + d = 0 \]

\[
\begin{bmatrix}
a & b & c \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} + d = ax + by + cz + d = 0
\]
Triangle Normal

\[ N = \text{normalize}( (v_1 - v_0) \times (v_2 - v_0) ) \]

Note: Order matters to point the normal in “front-facing” direction (CCW for right-handed systems)
Barycentric Coordinates

A set of points can be used to create an affine frame

Form a frame with an origin C and vectors from C to other vertices: \( \mathbf{u} = \mathbf{A} - \mathbf{C} \quad \mathbf{v} = \mathbf{B} - \mathbf{C} \quad \mathbf{t} = \mathbf{C} \)

Write \( \mathbf{p} \) in this coordinate frame: \( \mathbf{p} = \alpha \mathbf{u} + \beta \mathbf{v} + \mathbf{t} \)

Coordinates \((\alpha, \beta, \gamma)\) are called the barycentric coordinates of \( \mathbf{p} \) relative to \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \)
In-class Exercises

1. Create a 2D box with Euclidean coordinates. Now separately:
   1. Apply a uniform and non-uniform scaling to its vertices
   2. Apply reflection to its vertices
   3. Apply a shear to its vertices
   4. Apply a translation then a rotation
   5. Apply a rotation then a translation

2. Draw all of these transformations

3. Find points \( p' \) along an edge of this box using the parametric equation and \( t \) values 0.1, 0.4 and 0.7

**For all activities, show matrices**