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Deductive Inference for the Interiors and Exteriors of Horn Theories

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In this paper, we investigate deductive inference for interiors and exteriors of Horn knowledge bases, where interiors and exteriors were introduced by Makino and Ibaraki [Makino and Ibaraki 1996] to study stability properties of knowledge bases. We present a linear time algorithm for deduction for interiors and show that deduction is coNP-complete for exteriors. Under model-based representation, we show that the deduction problem for interiors is NP-complete while the one for exteriors is coNP-complete. As for Horn envelopes of exteriors, we show that it is linearly solvable under model-based representation, while it is coNP-complete under formula-based representation. We also discuss polynomially solvable cases for all the intractable problems.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems – Complexity of proof procedures, computations on discrete structures

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1. INTRODUCTION

Knowledge-based systems are commonly used to store the sentences as our knowledge for the purpose of having automated reasoning such as deduction applied to them (see e.g., [Brachman and Levesque 2004]). Deductive inference is a fundamental mode of reasoning, and usually abstracted as follows: Given a knowledge base KB, assumed to capture our knowledge about the domain in question, and a query χ that is assumed to capture the situation at hand, decide whether KB implies χ, denoted by $KB \models \chi$, which can be understood as the question: "Is χ necessarily true given the current state of knowledge?"

In this paper, we consider interiors and exteriors of knowledge bases. Formally, for a given positive integer α, α-interior of KB, denoted by $\sigma_{-\alpha}(KB)$, is a knowledge that consists of the models (or assignments) v satisfying that the α-neighbors of v are all models of KB, and α-exterior of KB, denoted by $\sigma_{\alpha}(KB)$, is a knowledge that consists of the models v satisfying that at least one of the α-neighbors of v is a model of

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KB [Makino and Ibaraki 1996]. Intuitively, the interior consists of the models v that *strongly* satisfy KB, since all neighbors of v are models of KB, while the exterior consists of the models v that *weakly* satisfy KB, since at least one of the α-neighbors of v is a model of KB. Here we note that v might not satisfy KB, even if we say that it weakly satisfies KB. As mentioned in [Makino and Ibaraki 1996], the interiors and exteriors of knowledge bases merit study in their own right, since they shed light on the structure of knowledge bases. Moreover, let us consider the situation in which knowledge base KB is *not perfect* in the sense that some sentences in KB are wrong and/or some are missing in KB (see also [Makino and Ibaraki 1996]).

Suppose that we use KB as a knowledge base for automated reasoning, say, deductive inference $KB \models \chi$. Since KB does not represent *real* knowledge KB^*, the reasoning result is no longer true. However, if we use the interior $\sigma_{-\alpha}(KB)$ of KB as a knowledge base and have $\sigma_{-\alpha}(KB) \not\models \chi$, then we can expect that the result is true for real knowledge KB^*, since $\sigma_{-\alpha}(KB)$ consists of models which strongly satisfy KB. On the other hand, if we use the exterior $\sigma_{\alpha}(KB)$ of KB as a knowledge base and have $\sigma_{\alpha}(KB) \models \chi$, then we can expect that the result is true for real knowledge KB^*, since $\sigma_{\alpha}(KB)$ consists of models which weakly satisfy KB. In this sense, the interiors and exteriors help to have *safe* reasoning.

In this paper, we restrict knowledge bases to be Horn. Note that Horn theories are ubiquitous in Computer Science, cf. [Makowsky 1987], and are of particular relevance in Artificial Intelligence and Databases. It is known that important reasoning problems like deductive inference and satisfiability checking, which are intractable for arbitrary propositional theories, are solvable in linear time for Horn theories (cf. [Dowling and Galliear 1983]). Because of these computational advantage of Horn theories, knowledge bases are sometimes approximated to Horn theories, even if the original knowledge base are not Horn [Kavvadias et al. 1993]. Thus it is important to study safe reasoning for Horn theories.

Main problems considered. In this paper, we study deductive inference for interiors and exteriors of propositional Horn theories. More precisely, we address the following problems:

- Given a Horn theory Σ, a clause c, and nonnegative integer α, we consider the problems of deciding if deductive queries hold for the α-interior and exterior of Σ, i.e., $\sigma_{-\alpha}(\Sigma) \models c$ and $\sigma_{\alpha}(\Sigma) \models c$. It is well-known [Dowling and Galliear 1983] that a deductive query for a Horn theory can be answered in linear time. Note that it is intractable to construct the interior and exterior for a Horn theory [Makino and Ibaraki 1996; Makino et al. 2003], and hence a direct method (i.e., first construct the interior (or exterior) and then check a deductive query) is not possible efficiently.
- We contrast traditional formula-based (syntactic) with model-based (semantic) representation of Horn theories. The latter form of representation has been proposed as an alternative form of representing and accessing a logical knowledge base, cf. [Dechter and Pearl 1992; Eiter et al. 1999; Eiter and Makino 2002; Kautz et al. 1993; 1995; Kavvadias et al. 1993; Khardon and Roth 1996; 1997]. In model-based reasoning, Σ is represented by a subset of its models $\mathcal{M}$, which are commonly called *characteristic models*. As shown by Kautz *et al.* [Kautz et al. 1993], deductive inference can be done in polynomial time, given its characteristic models.
- Finally, we consider Horn approximations for the exteriors of Horn theories. Note that the interiors of Horn theories are Horn, while the exteriors might not be Horn. We deal with the least upper bounds, called the *Horn envelopes* [Selman and Kautz 1991], for the exteriors of Horn theories.

	Interiors	Exteriors	Envelopes of Exteriors
Formula-Based	P	coNP-complete*	coNP-complete*
Model-Based	NP-complete†	coNP-complete‡	P

*: It is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(\log \|\Sigma\|)$.
†: It is polynomially solvable, if $\alpha = O(1)$.
‡: It is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(\log(n|char(\Sigma)|))$.

Fig. 1. Complexity of deduction for interiors and exteriors of Horn theories

Main results. We investigate the problems mentioned above from an algorithmical viewpoint. For all the problems, we provide either polynomial time algorithms or proofs of the intractability; thus, our work gives a complete picture of the tractability/intractability frontier of deduction for interiors and exteriors of Horn theories. Our main results can be summarized as follows (see Figure 1).

- We present a linear time algorithm for deduction for interiors of a given Horn theory, and show that it is coNP-complete for deduction for the exteriors. Thus, the positive result for ordinary deduction for Horn theories extends to the interiors, but does not to the exteriors. We also show that deduction for the exteriors is solvable in polynomial time, if α is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input size, where $N(c)$ corresponds to the set of negative literals in c.
- Under model-based representation, we show that the consistency problem and deduction for interiors of Horn theories are both coNP-complete. As for exteriors, we show that the deduction is coNP-complete. We also show that deduction for interiors is solvable in polynomial time if α is bounded by a constant, and so is for the exteriors, if α or $|P(c)|$ is bounded by a constant, or if $|N(c)|$ is bounded by a logarithm of the input size, where $P(c)$ corresponds to the set of positive literals in c.
- As for Horn envelopes of exteriors of Horn theories, we show that it is linearly solvable under model-based representation, while it is coNP-complete under formula-based representation. The former contrasts to the negative result for the exteriors. We also present a polynomial algorithm for formula-based representation, if α is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input size.

We remark that deduction for (envelopes of) exteriors for formula-based representations and exteriors for model-based representation is fixed-parameter tractable (FPT) with respect to $|N(c)|$.

The rest of the paper is organized as follows. In the next section, we review the basic concepts and fix notations. Sections 3 and 4 investigate deductive inference for the interiors and exteriors of Horn theories. Section 5 considers deductive inference for the envelopes of the exteriors of Horn theories.

2. PRELIMINARIES

Horn Theories

We assume a standard propositional language with atoms $At = \{x_1, x_2, \ldots, x_n\}$, where each x_i takes either value 1 (true) or 0 (false). A *literal* is either an atom x_i or its negation, which we denote by $\overline{x}_i$. The opposite of a literal ℓ is denoted by $\overline{\ell}$, and the opposite of a set of literals L by $\overline{L} = \{\overline{\ell} \mid \ell \in L\}$. Furthermore, $Lit = At \cup \overline{At}$ denotes the set of all literals.

A *clause* is a disjunction $c = \bigvee_{i \in P(c)} x_i \vee \bigvee_{i \in N(c)} \overline{x}_i$ of literals, where $P(c)$ and $N(c)$ are the sets of indices whose corresponding variables occur positively and negatively in c and $P(c) \cap N(c) = \emptyset$. Dually, a *term* is conjunction $t = \bigwedge_{i \in P(t)} x_i \wedge \bigwedge_{i \in N(t)} \overline{x}_i$ of literals, where $P(t)$ and $N(t)$ are similarly defined. We also view clauses and terms as sets of literals. A *conjunctive normal form* (*CNF*) is a conjunction of clauses. A clause c is *Horn*, if $|P(c)| \leq 1$. A *theory* Σ is any set of formulas; it is *Horn*, if it is a set of Horn clauses. As usual, we identify Σ with $\varphi = \bigwedge_{c\in\Sigma} c$, and write $c \in \varphi$, etc. It is known [Dowling and Galliear 1983] that the deductive problem for a Horn theory, i.e., deciding if $\Sigma \models c$ for a clause c is solvable in linear time.

We recall that Horn theories have a well-known semantic characterization. An *assignment* is a vector $v \in \{0,1\}^n$, whose i-th component is denoted by v_i. For an assignment v, let $ON(v) = \{i \mid v_i = 1\}$ and $OFF(v) = \{i \mid v_i = 0\}$. The value of a formula φ on an assignment v, denoted $\varphi(v)$, is inductively defined as usual; satisfaction of φ in v, i.e., $\varphi(v) = 1$, will be denoted by $v \models \varphi$. For a formula φ (resp., a theory Σ), an assignment v is called a *model* of φ (resp., Σ) if $\varphi(v) = 1$ (resp., $\bigwedge_{c\in\Sigma} c(v) = 1$). The set of models of a formula φ (resp., theory Σ), denoted by $mod(\varphi)$ (resp., $mod(\Sigma)$), and logical consequence $\varphi \models \psi$ (resp., $\Sigma \models \psi$) are defined as usual. For two assignments v and w, we denote by $v \leq w$ the usual componentwise ordering, i.e., $v_i \leq w_i$ for all $i = 1, 2, \ldots, n$, where $0 \leq 1$; $v < w$ means $v \neq w$ and $v \leq w$. Denote by $v \bigwedge w$ componentwise AND of assignments $v, w \in \{0,1\}^n$, and by $Cl_\wedge(\mathcal{M})$ the closure of $\mathcal{M} \subseteq \{0,1\}^n$ under $\bigwedge$. Then, a theory Σ is Horn representable if and only if $mod(\Sigma) = Cl_\wedge(mod(\Sigma))$ (see [Dechter and Pearl 1992; Khardon and Roth 1996]) for proofs).

EXAMPLE 2.1. Consider $\mathcal{M}_1 = \{(0101), (1001), (1000)\}$ and $\mathcal{M}_2 = \{(0101), (1001), (1000), (0001), (0000)\}$. Then, for $v = (0101)$, $w = (1000)$, we have $w, v \in \mathcal{M}_1$, while $v \bigwedge w = (0000) \notin \mathcal{M}_1$; hence $\mathcal{M}_1$ is not the set of models of a Horn theory. On the other hand, $Cl_\wedge(\mathcal{M}_2) = \mathcal{M}_2$, thus $\mathcal{M}_2 = mod(\Sigma_2)$ for some Horn theory Σ_2.

As discussed by Kautz *et al.* [Kautz et al. 1993], a Horn theory Σ is semantically represented by its characteristic models, where $v \in mod(\Sigma)$ is called *characteristic* (or *extreme* [Dechter and Pearl 1992]), if $v \notin Cl_\wedge(mod(\Sigma) \setminus \{v\})$. The set of all such models, the *characteristic set of* Σ, is denoted by $char(\Sigma)$. Note that $char(\Sigma)$ is unique. E.g., $(0101) \in char(\Sigma_2)$, while $(0000) \notin char(\Sigma_2)$; we have $char(\Sigma_2) = \mathcal{M}_1$. It is known [Kautz et al. 1993] that the deductive query for a Horn theory Σ from the characteristic set $char(\Sigma)$ can be done in linear time, i.e., $O(n|char(\Sigma)|)$ time.

Interior and Exterior of Theories

For an assignment $v \in \{0,1\}^n$ and a nonnegative integer α, its *α-neighborhood* is defined by

$$\mathcal{N}_\alpha(v) = \{w \in \{0,1\}^n \mid \ |ON(w) \bigtriangleup ON(v)| \leq \alpha\},$$

where $\bigtriangleup$ denotes the symmetric difference operator. Note that $|ON(w) \bigtriangleup ON(v)|$ denotes the Hamming distance between w and v, and $|\mathcal{N}_\alpha(v)| = \sum_{i=0}^{\alpha}\binom{n}{i} = O(n^{\alpha+1})$. For a theory Σ and a nonnegative integer α, α-interior and α-exterior of Σ, denoted by $\sigma_{-\alpha}(\Sigma)$ and $\sigma_\alpha(\Sigma)$ respectively, are theories defined by

$$mod(\sigma_{-\alpha}(\Sigma)) = \{v \in \{0,1\}^n \mid \mathcal{N}_\alpha(v) \subseteq mod(\Sigma)\} \quad (1)$$

$$mod(\sigma_\alpha(\Sigma)) = \{v \in \{0,1\}^n \mid \mathcal{N}_\alpha(v) \cap mod(\Sigma) \neq \emptyset\}. \quad (2)$$

By definition, $\sigma_0(\Sigma) = \Sigma$, $\sigma_\alpha(\Sigma) \models \sigma_\beta(\Sigma)$ for integers α and β with $\alpha < \beta$, and $\sigma_\alpha(\Sigma_1) \models \sigma_\alpha(\Sigma_2)$ holds for any integer α, if two theories Σ_1 and Σ_2 satisfy $\Sigma_1 \models \Sigma_2$.

EXAMPLE 2.2. Let us consider a Horn theory $\Sigma = \{\overline{x}_1 \vee x_3, \overline{x}_2 \vee x_3, \overline{x}_2 \vee x_4\}$ of 4 variables, where $mod(\Sigma)$ is given by

$$mod(\Sigma) = \{(1111), (1011), (1010), (0111), (0011), (0010), (0001), (0000)\}$$

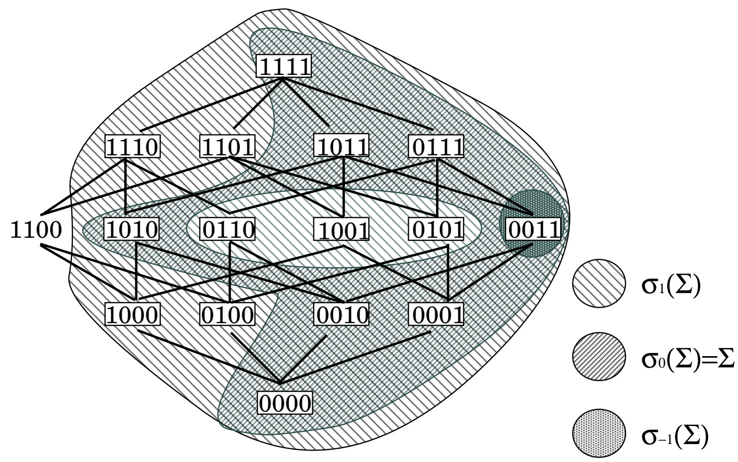

Fig. 2. A Horn theory and its interiors and exteriors

(See Figure 2). Then we have $\sigma_\alpha(\Sigma) = \{\emptyset\}$ for $\alpha \leq -2$, $\{\overline{x}_1, \overline{x}_2, x_3, x_4\}$ for $\alpha = -1$, Σ for $\alpha = 0$, $\{\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee x_4\}$ for $\alpha = 1$, and $\emptyset$ for $\alpha \geq 2$. For example, (0011) is the unique model of $mod(\sigma_{-1}(\Sigma))$, since $\mathcal{N}_1(0011) \subseteq mod(\Sigma)$ and $\mathcal{N}_1(v) \not\subseteq mod(\Sigma)$ holds for all the other assignments v. For the 1-exterior, we can see that all assignments v with $(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee x_4)(v) = 1$ satisfy $\mathcal{N}_1(v) \cap mod(\Sigma) \neq \emptyset$, and no other such assignment exists. For example, (0101) is a model of $\sigma_1(\Sigma)$, since $(0111) \in \mathcal{N}_1(0101) \cap mod(\Sigma)$. On the other hand, (1100) is not a model of $\sigma_1(\Sigma)$, since $\mathcal{N}_1(1100) \cap mod(\Sigma) = \emptyset$. Notice that $\sigma_{-1}(\Sigma)$ is Horn, while $\sigma_1(\Sigma)$ is not.

Makino and Ibaraki [Makino and Ibaraki 1996] introduced the interiors and exteriors to analyze stability of Boolean functions, and studied their basic properties and complexity issues on them (see also [Makino et al. 2003]). For example, it is known [Makino and Ibaraki 1996] that, for a theory Σ and nonnegative integers α and β, $\sigma_{-\alpha}(\sigma_{-\beta}(\Sigma)) = \sigma_{-\alpha-\beta}(\Sigma)$, $\sigma_\alpha(\sigma_\beta(\Sigma)) = \sigma_{\alpha+\beta}(\Sigma)$, and

$$\sigma_\alpha(\sigma_{-\beta}(\Sigma)) \;\models\; \sigma_{\alpha-\beta}(\Sigma) \;\models \sigma_{-\beta}(\sigma_\alpha(\Sigma)). \tag{3}$$

For a nonnegative integer α and two theories Σ_1 and Σ_2, we have

$$\sigma_{-\alpha}(\Sigma_1 \cup \Sigma_2) \;=\; \sigma_{-\alpha}(\Sigma_1) \cup \sigma_{-\alpha}(\Sigma_2) \tag{4}$$

$$\sigma_\alpha(\Sigma_1 \cup \Sigma_2) \;\models\; \sigma_\alpha(\Sigma_1) \cup \sigma_\alpha(\Sigma_2), \tag{5}$$

where $\sigma_\alpha(\Sigma_1 \cup \Sigma_2) \neq \sigma_\alpha(\Sigma_1) \cup \sigma_\alpha(\Sigma_2)$ holds in general.

As demonstrated in Example 2.2, it is not difficult to see that interiors of any Horn theory are Horn, which is, for example, proved by (4) and Lemma 3.1, while the exteriors might be not Horn.

3. DEDUCTIVE INFERENCE FROM HORN THEORIES

In this section, we investigate deductive inference for the interiors and exteriors of a given Horn theory.

3.1. Interiors

Let us first consider deduction for α-interiors of a Horn theory: Given a Horn theory Σ, a clause c, and a positive integer α, decide if $\sigma_{-\alpha}(\Sigma) \models c$ holds. We show that the problem is solvable in linear time after showing a series of lemmas.

The following lemma is a basic property of interiors of a theory, where we regard c as a set of literals.

LEMMA 3.1. *Let c be a clause. Then for a nonnegative integer α, we have $\sigma_{-\alpha}(c) = \bigvee_{\substack{S\subseteq c:\\ |S|=\alpha+1}} (\bigwedge_{\ell\in S} \ell) = \bigwedge_{\substack{S\subseteq c:\\ |S|=|c|-\alpha}} (\bigvee_{\ell\in S} \ell)$ if $\alpha < |c|$, and 0 (i.e., always false), otherwise.*

For example, let us consider $c = x_1 \vee x_2 \vee \overline{x}_3 \vee \overline{x}_4$, $\alpha = 2$. Then we have $\sigma_{-\alpha}(c) = x_1x_2\overline{x}_3 \vee x_1x_2\overline{x}_4 \vee x_1\overline{x}_3\overline{x}_4 \vee x_2\overline{x}_3\overline{x}_4 = (x_1 \vee x_2)(x_1 \vee \overline{x}_3)(x_1 \vee \overline{x}_4)(x_2 \vee \overline{x}_3)(x_2 \vee \overline{x}_4)(\overline{x}_3 \vee \overline{x}_4)$.

This lemma, together with (4), implies that for a CNF φ and a nonnegative integer α, we have

$$\sigma_{-\alpha}(\varphi) = \bigwedge_{c\in\varphi}\Big(\bigvee_{\substack{S\subseteq c:\\ |S|=\alpha+1}} (\bigwedge_{\ell\in S} \ell)\Big) = \bigwedge_{c\in\varphi}\Big(\bigwedge_{\substack{S\subseteq c:\\ |S|=|c|-\alpha}} (\bigvee_{\ell\in S} \ell)\Big),$$

if all $c \in \varphi$ satisfy $\alpha < |c|$, and 0 (i.e., always false), otherwise.

LEMMA 3.2. *Let Σ be a Horn theory, and let c be a clause. For a nonnegative integer α, if there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N(c)| \leq \alpha - 1$ or $(|N(d) \setminus N(c)| = \alpha$ and $P(d) \subseteq P(c))$, then we have $\sigma_{-\alpha}(\Sigma) \models c$.*

PROOF. If Σ has a clause d such that $|N(d) \setminus N(c)| \leq \alpha - 1$, then $|(N(d) \setminus N(c)) \cup P(d)| \leq \alpha$ holds. Thus by Lemma 3.1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i\in N(c)\cap N(d)} \overline{x}_i \models c$. Therefore, by (4), $\sigma_{-\alpha}(\Sigma) \models c$ holds.

On the other hand, if Σ has a clause d such that $|N(d) \setminus N(c)| = \alpha$ and $P(d) \subseteq P(c)$, then by Lemma 3.1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i\in P(d)} x_i \vee \bigvee_{i\in N(c)\cap N(d)} \overline{x}_i \models c$. Therefore, by (4), $\sigma_{-\alpha}(\Sigma) \models c$ holds. □

LEMMA 3.3. *Let Σ be a Horn theory, and let c be a clause. For a nonnegative integer α, if* (i) *$|N(d) \setminus N(c)| \geq \alpha$ holds for all $d \in \Sigma$ and* (ii) *$\emptyset \neq P(d) \subseteq N(c)$ holds for all $d \in \Sigma$ with $|N(d) \setminus N(c)| = \alpha$, then we have $\sigma_{-\alpha}(\Sigma) \not\models c$.*

PROOF. Let v be the unique minimal assignment that does not satisfy c, i.e., $v_i = 1$ if $\overline{x}_i \in c$ and 0, otherwise. We show that $v \models \sigma_{-\alpha}(\Sigma)$, which implies $\sigma_{-\alpha}(\Sigma) \not\models c$.

Let d be a clause in Σ with $|N(d) \setminus N(c)| \geq \alpha + 1$, and let t be a term obtained by conjuncting arbitrary $\alpha + 1$ literals in $N(d) \setminus N(c)$. Then we have $t(v) = 1$ and $t \models \sigma_{-\alpha}(d)$ by Lemma 3.1. On the other hand, for a clause d in Σ with $|N(d) \setminus N(c)| = \alpha$, let t be a term obtained by conjuncting all literals in $(N(d) \setminus N(c)) \cup P(d)$. Then we have $|t| = \alpha + 1$ and $t \models \sigma_{-\alpha}(d)$ by Lemma 3.1. Moreover, it holds that $t(v) = 1$ by $P(d) \subseteq N(c)$. Therefore, by (4), we have $v \models \sigma_{-\alpha}(\Sigma)$. □

By Lemmas 3.2 and 3.3, we can easily answer the deductive queries, if Σ satisfies certain conditions mentioned in them. In the remaining case, we have the following lemma.

LEMMA 3.4. *For a Horn theory Σ that satisfies none of the conditions in Lemmas* 3.2 *and* 3.3*, let d be a clause in Σ such that $|N(d) \setminus N(c)| = \alpha$, and $P(d) = P(d) \setminus (P(c) \cup N(c)) = \{j\}$. Then $\sigma_{-\alpha}(\Sigma) \models c \vee x_j$ holds.*

PROOF. By Lemma 3.1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i\in N(c)\cap N(d)} \overline{x}_i \vee x_j \models c \vee x_j$. This implies $\sigma_{-\alpha}(\Sigma) \models c \vee x_j$ by (4). □

From this lemma, we have only to check a deductive query $\sigma_{-\alpha}(\Sigma) \models c \vee \overline{x}_j$, instead of $\sigma_{-\alpha}(\Sigma) \models c$. Since $|c| < |c \vee \overline{x}_j| \leq n$, we can answer the deduction by checking the conditions in Lemmas 3.2 and 3.3 at most n times. Formally, this procedure is described as Algorithm 1 below.

Algorithm 1 DEDUCTION-INTERIOR-FROM-HORN-THEORY

Input:. A Horn theory Σ, a clause c and a nonnegative integer α.

Output:. Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

Step 0.. Let $N := N(c)$ and $P := P(c)$.

Step 1.. /* Check the condition in Lemma 3.2. */
If there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N| \leq \alpha - 1$ or ($|N(d) \setminus N| = \alpha$ and $P(d) \subseteq P$), **then** output Yes and halt.

Step 2.. /* Check the condition in Lemma 3.3. */
If $P(d) \subseteq N$ holds for all $d \in \Sigma$ with $|N(d) \setminus N| = \alpha$, **then** output No and halt.

Step 3.. /* Update N by Lemma 3.4. */
For a clause d in Σ such that $|N(d) \setminus N| = \alpha$ and $P(d) = P(d) \setminus (P \cup N) = \{j\}$, update $N := N \cup \{j\}$ and return to Step 1. □

We can see that a straightforward implementation of the algorithm requires $O(n(\|\Sigma\| + |c|))$ time, where $\|\Sigma\|$ denotes the length of Σ, i.e., $\|\Sigma\| = \sum_{d \in \Sigma} |d|$. The running time can be improved by properly maintaining $N(d) \setminus N$ for all $d \in \varphi$. For each variable x_i, we prepare a pointer to clause d with $i \in N(d)$. Then if N is updated to $N := N \cup \{j\}$, then $N(d) \setminus N$ can also be updated in $O(1)$ time for each d with $i \in N(d)$. This means that the total time to update $N(d) \setminus N$ for all $d \in \varphi$ is linear in $\|\Sigma\|$. Moreover, if we additionally keep the size of $N(d) \setminus N$ for each $d \in \varphi$, the conditions in Steps 1 and 2 can be checked in $O(1)$ time. Thus Algorithm 1 requires $O(\|\Sigma\| + |c|)$ time.

THEOREM 3.5. *The problem of deciding, given a Horn theory Σ, a clause c and a nonnegative integer α, whether $\sigma_{-\alpha}(\Sigma) \models c$ holds can be solved in linear time, i.e., $O(\|\Sigma\| + |c|)$ time.*

EXAMPLE 3.6. Let us consider a clause $c = x_1 \vee x_2 \vee \bar{x}_3 \vee x_4 \vee x_5$ and a Horn theory $\Sigma = \{d_1 = \bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4 \vee x_6, d_2 = \bar{x}_1 \vee x_2 \vee \bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_6\}$. We apply Algorithm 1 to c, Σ, and $\alpha = 2$. Step 0 initializes $P = \{1, 2, 4, 5\}$ and $N = \{3\}$. In Step 1, we have $N(d_1) \setminus N = \{1, 4\}, P(d_1) \not\subseteq P$ and $N(d_2) \setminus N = \{1, 4, 6\}$; no clause d in Σ satisfies $|N(d) \setminus N| \leq \alpha - 1\,(= 1)$ or ($|N(d) \setminus N| = \alpha\,(= 2)$ and $P(d) \subseteq P$). In Step 2, d_1 satisfies $|N(d_1) \setminus N| = 2$ and $P(d_1) \not\subseteq N$. Step 3 updates $N := N \cup P(d_1) = \{3, 6\}$ and returns to Step 1. In Step 1, we have $N(d_2) \setminus N = \{1, 4\}$ (i.e., $|N(d_2) \setminus N| = 2$) and $P(d_2) = \{2\} \subseteq P$, and hence we output "Yes" and halt. This answer can be also verified by $\sigma_{-2}(\Sigma) = (\bar{x}_1 \vee \bar{x}_3)(\bar{x}_1 \vee \bar{x}_4)(\bar{x}_3 \vee \bar{x}_4)(\bar{x}_1 \vee x_2)(\bar{x}_3 \vee x_2)(\bar{x}_4 \vee x_2)(\bar{x}_1 \vee x_6)(\bar{x}_3 \vee x_6)(\bar{x}_4 \vee x_6)$, which is obtained by Lemma 3.1 and a few resolution steps.

3.2. Exteriors

Let us next consider deduction for α-exteriors of a Horn theory. In contrast to the interior case, we have the following negative result.

THEOREM 3.7. *The problem of deciding, given a Horn theory Σ, a clause c and a positive integer α, whether $\sigma_{\alpha}(\Sigma) \models c$ holds is* coNP*-complete, even if $P(c) = \emptyset$ and Σ is restricted to be both negative and bijunctive.*

PROOF. By definition, $\sigma_\alpha(\Sigma) \not\models c$ if and only if there exists a model v of Σ such that some assignment in $\mathcal{N}_\alpha(v)$ does not satisfy c. The latter is equivalent to the condition that there exists a model v of Σ such that $|ON(v) \cap P(c)| + |OFF(v) \cap N(c)| \leq \alpha$, which can be checked in polynomial time. Thus the problem is in coNP.

We then show the hardness by reducing a well-known NP-complete problem INDEPENDENT SET to the complement of our problem. INDEPENDENT SET is the problem of deciding if a given graph $G = (V, E)$ has an independent set $W \subseteq V$ such that $|W| \geq k$ for a given integer k. Here we call a subset $W \subseteq V$ an *independent set* of G if $|W \cap e| \leq 1$ for all edges $e \in E$. For a problem instance $G = (V = \{1, 2, \ldots, n\}, E)$ and k of INDEPENDENT SET, let us define a Horn theory Σ_G over $At = \{x_1, x_2, \ldots, x_n\}$ by

$$\Sigma_G = \{(\bar{x}_i \vee \bar{x}_j) \mid \{i, j\} \in E\}.$$

Let $c = \bigvee_{i=1}^n \overline{x}_i$ and $\alpha = n - k$. Note that $(11 \cdots 1)$ is the unique assignment that does not satisfy c. Thus $\sigma_\alpha(\Sigma) \not\models c$ if and only if $\sigma_\alpha(\Sigma)(11 \cdots 1) = 1$. Since W is an independent set of G if and only if Σ_G contains a model w defined by $ON(w) = W$, $\sigma_\alpha(\Sigma_G)(11 \cdots 1) = 1$ is equivalent to the condition that G has an independent set of size at least $k (= n - \alpha)$. This completes the proof. □

We remark that this result can also be derived from the ones in [Makino and Ibaraki 1996].

However, by using the next lemma, a deductive query can be answered in polynomial time, if α or $N(c)$ is small.

LEMMA 3.8. *Let Σ_1 and Σ_2 be theories. For a nonnegative integer α, Then $\sigma_\alpha(\Sigma_1) \models \Sigma_2$ if and only if $\Sigma_1 \models \sigma_{-\alpha}(\Sigma_2)$.*

PROOF. For the if part, if $\Sigma_1 \models \sigma_{-\alpha}(\Sigma_2)$, then we have $\sigma_\alpha(\Sigma_1) \models \sigma_\alpha(\sigma_{-\alpha}(\Sigma_2)) \models \Sigma_2$ by (3). On the other hand, if $\sigma_\alpha(\Sigma_1) \models \Sigma_2$, then we have $\Sigma_1 \models \sigma_{-\alpha}(\sigma_\alpha(\Sigma_1)) \models \sigma_{-\alpha}(\Sigma_2)$ by (3). □

From Lemma 3.8, the deductive query for the α-exterior of a theory Σ, i.e., $\sigma_\alpha(\Sigma) \models c$ for a given clause c is equivalent to the condition that $\Sigma \models \sigma_{-\alpha}(c)$. Since we have $\sigma_{-\alpha}(c) = \bigwedge_{\substack{S \subseteq c: \\ |S| = |c| - \alpha}} \left(\bigvee_{\ell \in S} \ell\right)$ by Lemma 3.1, the deductive query for the α-interior can be done by checking $\binom{|c|}{\alpha}$ deductions for Σ. More precisely, we have the following lemma.

LEMMA 3.9. *Let Σ be a Horn theory, let c be a clause, and α be a nonnegative integer. Then $\sigma_\alpha(\Sigma) \models c$ holds if and only if, for each subset S of $N(c)$ such that $|S| \geq |N(c)| - \alpha$, at least $(\alpha - |N(c)| + |S| + 1)$ j's in $P(c)$ satisfy $\Sigma \models \bigvee_{i \in S} \overline{x}_i \vee x_j$.*

PROOF. From Lemmas 3.1 and 3.8, $\sigma_\alpha(\Sigma) \models c$ if and only if $\Sigma \models \bigwedge_{\substack{d \subseteq c: \\ |d| = |c| - \alpha}} d = \bigwedge_{\substack{S \subseteq N(c): \\ |S| \geq |N(c)| - \alpha}} \Phi_S$, where

$$\Phi_S = \bigwedge_{\substack{N(d) = S \\ P(d) \subseteq P(c) \\ |P(d)| = |c| - \alpha - |S|}} d.$$

It is known that for a Horn theory Σ and clause d, $\Sigma \models d$ if and only if $\Sigma \models \bigvee_{i \in N(d)} \overline{x}_i \vee x_j$ holds for some $j \in P(d)$ (i.e., all the prime implicates of Horn theory are Horn). Therefore, for each $S \subseteq N(c)$ with $|S| \geq |N(c)| - \alpha$, $\Sigma \models \Phi_S$ if and only if there exists a set $S' \subseteq P(c)$ such that $|S'| = \alpha - |N(c)| + |S| + 1$ and $\Sigma \models \bigvee_{i \in S} \overline{x}_i \vee x_j$ holds for all $j \in S'$. This proves the lemma. □

This lemma implies that the deductive query can be answered by checking the number of j's in $P(c)$ that satisfy $\Sigma \models \bigvee_{i \in S} \overline{x}_i \vee x_j$ for each S. Since we can check this condition in linear time and there are $\sum_{p=0}^{\alpha}\binom{|N(c)|}{p}$ such S's, we have the following result, which complements Theorem 3.7 that the problem is intractable, even if $P(c)=\emptyset$.

PROPOSITION 3.10. *The problem of deciding, given a Horn theory Σ, a clause c and a nonnegative integer α, whether $\sigma_{\alpha}(\Sigma)\models c$ can be solved in $O\Big(\sum_{p=0}^{\alpha}\binom{|N(c)|}{p}\,\|\Sigma\| + |P(c)|\Big)$ time. In particular, it is polynomially solvable, if $\alpha=O(1)$ or $|N(c)|=O(\log\|\Sigma\|)$.*

4. DEDUCTIVE INFERENCE FROM CHARACTERISTIC SETS

In this section, we consider the case when Horn knowledge bases are represented by characteristic sets. Contrary to formula-based representation, deductions for interiors and exteriors are both intractable, unless P=NP.

4.1. Interiors

We first present an algorithm to solve the deduction problem for the interiors of Horn theories. The algorithm requires exponential time in general, but it is polynomial when α is small.

Let Σ be a Horn theory given by its characteristic set $char(\Sigma)$, and let c be a clause. Then for a nonnegative integer α, we have

$$\sigma_{-\alpha}(\Sigma)\models c \text{ if and only if } \sigma_{-\alpha}(\Sigma)\wedge\overline{c}\equiv 0. \quad (6)$$

Let v^* be the unique minimal assignment such that $c(v^*)=0$ (i.e., $\overline{c}(v^*)=1$). By the definition of interiors, v^* is a model of $\sigma_{-\alpha}(\Sigma)$ if and only if all v's in $\mathcal{N}_{\alpha}(v^*)$ are models of Σ. Therefore, for each assignment v in $\mathcal{N}_{\alpha}(v^*)$, we check if $v\in mod(\Sigma)$, which is equivalent to

$$\bigwedge_{\substack{w\in char(\Sigma)\\ w\geq v}} w \;=\; v. \quad (7)$$

If (7) holds for all assignments v in $\mathcal{N}_{\alpha}(v^*)$, then we can immediately conclude by (6) that $\sigma_{-\alpha}(\Sigma)\not\models c$. On the other hand, if there exists an assignment v in $\mathcal{N}_{\alpha}(v^*)$ such that (7) does not hold, let $J=ON(\bigwedge_{\substack{w\in char(\Sigma)\\ w\geq v}} w)\setminus ON(v)$. By definition, we have $J\neq\emptyset$, and $\Sigma\wedge\bigwedge_{i\in ON(v)}x_i\wedge\overline{x}_j\equiv 0$ for all $j\in J$, that is,

$$\Sigma \;\models\; \bigvee_{i\in ON(v)}\overline{x}_i\vee x_j \quad \text{for all } j\in J. \quad (8)$$

If $J\cap N(c)\neq\emptyset$, then by Lemma 3.1 and (8), we have $\sigma_{-\alpha}(\Sigma)\models\bigvee_{i\in ON(v)\cap N(c)}\overline{x}_i$, since $|ON(v)\setminus N(c)|\leq\alpha-1$. This implies $\sigma_{-\alpha}(\Sigma)\models c$. On the other hand, if $J\cap N(c)=\emptyset$, then by Lemma 3.1 and (8), we have

$$\sigma_{-\alpha}(\Sigma)\models\bigvee_{i\in ON(v)\cap N(c)}\overline{x}_i\vee x_j\models\bigvee_{i\in N(c)}\overline{x}_i\vee x_j \quad \text{for all } j\in J.$$

Thus, if J contains an index in $P(c)$, then we can conclude that $\sigma_{-\alpha}(\Sigma)\models c$; Otherwise, we check the condition $\sigma_{-\alpha}(\Sigma)\models c\vee\bigvee_{j\in J}\overline{x}_j$, instead of $\sigma_{-\alpha}(\Sigma)\models c$. Since a new clause $d=c\vee\bigvee_{j\in J}\overline{x}_j$ is longer than c, after at most n iterations, we can answer the deductive query. Formally, our algorithm can be described as Algorithm 2.

PROPOSITION 4.1. *The problem of deciding, given the characteristic model $char(\Sigma)$ of a Horn theory Σ, a clause c and a nonnegative integer α, whether $\sigma_{-\alpha}(\Sigma)\models c$ can be solved in $O(n^{\alpha+2}|char(\Sigma)|)$ time. In particular, it is polynomially solvable, if $\alpha=O(1)$.*

Algorithm 2 DEDUCTION-INTERIOR-FROM-CHARSET

Input:. The characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c and a nonnegative integer α.

Output:. Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

Step 0.. Let $N := N(c)$, $d^{(1)} := c$ and $q := 1$.

Step 1.. Let u be the unique minimal assignment such that $d^{(q)}(u) = 0$.

Step 2.. **For** each v in $\mathcal{N}_\alpha(u)$ **do**

  If (7) does not hold,

    then let $v^{(q)} := v$, $J := ON(\bigwedge_{\substack{w \in char(\Sigma) \\ w \geq v}} w) \setminus ON(v)$ and $q := q + 1$

    If $J \cap (N \cup P(c)) \neq \emptyset$, **then** output yes and halt.

    Let $N := N \cup J$ and $d^{(q)} := \bigvee_{i \in N} \overline{x}_i \vee \bigvee_{i \in P(c)} x_i$.

    Go to Step 1.

  end{for}

Step 3.. Output No and halt. □

PROOF. Since we can see algorithm DEDUCTION-INTERIOR-FROM-CHARSET correctly answers a deductive query from the discussion before the description, we only estimate the running time of the algorithm.

Steps 0, 1 and 3 require $O(n)$ time. Step 2 requires $O(n^{\alpha+1}|char(\Sigma)|)$ time, since (7) can be checked in $O(n|char(\Sigma)|)$ time. Since we have at most n iterations between Steps 1 and 2, the algorithm requires $O(n^{\alpha+2}|char(\Sigma)|)$ time. □

EXAMPLE 4.2. Let us consider a Horn theory Σ with $char(\Sigma) = \{(1111), (1011), (1010), (0111), (0001)\}$ given in Example 2.2 (see Figure 2), and let $c_1 = \bar{x}_1 \vee x_2 \vee x_3$ and $c_2 = x_1 \vee x_2 \vee \bar{x}_3$. It is easy to see that $\sigma_{-1}(\Sigma) \models c_1$ and $\sigma_{-1}(\Sigma) \not\models c_2$ by $\sigma_{-1}(\Sigma) = \bar{x}_1\bar{x}_2x_3x_4$. Here, we execute Algorithm 2 for c_1 and c_2. For c_1, let $N = \{1\}$ and $d^{(1)} = \bar{x}_1 \vee x_2 \vee x_3$ in Step 0. Note that (1000) is the unique minimal assignment of $d^{(1)} = 0$. In Step 2, we see that $v = (1100) \in N_1(1000)$ does not satisfy (7), because $\bigvee_{\substack{w \in char(\Sigma) \\ w \geq v}} w = (1111) \neq v$. By $J = \{3, 4\}$, we have $J \cap (N \cup P(c_1)) = \{3\} \neq \emptyset$, and hence the algorithm answers "Yes" and halts. For c_2, let $N = \{3\}$ and $d^{(1)} = x_1 \vee x_2 \vee \bar{x}_3$ in Step 0. Note that (0010) is the unique minimal assignment of $d^{(1)} = 0$. In Step 2, we see that $v = (0110) \in N_1(0010)$ does not satisfy (7), because $\bigvee_{\substack{w \in char(\Sigma) \\ w \geq v}} w = (0111) \neq v$. By $J = \{4\}$, we have $J \cap (N \cup P(c)) = \emptyset$. Thus we update N to $N = \{3, 4\}$ and let $d^{(2)} = x_1 \vee x_2 \vee \bar{x}_3 \vee \bar{x}_4$. Again, Step 1 computes the unique minimal assignment (0011) of $d^{(2)} = 0$. Then any vector in $N_1(0011)$ satisfies (7) and hence we output "No" and halt.

However, in general, the problem is intractable, which contrasts with the formula-model representation.

THEOREM 4.3. *The problem of deciding, given the characteristic set $char(\Sigma)$ of a Horn theory Σ and a positive integer α, whether $\sigma_{-\alpha}(\Sigma)$ is consistent, i.e., $mod(\sigma_{-\alpha}(\Sigma)) \neq \emptyset$, is* coNP*-complete.*

PROOF. Let us first show that the problem belongs to coNP. Apply Algorithm DEDUCTION-INTERIOR-FROM-CHARSET to the instance ($char(\Sigma)$, $c = \emptyset$, α). If $\sigma_{-\alpha}(\Sigma)$

is not consistent, then the algorithm constructs a series of vectors, $v^{(1)}, \dots, v^{(k)}$, $k \leq n$, in Step 2. We can see that these vectors form a polynomial-size witness to the inconsistency of $\sigma_{-\alpha}(\Sigma)$. In fact, if we are given these vectors, we can compute clauses $d^{(1)}, d^{(2)}, \dots, d^{(k)}$ and reduce the deduction problem $\sigma_{-\alpha}(\Sigma) \models c = d^{(1)}$ to $\sigma_{-\alpha}(\Sigma) \models d^{(2)}$, $\sigma_{-\alpha}(\Sigma) \models d^{(3)}, \cdots, \sigma_{-\alpha}(\Sigma) \models d^{(k)}$, and conclude the inconsistency. Since all the computation can be done in polynomial time, the problem belongs to coNP.

We show the coNP-hardness by reducing INDEPENDENT SET to our problem. Given a problem instance $G = (V = \{1, 2, \dots, n\}, E)$ and k of INDEPENDENT SET, let us define a Horn theory Σ_G over $At = \{x_1, x_2, \dots, x_n\}$ by

$$char(\Sigma_G) = \{v^{(i,j)}, v^{(i,j,l)} \mid \{i, j\} \in E, l \in V \setminus \{i, j\}\},$$

where $v^{(i,j)}$ and $v^{(i,j,l)}$ are respectively the vectors defined by $OFF(v^{(i,j)}) = \{i, j\}$ and $OFF(v^{(i,j,l)}) = \{i, j, l\}$. Let $\alpha = n - k$. Note that Σ_G is a negative theory, and hence $\sigma_{-\alpha}(\Sigma_G)$ is consistent if and only if $(00 \cdots 0)$ is a model of $\sigma_{-\alpha}(\Sigma_G)$. Moreover, the latter condition is equivalent to the one that G has no independent set of size at least $k\,(= n - \alpha)$. This completes the proof. □

This result immediately implies the following corollary.

COROLLARY 4.4. *The problem of deciding, given the characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c and a positive integer α, whether $\sigma_{-\alpha}(\Sigma) \models c$ holds is* NP*-complete, even if $c = \emptyset$.*

Note that, different from the other hardness results, the hardness does not require c to be large enough.

4.2. Exteriors

Let us consider the exteriors. Similarly to the formula-based representation, we have the following negative result.

THEOREM 4.5. *The problem of deciding, given the characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c and a positive integer α, whether $\sigma_\alpha(\Sigma) \models c$ holds is* coNP-*complete.*

PROOF. From Lemmas 3.1 and 3.8, $\sigma_\alpha(\Sigma) \not\models c$ if and only if there exists a subclause d of c such that $|d| = |c| - \alpha$ and $\Sigma \not\models d$. This d is a witness that the problem belongs to coNP.

We then show the hardness by a reduction from VERTEX COVER which is known to be NP-hard. VERTEX COVER is the problem to decide if a given graph $G = (V, E)$ has a vertex cover U such that $|U| \leq k$ for a given integer $k\,(< |V|)$. Here $U \subseteq V$ is called *vertex cover* if $U \cap e \neq \emptyset$ holds for all $e \in E$. For this problem instance, we construct our problem instance. For each $e \in E$, let $W_e = \{e_1, e_2, \dots, e_{|V|}\}$, and let $W = \bigcup_{e \in E} W_e$. Let $m^{(v)}$, $v \in V$, be an assignment over $V \cup W$ such that

$$ON(m^{(v)}) = (V \setminus \{v\}) \cup \bigcup_{v \notin e} W_e,$$

and let $char(\Sigma)$ be the characteristic set for some Horn theory Σ defined by $char(\Sigma) = \{m^{(v)} \mid v \in V\}$. We define c and α by

$$c = \bigvee_{i \in V} \overline{x}_i \vee \bigvee_{i \in W} x_i \ \text{ and } \ \alpha = k,$$

respectively. For this instance, we show that $\sigma_\alpha(\Sigma) \not\models c$ if and only if the corresponding G has a vertex cover U of size at most $k\,(= \alpha)$.

For the if part, let U be such a vertex cover of G. For this U, we consider assignment $m^{(U)} \stackrel{\text{def}}{=} \bigwedge_{v\in U} m^{(v)}$, which is a model of Σ by the intersection property of a Horn theory. Note that $m^{(U)}$ does not satisfy a clause $d = \bigvee_{i\in V\setminus U} \overline{x}_i \vee \bigvee_{i\in W} x_i$. Since d is a subclause of c of length at least $|c|-\alpha$, $m^{(U)}$ is not a model of $\sigma_{-\alpha}(c)$ by Lemma 3.1. This completes the if part by Lemma 3.8.

For the only-if part, let us assume that $\sigma_\alpha(\Sigma) \not\models c$. Then by Lemmas 3.1 and 3.8, there exists a subclause d of c such that $|d| = |c| - \alpha$ and $\Sigma \not\models d$. This implies that $\Sigma \wedge \overline{d}$ contains a model m. By $\alpha < |V|$, for each $e \in E$, there exist an index j in W_e such that $m_j = 0$. Since any model m' in Σ satisfy either $m'_i = 0$ or $m'_i = 1$ for all $i \in W_e$, we have $m_i = 0$ for all $i \in W$. This means that $V \setminus ON(m)$ is a vertex cover of G, and since $|V \setminus ON(m)| \leq k$, we have the only-if part. □

By using Lemma 3.9, we can see that the problem can be solved in polynomial time, if α or $|N(c)|$ is small. Namely, for each subset S of $N(c)$ such that $|S| \geq |N(c)| - \alpha$, let v^S denotes the assignment such that $ON(v^S) = S$. Then $w^S = \bigwedge_{\substack{w\in char(\Sigma):\\ w\geq v^S}} w$ is the unique minimal model of Σ such that $ON(w^S) \supseteq S$, and hence it follows from Lemma 3.9 that it is enough to check if $|ON(w^s) \cap P(c)| \geq \alpha - |N(c)| + |S| + 1$. Clearly, this can be done in in $O\left(\sum_{p=0}^{\alpha} \binom{|N(c)|}{p} n|char(\Sigma)|\right)$ time.

Moreover, if $|P(c)|$ is small, then the problem also becomes tractable, which contrasts with Theorem 3.7.

LEMMA 4.6. *Let Σ be a theory, let c be a clause, and α be a nonnegative integer. Then $\sigma_\alpha(\Sigma) \models c$ holds if and only if each $S \subseteq P(c)$ such that $|S| \geq |P(c)| - \alpha$ satisfies*

$$|OFF(w) \cap N(c)| \geq \alpha - |P(c)| + |S| + 1 \tag{9}$$

for all models w of Σ such that $OFF(w) \cap P(c) = S$.

Note that the lemma holds for any theory Σ, and (9) is monotone in the sense that, if an assignment w satisfies (9), then all assignments v with $v < w$ also satisfy it. Thus it is sufficient to check if (9) holds for all *maximal* models w of Σ such that $OFF(w) \cap P(c) = S$. If Σ is Horn, then such maximal models w can be obtained from $w^{(i)}$ $(i \in S)$ with $i \in OFF(w^{(i)}) \cap P(c) \subseteq S$ by their intersection $w = \bigwedge_{i\in S} w^{(i)}$. Thus we can answer the deduction problem in $O\left(n \sum_{p=|P(c)|-\alpha}^{|P(c)|} \binom{|P(c)|}{p} |char(\Sigma)|^p\right)$ time.

PROPOSITION 4.7. *The problem of deciding, given the characteristic set $char(\Sigma)$ of a Horn theory, a clause c and a nonnegative integer α, whether $\sigma_\alpha(\Sigma) \models c$ holds can be solved in $O\Big(n \min\{\sum_{p=0}^{\alpha} \binom{|N(c)|}{p} |char(\Sigma)|, \sum_{p=|P(c)|-\alpha}^{|P(c)|} \binom{|P(c)|}{p} |char(\Sigma)|^p\}\Big)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(\log(n|char(\Sigma)|))$.*

5. DEDUCTIVE INFERENCE FOR ENVELOPES OF THE EXTERIORS OF HORN THEORIES

We have considered deduction for interiors and exteriors of Horn theories. As mentioned before, the interiors of Horn theories are also Horn, while this does not hold for the exteriors. This means that the exteriors of Horn theories might lose beneficial properties of Horn theories. One of the ways to overcome such a hurdle is *Horn Approximation*, that is, approximating a theory by a Horn theory [Selman and Kautz 1991]. There are several methods for approximation, but one of the most natural ones is to approximate a theory by its *Horn envelope*. For a theory Σ, its *Horn envelope* is the Horn theory Σ_e such that $mod(\Sigma_e) = Cl_\wedge(mod(\Sigma))$. Since Horn theories are closed under intersection, the Horn envelope is the least Horn upper bound for Σ, i.e., $char(\Sigma_e) \supseteq char(\Sigma)$

and there exists no Horn theory Σ^* such that $char(\Sigma_e) \supsetneq char(\Sigma^*) \supseteq char(\Sigma)$. In this section, we consider deduction for Horn envelopes of exteriors of Horn theories, i.e., $\sigma_\alpha(\Sigma)_e \models c$.

5.1. Model-Based Representations

Let us first consider the case in which knowledge bases are represented by characteristic sets.

LEMMA 5.1. *Let Σ be a Horn theory, and let α be a nonnegative integer. Then we have*

$$mod(\sigma_\alpha(\Sigma)_e) \ = \ Cl_\wedge(\bigcup_{v \in char(\Sigma)} \mathcal{N}_\alpha(v)). \tag{10}$$

PROOF. By definition, $mod(\sigma_\alpha(\Sigma)_e) \ = \ Cl_\wedge(mod(\sigma_\alpha(\Sigma))) \ \supseteq \ Cl_\wedge(\bigcup_{v\in char(\Sigma)} \mathcal{N}_\alpha(v))$ holds. For the converse direction, let v^* be a model of Horn envelope of the α-exterior, i.e., $v^* \in mod(\sigma_\alpha(\Sigma)_e)$. Then v^* can be represented by $v^* = \bigwedge_{w\in W} w$ for some $W \subseteq mod(\sigma_\alpha(\Sigma))$. For $w \in W$, let u be a model of Σ such that w is contained in $\mathcal{N}_\alpha(u)$. Since such a u can be represented by $u = \bigwedge_{v\in S_u} v$ for some $S_u \subseteq char(\Sigma)$, w is represented by $w = \bigwedge_{v\in S_u} \tilde{v}$, where $\tilde{v}$ is defined by $\tilde{v}_i = 0$ if $i \in ON(u) \setminus ON(w)$, 1 if $i \in ON(w) \setminus ON(u)$, and v_i otherwise. Note that $|ON(v)\Delta ON(\tilde{v})| \leq |ON(w)\Delta ON(u)| \leq \alpha$, that is, $\tilde{v} \in \mathcal{N}_\alpha(v)$, and w belongs to $Cl_\wedge(\bigcup_{v\in S_u} \mathcal{N}_\alpha(v))$. This, together with $v^* = \bigwedge_{w\in W} w$, implies that v^* also belongs to $Cl_\wedge(\bigcup_{v\in char(\Sigma)} \mathcal{N}_\alpha(v))$.

□

For a clause c, let v^* be the unique minimal assignment such that $c(v^*) = 0$. We recall that, for a Horn theory Φ,

$$\Phi \models c \ \text{ if and only if } \ c(\bigwedge_{\substack{v\in char(\Phi)\\ v\geq v^*}} v) = 1. \tag{11}$$

Therefore, Lemma 5.1 immediately implies an algorithm for deduction for $\sigma_\alpha(\Sigma)_e$ from $char(\Sigma)$, since we have $char(\sigma_\alpha(\Sigma)_e) \subseteq \bigcup_{v\in char(\Sigma)} \mathcal{N}_\alpha(v) \subseteq \sigma_\alpha(\Sigma)_e$. However, for a general α, $\bigcup_{v\in char(\Sigma)} \mathcal{N}_\alpha(v)$ is exponentially larger than $char(\Sigma)$, and hence this direct method is not efficient. The following lemma helps developing a polynomial time algorithm.

LEMMA 5.2. *Let Σ be a Horn theory, let c be a clause, and let α be a nonnegative integer. Then $\sigma_\alpha(\Sigma)_e \models c$ holds if and only if the following two conditions are satisfied.*

(i). $|OFF(v) \cap N(c)| \geq \alpha$ *holds for all* $v \in char(\Sigma)$.
(ii). *If $S = \{v \in char(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\} \neq \emptyset$, $P(c)$ is not covered with $OFF(v)$ for models v in S, i.e., $P(c) \not\subseteq \bigcup_{\substack{v\in char(\Sigma)\\ |OFF(v)\cap N(c)|=\alpha}} OFF(v)$.*

PROOF. To show the if part, let us first assume that (i) and (ii) in the lemma holds. Let v be a model in $char(\Sigma)$ such that $|OFF(v) \cap N(c)| > \alpha$. Then all assignments w in $\mathcal{N}_\alpha(v)$ satisfy $OFF(w) \cap N(c) \neq \emptyset$. Therefore, if all the models v in $char(\Sigma)$ satisfy $|OFF(v) \cap N(c)| > \alpha$, then by Lemma 5.1, we have $OFF(w) \cap N(c) \neq \emptyset$ for any model w of $\sigma_\alpha(\Sigma)_e$. This implies $\sigma_\alpha(\Sigma)_e \models c$. Therefore, let us consider the case when $S = \{v \in char(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\}$ is not empty. Let v^* be the unique minimal assignment such that $c(v^*) = 0$. Then by Lemma 5.1, we have

$$\{v \in char(\sigma_\alpha(\Sigma)_e) \mid v \geq v^*\}$$

$$\subseteq \{w \mid ON(w) = ON(v) \cup N(c) \text{ for some } v \in S\}. \quad (12)$$

Since $P(c)$ is not covered with $OFF(v)$ for models v in S, this, together with (11) implies $\sigma_\alpha(\Sigma)_e \models c$.

Let us next show the only-if part. Assume that (i) is satisfied, but (ii) is not. Then (11) and (12) imply $\sigma_\alpha(\Sigma)_e \not\models c$. On the other hand, if (i) is not satisfied, i.e., there exists a $v \in char(\Sigma)$ such that $|OFF(v) \cap N(c)| < \alpha$, let $w^{(i)}$, $i \in P(c)$, be an assignment in $\mathcal{N}_\alpha(v)$ such that $ON(w^{(i)}) \supseteq N(c)$ and $OFF(w^{(i)}) \supseteq \{i\}$, and let $w^* = \bigwedge_{i \in P(c)} w^{(i)}$. Then we have $c(w^*) = 0$ and $w^* \in mod(\sigma_\alpha(\Sigma)_e)$ by Lemma 5.1. This implies $\sigma_\alpha(\Sigma)_e \not\models c$. □

The lemma immediately implies the following theorem.

THEOREM 5.3. *The problem of deciding, given the characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c and a nonnegative integer α, whether $\sigma_\alpha(\Sigma)_e \models c$ holds can be solved in linear time.*

We remark that this contrasts with Corollary 4.4. Namely, if we are given the characteristic set $char(\Sigma)$ of a Horn theory Σ, $\sigma_\alpha(\Sigma)_e \models c$ is polynomially solvable, while it is coNP-complete to decide if $\sigma_\alpha(\Sigma) \models c$.

5.2. Formula-Based Representation

Recall that *negative* theories (i.e., theories consisting of clauses with no positive literal) are Horn and the exteriors of negative theories are also negative, and hence Horn. This means that, for a negative theory Σ, we have $\sigma_\alpha(\Sigma)_e = \sigma_\alpha(\Sigma)$. Therefore, we can again make use of the reduction in the proof of Theorem 3.7, since the reduction uses negative theories.

THEOREM 5.4. *The problem of deciding, given a Horn theory Σ, a clause c, and a nonnegative integer α, whether $\sigma_\alpha(\Sigma)_e \models c$ holds is* coNP*-complete, even if $P(c) = \emptyset$.*

PROOF. Since the hardness is proved similarly to Theorem 3.7, we show that the problem belongs to coNP.

Note that $\sigma_\alpha(\Sigma)_e \not\models c$ if and only if there exists a model v of $\sigma_\alpha(\Sigma)_e$ such that $c(v) = 0$. A model v of $\sigma_\alpha(\Sigma)$ can be represented by $v = \bigwedge_{w \in W} w$ for some $W \subseteq char(\sigma_\alpha(\Sigma))$. In order to have such a representation, for each $j \in OFF(v)$, there exists a model $u^{(j)}$ in $char(\sigma_\alpha(\Sigma))$ such that $u_j = 0$ and $u \geq v$. This implies that there exists a W with $|W| \leq n$. Since $char(\sigma_\alpha(\Sigma)) \subseteq \bigcup_{w \in char(\Sigma)} \mathcal{N}_\alpha(w)$ by Lemma 5.1, each $w \in W$ can be represented as a neighbor of some model of $char(\Sigma)$. By this representation of w, we have a representation of v with a polynomial size, and we can check in polynomial time if v is a model of $\sigma_\alpha(\Sigma)$. This implies that the problem belongs to coNP. □

However, if α or $N(c)$ is small, the problem becomes tractable by algorithm DEDUCTION-ENVELOPE-EXTERIOR-FROM-HORN-THEORY (Algorithm 3).

The algorithm is based on a necessary and sufficient condition for $\sigma_\alpha(\Sigma)_e \models c$, which is obtained from Lemma 5.2 by replacing all $char(\Sigma)$'s with $mod(\Sigma)$'s. It is not difficult to see that such a condition holds from the proof of Lemma 5.2.

PROPOSITION 5.5. *The problem of deciding, given a Horn theory Σ, a clause c and a nonnegative integer α, whether $\sigma_\alpha(\Sigma)_e \models c$ holds can be solved in $O\Big(\big(\binom{|N(c)|}{\alpha-1} + \binom{|N(c)|}{\alpha}\big) \parallel \Sigma \parallel + |P(c)|\Big)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(\log \|\Sigma\|)$.*

PROOF. The correctness of the algorithm follows from the discussion after its description. For the time complexity, it is known [Dowling and Galliear 1983] that the

Algorithm 3 DEDUCTION-ENVELOPE-EXTERIOR-FROM-HORN-THEORY

Input:. A Horn theory Σ, a clause c and a nonnegative integer α.

Output:. Yes, if $\sigma_\alpha(\Sigma)_e \models c$; Otherwise, No.

Step 1.. /* Check if there exists a model v of Σ such that $|OFF(v) \cap N(c)| < \alpha$. */

For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha + 1$ **do**

Check if the theory obtained from Σ by assigning $x_i = 1$ for $i \in N$ is satisfiable.

If so, **then** output No and halt.

end{for}

Step 2.. /* Check if there exists a set $S = \{v \in mod(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\}$ such that $\bigcup_{v \in S} OFF(v) \supseteq P(c)$. */

Let $J := \emptyset$.

For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha$ **do**

Compute the unique minimal satisfiable model v for the theory obtained from Σ by assigning $x_i = 1$ for $i \in N$.

If such a model v exists, update $J := J \cup \{j \in P(c) \mid v_j = 0\}$.

end{for}

If $J = P(c)$, **then** output NO and halt.

Step 3.. Output Yes and halt. □

satisfiability problem, together with computing the unique minimal model of a Horn theory, is possible in linear time. Since the number of the iterations of for-loops in Steps 2 and 3 are bounded by $\binom{|N(c)|}{\alpha-1}$ and $\binom{|N(c)|}{\alpha}$, respectively, the algorithm requires $O\left(\left(\binom{|N(c)|}{\alpha-1} + \binom{|N(c)|}{\alpha}\right) \|\Sigma\| + |P(c)|\right)$ time. □

EXAMPLE 5.6. Let us consider $\Sigma = \{\bar{x}_1 \vee x_3, \bar{x}_2 \vee x_4, \bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4\}$ and $c = x_1 \vee \bar{x}_2 \vee \bar{x}_3$. Since $\sigma_1(\Sigma) = \{\bar{x}_1 \vee \bar{x}_2 \vee x_3 \vee x_4\}$, $\sigma_1(\Sigma)_e$ is a tautology. Thus we have $\sigma_1(\Sigma)_e \not\models c$. We apply Algorithm 3 to Σ, c and $\alpha = 1$.

In Step 1, by $N(c) = \{2, 3\}$ and $|N(c)| - \alpha + 1 = 2$, we check the theory obtained from Σ by fixing $x_2 = x_3 = 1$. Since it is an unsatisfiable theory $\{x_4, \bar{x}_4\}$, we go to Step 2. In Step 2, we consider $\{2\}$ and $\{3\}$ as N by $|N(c)| - \alpha = 1$. For $N = \{2\}$, we compute the unique minimal model (0001) of the obtained theory $\{\bar{x}_1 \vee x_3, x_4, \bar{x}_3 \vee \bar{x}_4\}$, and let $J = \{1\}$. For $N = \{3\}$, we compute the unique minimal model (0000) of the obtained theory $\{\bar{x}_2 \vee x_4, \bar{x}_2 \vee \bar{x}_4\}$, and we again have $J = \{1\}$. Since $J = P(c) = \{1\}$, we output "No" and halt.

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