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# Annotated Probabilistic Temporal Logic: Approximate Fixpoint Implementation

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# A. COMPLEXITY PROOFS (SECTION 3)

# A.1 Small-Model Lemma for APT-Logic

The following lemmas are not part of the main text, but are needed to prove some of the theorems.

Let us define the "size" of a rational number  $\frac{a}{b}$  (where a, b are relatively prime) as the number of bits it takes to represent a and b. As stated earlier, for both the probability bound of rules, as well as the values returned by frequency functions, we assume that this is a fixed quantity. In [Fagin et al. 1990], the authors provide another result we can leverage to ensure that there is a solution to a linear program where the solution can be represented with a polynomial number of bits.

LEMMA A.1. If a system of r linear inequalities and/or equalities with integer coefficients of length at most l has a nonnegetive solution, then it has a nonnegetive solution with at most r entries positive, and where the size of each solution is  $O(r \cdot l + r \cdot \log(l))$ . (Lemma 2.7 in [Fagin et al. 1990]).

LEMMA A.2. APT-program  $\mathcal{K}$  is consistent iff it has an interpretation that only assigns non-zero probabilities to at most  $2 \cdot |\mathcal{K}| + 1$  threads and the probability assigned to each thread can be represented with  $O(|\mathcal{K}| \cdot size + |\mathcal{K}| \cdot \log(size))$  bits (where size is the maximum number of bits required to represent the result of a frequency function of probability bounds of a rule).

PROOF. By Proposition 3.9 of [Shakarian et al. 2011], an APT-program is consistent iff there is a solution to the SLC constraints. By Remark 3.10 of [Shakarian et al. 2011], there are  $2 \cdot |\mathcal{K}| + 1$  constraints in SLC. Hence, by Theorem 3.3, if there is a solution to the SLC constraints, then there exists a solution where only  $2 \cdot |\mathcal{K}| + 1$  are given positive values. The second part of the satement follows directly from Lemma A.1. The statement of the theorem follows.  $\Box$ 

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#### A.2 Proof of Theorem 3.4

Deciding if APT-program  $\mathcal{K}$  is consistent is NP-Complete if  $|\mathcal{K}|$  is a polynomial in terms of  $|B_{\mathcal{L}}|$ .

PROOF. NP-Hardness by Theorem 3.4 of [Shakarian et al. 2011]. By Lemma A.2, every consistent APT-program must be associated with a set  $\mathcal{T}'$  of threads, where  $|\mathcal{T}'| \leq 2 \cdot |\mathcal{K}| + 1$  and that there exists an interpretation I' which only assigns non-zero probabilities to threads in  $\mathcal{T}'$  and satisfies  $\mathcal{K}$ . Hence, we use  $\mathcal{T}'$  as a witness. We can check the witness in polynomial time by setting up SLC constraints using only threads in  $\mathcal{T}'$  rather than  $\mathcal{T}$ . By the statement, such a linear program will have a polynomial number of variables. Hence,  $\mathcal{K}$  is consistent iff there is a solution to this linear program (which can be checked in PTIME). The statement follows.  $\Box$ 

#### A.3 Proof of Theorem 3.5

Deciding if APT-rule r is entailed by APT-program  $\mathcal{K}$  is coNP-Complete if  $|\mathcal{K}|$  is a polynomial in terms of  $|B_{\mathcal{L}}|$ .

PROOF. coNP-hardness by Theorem 4.2 of [Shakarian et al. 2011]. Let  $[\ell, u]$  be the probability bounds associated with r. Let  $num \in [0, 1]$  be a real number that is outside of  $[\ell, u]$ . Create new rule r' that is the same as r except the probability bounds are [num, num]. Create APT-program  $\mathcal{K}' = \mathcal{K} \cup \{r'\}$ . Note that if  $\mathcal{K}'$  is consistent, then r is not entailed. Hence, we can check the consistency of  $\mathcal{K}'$  using a witness  $\mathcal{T}'$  as described in Theorem 3.4 as well as num. Note that this check can still be performed in PTIME. The statement follows.  $\Box$ 

# A.4 Proof of Theorem 3.6

Given APT-program  $\mathcal{K}$ , interpretation I, and ptf  $\phi$ , determining the maximum  $\ell$ and minimum u such that  $\phi : [\ell, u]$  is entailed by  $\mathcal{K}$  and is satisfied by I is #P-hard. Further, for constant  $\epsilon > 0$ , approximating either the maximum  $\ell$  and/or minimum u within  $2^{|\mathcal{B}_{\mathcal{L}}|^{1-\epsilon}}$  is NP-Hard.

For ease of readability, we divide the above theorem into three leammas. The statement of the theorem follows directly from Lemmas A.3 and A.4. Throughout the proof, we shall define the problem **APT-OPT-ENT** as follows:

# APT-OPT-ENT

INPUT: APT-program  $\mathcal{K}$ , interpretation I, and ptf  $\phi$ OUTPUT: maximum  $\ell$  and minimum u such that  $\phi : [\ell, u]$  is entailed by  $\mathcal{K}$  and is satisfied by I.

LEMMA A.3. APT-OPT-ENT is #P-hard.

PROOF. Intuition Given an instance of #SAT (known to be #P-complete), we can an instance of **APT-ENT-OPT** and such that #SAT  $\leq_p$  **APT-ENT-OPT**.

Definition of #SAT: INPUT: Set of atoms  $B_{\mathcal{L}}$ , formula f. OUTPUT: Number of worlds in  $2^{B_{\mathcal{L}}}$  that satisfy f.

#### CONSTRUCTION:

- (1) Set F to be f.
- (2) Set t = 1.
- (3) For each  $a \in B_{\mathcal{L}}$ , add a : [1, 0.5, 0.5] to  $\mathcal{K}$ .
- (4) Set  $t_{max} = 1$ .
- (5) We will consider  $B_{\mathcal{L}}$  (the set of atoms from the input of #SAT) as the set of atoms used for the
- (6) input of **APT-ENT-OPT**.
- (7) Set  $IC \equiv \emptyset$ .
- (8) Interpretation  $I_{uniform}$  sets each thread in  $\mathcal{T}$  a probability of  $\frac{1}{|\mathcal{T}|}$

For this construction, we shall denote the set of all threads formed with  $t_{max} = 1$  on set of atoms  $B_{\mathcal{L}}$  as  $\mathcal{T}$ .

As step 3 is the only step of the construction that cannot be done in constant time, but requires  $O(|B_{\mathcal{L}}|)$  time, so the construction is polynomial.

CLAIM 1: Interpretation  $I_{uniform}$  satisfies  $\mathcal{K}$ .

Each thread in  $\mathcal{T}$  consists of only one world. For some atom  $a \in B_{\mathcal{L}}$ , half of all possible worlds satisfy a. Hence, as  $I_{uniform}$  is a uniform probability distribution among threads, the sum of probabilities for all threads that satisfy a in the first (and only) time point is 0.5. By the construction of  $\mathcal{K}$  in step 3 in the construction, the claim follows.

CLAIM 2: For any annotated formula  $F : [t, \ell, u]$  that is entailed by  $\mathcal{K}$  and satisfied by  $I_{uniform}$ ,  $\ell$  must equal u.

As  $\mathcal{K}$  is satisfied by exactly one interpretation,  $I_{uniform}$ , the sum of probabilities for all threads that satisfy F at time t is bounded above and below by the same number.

CLAIM 3: If f is satisfied by exactly m worlds, then  $f : [1, \frac{m}{2^{|B_{\mathcal{L}}|}}, \frac{m}{2^{|B_{\mathcal{L}}|}}]$  is entailed by  $\mathcal{K}$ .

Let  $W_1, \ldots, W_m$  be the worlds that satisfy f. Let  $Th_1, \ldots, Th_m$  be all the threads in  $\mathcal{T}$  where  $Th_i \equiv W_i$  ( $W_i$  is the *i*th world that satisfies f). As we have only one time point, and our threads are created using  $B_{\mathcal{L}}$ , we know that the following holds:

$$\sum_{i=1}^{m} I_{uniform}(Th_i) = \frac{m}{2^{|B_{\mathcal{L}}|}}$$

This is equivalent to the following:

$$\sum_{\substack{Th\in\mathcal{T}\\Th(1)\models f}} I_{uniform}(Th)$$

Hence, by claims 1-2 and the definition of satisfaction, the claim follows.

CLAIM 4: If  $f : [1, \frac{m}{2^{|B_{\mathcal{L}}|}}, \frac{m}{2^{|B_{\mathcal{L}}|}}]$  is entailed by  $\mathcal{K}$ , then f is satisfied by exactly m worlds.

By claims 1-3 and the definition of satisfaction, there are exactly m threads that satisfy f in the first time point. As there is only one time point per threads, there are also m worlds that satisfy f. Since  $B_{\mathcal{L}}$  is the set of atoms for both the instance

of #SAT and APT-ENT-OPT, the statement follows.

The proof of the theorem follows directly from claims 3-4.  $\Box$ 

LEMMA A.4. For constant  $\epsilon > 0$ , approximating **APT-ENT-OPT** (i.e. approximating outputs  $\ell$  and/or u) within  $2^{|B_{\mathcal{L}}|^{1-\epsilon}}$  is NP-Hard.

PROOF. Suppose, by way of contradiction, that approximating a solution within  $2^{|B_{\mathcal{L}}|^{1-\epsilon}}$  is easier than NP-Hard. Then, using the construction from the proof of Theorem A.3, we could approximate #SAT within  $2^{|B_{\mathcal{L}}|^{1-\epsilon}}$ . However, by [Roth 1996] (Theorem 3.2), approximating #2MONCNF, a more restricted version of #SAT, within  $2^{|B_{\mathcal{L}}|^{1-\epsilon}}$  is NP-hard. The statement follows.  $\Box$ 

# B. SUPPLEMENTARY INFORMATION FOR SECTION 4

#### B.1 Proof of Proposition 4.1

If  $F_1 : t_1 \land \ldots \land F_n : t_n \land F_{n+1} : t'_1 \land \ldots \land F_{n+m} : t'_m$  and  $G_1 : t_1 \land \ldots \land G_n : t_n \land G_{n+1} : t''_1 \land \ldots \land G_{n+m} : t''_m$  are time conjunctions, then

$$(F_1 \wedge G_1) : t_1 \wedge \ldots \wedge (F_n \wedge G_n) : t_n \wedge F_{n+1} : t'_1 \wedge \ldots \wedge F_{n+m} : t'_m \wedge G_{n+1} : t''_1 \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_n \wedge \ldots \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_m \wedge G_{n+1} : t''_m \wedge G_{n+m} : t''_m \wedge G_{n+1} : t''_m \wedge G_{n$$

is also a time conjunction.

PROOF. Straightforward from the definitions of satisfaction and time conjunction.  $\Box$ 

# B.2 Proof of Proposition 4.4

For formulas F, G, time  $\Delta t$ , and time conjunction  $\phi$ ,  $EFR(F, G, \Delta t, \phi) \subseteq$ 

 $\left[\frac{cnt(\phi, F, G, \Delta t) + end(\phi, F, G, \Delta t)}{denom(\phi, F, G, \Delta t) + end(\phi, F, G, \Delta t)}, \frac{poss(\phi, F, G, \Delta t) + endposs(\phi, F, G, \Delta t)}{denom(\phi, F, G, \Delta t) + endposs(\phi, F, G, \Delta t)}\right]$ 

**PROOF.** Straightforward from definitions.  $\Box$ 

# B.3 Proof of Lemma 4.5

- (1) If  $I \models \phi : [\ell, u]$  and  $\rho : [\ell', u']$ , then  $I \models \phi \land \rho : [\max(0, \ell + \ell' 1), \min(u, u')]$
- (2) If  $I \models \phi : [\ell, u]$  and  $\rho : [\ell', u']$ , then  $I \models \phi \lor \rho : [\max(\ell, \ell), \min(1, u + u')]$
- (3) If  $I \models \phi : [\ell, u]$  and  $\phi \Rightarrow \rho$  then  $I \models \rho : [\ell, 1]$
- (4) If  $I \models \phi : [\ell, u]$  and  $\rho \Rightarrow \phi$  then  $I \models \rho : [0, u]$
- (5) If  $I \models \phi : [\ell, u]$  then  $I \models \neg \phi : [1 u, 1 \ell]$

PROOF. Adapted from Theorem 1 of [Ng and Subrahmanian 1992] and Definition 2.8, except case 5: Suppose, BWOC,  $I \models \phi : [\ell, u]$  and  $I \not\models \neg \phi : [1 - u, 1 - \ell]$ . By the definition of satisfaction:

$$\ell \le \sum_{\substack{Th \in \mathcal{T} \\ Th \models \phi}} I(Th) \le u$$

By the definitoin of negation, we know that:

$$\sum_{\substack{Th \in \mathcal{T} \\ Th \models \neg \phi}} I(Th) = 1 - \sum_{\substack{Th \in \mathcal{T} \\ Th \models \phi}} I(Th)$$

Hence,

$$\ell \leq \sum_{\substack{Th \in \mathcal{T} \\ Th \models \neg \phi}} I(Th) \leq u$$

Which, by the definition of satisfaction, gives a contradiction.  $\Box$ 

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#### B.4 Proof of Theorem 4.6

If interpretation  $I \models \phi : [1,1]$  where  $EFR(F,G,\Delta t,\phi) \subseteq [\alpha,\beta], I \models F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \alpha, \beta].$ 

PROOF. **CLAIM 1:** If interpretation *I* satisfies  $\phi : [\ell, u]$  and  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha, \beta]$ , then  $I \models F \stackrel{efr}{\hookrightarrow} G : [\Delta t, \ell, 1, \alpha, \beta]$ .

Suppose, BWOC, there exists interpretation I s.t.  $I \models \phi : [\ell, u]$  and  $I \not\models F \stackrel{efr}{\hookrightarrow} G : [\Delta t, \ell, 1, \alpha, \beta]$ . By the definition of satisfaction, we know that:

$$\ell \leq \sum_{\substack{Th \in \mathcal{T} \\ Th \models \phi}} I(Th) \leq u$$

As  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha, \beta]$ , we know that:

$$\sum_{\substack{Th \in \mathcal{T} \\ Th \models \phi}} I(Th) \le \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha, \beta]}} I(Th)$$

Hence,

$$\ell \leq \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha, \beta]}} I(Th) \leq 1$$

So, by the definition of satisfaction,  $I \models F \xrightarrow{efr} G : [\Delta t, \ell, 1, \alpha, \beta]$  – a contradiction. **CLAIM 1.1:** If  $I \models \phi[1, 1]$ , then  $I \models F \xrightarrow{efr} G : [\Delta t, 1, 1, \alpha, \beta]$  (directly from claim 1).

**CLAIM 2:** If interpretation I satisfies  $F \stackrel{efr}{\hookrightarrow} G : [\Delta t, \ell, u, \alpha, \beta]$ , then  $I \models F \stackrel{efr}{\hookrightarrow} G : [\Delta t, \alpha \cdot \ell, 1]$ . Suppose, BWOC, there exists interpretation I s.t.  $I \models F \stackrel{efr}{\hookrightarrow} G : [\Delta t, \ell, u, \alpha, \beta]$  and  $I \not\models F \stackrel{efr}{\hookrightarrow} G : [\Delta t, \alpha \cdot \ell, 1]$ . By the definition of satisfaction,

$$\ell \leq \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha, \beta]}} I(Th) \leq u$$

We multiply through by  $\alpha$ :

$$\alpha \cdot \ell \leq \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha, \beta]}} \alpha \cdot I(Th)$$

It follows that:

$$\alpha \cdot \ell \leq \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha, \beta]}} \alpha \cdot I(Th) + \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \notin [\alpha, \beta]}} efr(Th, F, G, \Delta t) \cdot I(Th)$$

and

ef

$$\sum_{\substack{Th\in\mathcal{T}\\r(Th,F,G,\Delta t)\in[\alpha,\beta]}} \alpha \cdot I(Th) \leq \sum_{\substack{Th\in\mathcal{T}\\efr(Th,F,G,\Delta t)\in[\alpha,\beta]}} efr(Th,F,G,\Delta t) \cdot I(Th)$$

Hence, it follows that:

$$\alpha \cdot \ell \leq \sum_{Th \in \mathcal{T}} efr(Th, F, G, \Delta t) \cdot I(Th) \leq 1$$

So, by the definition of satisfaction,  $I \models F \stackrel{efr}{\leadsto} G : [\Delta t, \alpha \cdot \ell, 1]$  – which is a contradiction. **CLAIM 2.1:** If  $I \models \phi[1, 1]$ , then  $I \models F \stackrel{efr}{\leadsto} G : [\Delta t, \alpha, 1]$ . (follows directly from claims 1.1 and 2).

**CLAIM 3:** If interpretation I satisfies  $F \xrightarrow{efr} G : [\Delta t, 1, 1, \alpha, \beta]$ , then  $I \models F \xrightarrow{efr} G : [\Delta t, 0, \beta]$ . Suppose, BWOC,  $I \models \xrightarrow{efr} G : [\Delta t, 1, 1, \alpha, \beta]$  and  $I \not\models F \xrightarrow{efr} G : [\Delta t, 0, \beta]$ . By the definiton of satisfaction:

$$\sum_{\substack{Th\in\mathcal{T}\\fr(Th,F,G,\Delta t)\in[\alpha,\beta]}} I(Th) = \sum_{Th\in\mathcal{T}} I(Th) = 1$$

Hence,

$$\sum_{Th\in\mathcal{T}}\beta\cdot I(Th)=\beta$$

We know that:

e

$$0 \leq \sum_{Th \in \mathcal{T}} efr(Th, F, G, \Delta t) \cdot I(Th) \leq \sum_{Th \in \mathcal{T}} \beta \cdot I(Th)$$

Which leads to:

$$0 \leq \sum_{\mathit{Th} \in \mathcal{T}} efr(\mathit{Th},\mathit{F},\mathit{G},\Delta t) \cdot I(\mathit{Th}) \leq \beta$$

Which, by the definition of satisfaction, gives us a contradiction. **PROOF OF THEOREM:** Follows directly from claims 2.1 and 3.  $\Box$ 

# B.5 Proof of Corollary 4.7

If interpretation  $I \models \phi : [\ell, u]$  where  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha, \beta], I \models F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \alpha \cdot \ell, 1].$ 

**PROOF.** Follows directly from the first two claims of Theorem 4.6.  $\Box$ 

#### B.6 Proof of Theorem 4.8

Given time formulas  $\phi, \rho$  s.t.  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha_1, \beta_1]$  and  $EFR(F, G, \Delta t, \phi \land \rho) \subseteq [\alpha_2, \beta_2]$  and interpretation I that satisfies  $\phi : [1, 1]$  (see note<sup>10</sup>) and  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell, u]$ :

(1) If  $\beta_2 < \beta_1$ , then  $I \models \rho : [0, \min(\frac{\ell - \beta_1}{\beta_2 - \beta_1}, 1)]$ 

(2) If  $\alpha_2 > \alpha_1$ , then  $I \models \rho : [0, \min(\frac{u-\alpha_1}{\alpha_2-\alpha_1}, 1)]$ 

<sup>10</sup>Note that Theorem 4.6 requires  $\ell \leq \beta_1$  and  $\alpha_1 \leq u$ 

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PROOF. **CLAIM 1:** Given time formulas  $\phi, \rho$  s.t.  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha_1, \beta_1]$ and  $EFR(F, G, \Delta t, \phi \land \rho) \subseteq [\alpha_2, \beta_2]$  (where  $\beta_2 < \beta_1$ ) and interpretation I that satisfies  $\phi : [1, 1]$  and  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell, u]$  ( $\ell \leq \beta_1$ ),  $I \models \rho : [0, \min(\frac{\ell - \beta_1}{\beta_2 - \beta_1}, 1)]$ .

Assume, BWOC,  $I \not\models \rho : [0, \frac{\ell - \beta_1}{\beta_2 - \beta_1}]$ . By the definition of satisfaction, we know that:

$$\ell \leq \sum_{Th \in \mathcal{T}} efr(Th, F, G, \Delta t) \cdot I(Th)$$

As  $I \models \phi : [1, 1]$  and  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha_1, \beta_1]$ , we have:

$$\ell \leq \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1]}} efr(Th, F, G, \Delta t) \cdot I(Th)$$

We note that all threads either satisfy  $\rho$  or not. Hence, we have:

$$\sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \models \rho}} I(Th) + \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \not\models \rho}} I(Th) = 1$$

Therefore:

$$\ell \leq \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \models \rho}} \beta_2 \cdot I(Th) + \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \not\models \rho}} \beta_1 \cdot I(Th)$$

and:

$$\ell \leq \beta_2 \cdot \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \models \rho}} I(Th) + \beta_1 \cdot (1 - \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \models \rho}} I(Th))$$

$$\ell - \beta_1 \leq \beta_2 \cdot \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \models \rho}} I(Th) - \beta_1 \cdot \sum_{\substack{Th \in \mathcal{T} \\ efr(Th, F, G, \Delta t) \in [\alpha_1, \beta_1] \\ Th \models \rho}} I(Th))$$

Notice that  $\ell - \beta_1 \leq 0$  as  $\ell \leq \beta_1$  by the statement. Also, we know that  $\beta_2 < \beta_1$ , the quantity  $\beta_2 - \beta_1$  is negative. We have the following:

$$\frac{\ell - \beta_1}{\beta_2 - \beta_1} \ge \sum_{\substack{Th \in \mathcal{T} \\ Th \models \rho}} I(Th)$$

By the definition of satisfaction, this gives us a contradiction.

**CLAIM 2:** Given time formulas  $\phi$ ,  $\rho$  s.t.  $EFR(F, G, \Delta t, \phi) \subseteq [\alpha_1, \beta_1]$  and  $EFR(F, G, \Delta t, \phi \land \rho) \subseteq [\alpha_2, \beta_2]$  ( $\alpha_2 > \alpha_1$ ) and interpretation I that satisfies  $\phi : [1, 1]$  and  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell, u]$  ( $\alpha_1 \leq u$  or inconsistent),  $I \models \rho : [0, \min(\frac{u - \alpha_1}{\alpha_2 - \alpha_1}, 1)].$ 

Assume, BWOC,  $I \not\models \rho : [0, \frac{u-\alpha_1}{\alpha_2-\alpha_1}]$ . By the definition of satisfaction, we know that:

$$\sum_{Th\in\mathcal{T}} efr(Th, F, G, \Delta t) \cdot I(Th) \le u$$

Hence, as all threads either satisfy  $\rho$  or not, and as  $I \models \phi : [1, 1]$ , we know that all threads must also have a  $\alpha_1$  lower bound for the frequency function, and that the threads satisfying  $\rho$  must have  $\alpha_2$  as a lower bound. So, we have the following:

$$\sum_{\substack{Th \in \mathcal{T} \\ Th \models \rho}} \alpha_2 \cdot I(Th) + \sum_{\substack{Th \in \mathcal{T} \\ Th \neq \rho}} \alpha_1 \cdot I(Th) \le u$$

As we know the sum of all theads must be 1, we have the following:

$$\alpha_{2} \cdot \sum_{\substack{Th \in \mathcal{T} \\ Th \models \rho}} I(Th) + \alpha_{1} \cdot (1 - \sum_{\substack{Th \in \mathcal{T} \\ Th \models \rho}} I(Th)) \le u$$
$$(\alpha_{2} - \alpha_{1}) \cdot \sum_{\substack{Th \in \mathcal{T} \\ Th \models \rho}} I(Th) \le u - \alpha_{1}$$

As, by the statement, we know the quantities  $\alpha_2 - \alpha_1$  and  $u - \alpha_1$  are positive, we have the following:

$$\sum_{\substack{Th \in \mathcal{T} \\ Th \models \rho}} I(Th) \le \frac{u - \alpha_1}{\alpha_2 - \alpha_1}$$

Which, by the definition of satisfaction, gives us a contradiction. **Proof of theorem:** Follows directly from claims 1-2.  $\Box$ 

#### B.7 Proof of Proposition 4.10

If for atoms  $A_i$  and program  $\mathcal{K}$ , if  $\mathsf{BLK}(A_i) :< \mathsf{blk}_i \in \mathcal{K}$  and if there exists a ptf  $\phi : [1,1] \in \mathcal{K}$  such that  $\phi \Rightarrow A_i : t - \mathsf{blk}_i + 1 \land A_i : t - \mathsf{blk}_i + 2 \land \ldots \land A_i : t - 1$  then  $\mathcal{K}$  entails A : t : [0,0].

PROOF. Suppose, BWOC, there exists interpretation I s.t.  $I \models \mathcal{K}$  and  $I \not\models \mathcal{A}$ : t : [0,0]. As  $I \models \mathcal{K}$ , we know  $I \models \mathsf{BLK}(A_i) :< \mathsf{blk}_i$ . Hence, for all therads s.t.  $I(Th) \neq 0$ , there does not exist a series of  $\mathsf{blk}_i$  or more consecutive worlds in Thsatisfying atom  $A_i$ . We note that as  $I \models \phi : [1,1]$ , then  $I \models A_i : t - \mathsf{blk}_i + 1 \land A_i :$  $t - \mathsf{blk}_i + 2 \land \ldots \land A_i : t - 1 : [1,1]$  by the statement. Hence, there is a sequence of  $\mathsf{blk}_i - 1$  consecutive worlds satisfying  $A_i$  in every thread assigned a non-zero probability by I. So, by the definition of satisfaction, we have a contradiction.  $\Box$ 

# B.8 Proof of Proposition 4.11

If for atoms  $A_i$  and program  $\mathcal{K}$ , if  $\mathsf{OCC}(A_i) : [\mathsf{lo}_i, \mathsf{up}_i] \in \mathcal{K}$  and if there exists a ptf  $\phi : [1,1] \in \mathcal{K}$  such that there are numbers  $t_1, \ldots, t_{\mathsf{up}_i} \in \{1, \ldots, t_{max}\}$  where  $\phi \Rightarrow A_i : t_1 \land \ldots \land A_i : t_{\mathsf{up}_i}$  then for any  $t \notin \{t_1, \ldots, t_{\mathsf{up}_i}\}$   $\mathcal{K}$  entails A : t : [0,0].

PROOF. Suppose, BWOC, there exists interpretation I s.t.  $I \models \mathcal{K}$  and  $I \not\models A$ : t : [0,0]. As  $I \models \mathcal{K}$ , we know  $I \models \mathsf{OCC}(A_i) : [\mathsf{lo}_i, \mathsf{up}_i]$ . Hence, for all therads s.t.

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 $I(Th) \neq 0$ , there does not exist more than  $up_i$  worlds in Th satisfying atom  $A_i$ . We note that as  $I \models \phi : [1, 1]$ , then  $I \models A_i : t_1 \land \ldots \land A_i : t_{up_i} : [1, 1]$  by the statement. Hence, there are  $up_i$  worlds satisfying  $A_i$  in every thread assigned a non-zero probability by I. So, by the definition of satisfaction, we have a contradiction.  $\Box$ 

#### B.9 Proof of Proposition 4.16

Given APT-program  $\mathcal{K}$ , the following are true:

- $\forall I \text{ s.t. } I \models \mathcal{K}, I \models \Gamma(\mathcal{K})$
- $\forall I \text{ s.t. } I \models \Gamma(\mathcal{K}), I \models \mathcal{K}$

PROOF. Follows directly from Theorems 4.6-4.8 and Corollary 4.7.  $\Box$ 

#### B.10 Proof of Proposition 4.17

One iteration of  $\Gamma$  can be performed in time complexity  $O(|\mathcal{K}|^2 \cdot CHK)$  where CHK is the bound on the time it takes to check (for arbitrary time formulas  $\phi, \rho$  if  $\phi \models \rho$  is true.

PROOF. To compare a given element of  $\mathcal{K}$  with every other element (not conjuncts of elements) - we obviously need  $O(|\mathcal{K}| \cdot CHK)$  time. As we do this for every element in  $\mathcal{K}$ , the statement follows.  $\Box$ 

#### B.11 Proof of Lemma 4.21

Given  $\bot \equiv \{\}$  and  $\top \equiv inconsistent$ , then  $\langle PROG_{B_{\mathcal{L}}, t_{max}}, \sqsubseteq \rangle$  is a complete lattice.

PROOF. We must show that for any subset PROG' of  $PROG_{B_{\mathcal{L}},t_{max}}$ , that inf(PROG') and sup(PROG') exist in  $PROG_{B_{\mathcal{L}},t_{max}}$ . We show this for  $PROG_{B_{\mathcal{L}},t_{max}}$  as a set of APT-programs, and the result obviously extends for  $PROG_{B_{\mathcal{L}},t_{max}}$  as a set of equivalence classes of APT-programs.

CLAIM 1: For a set PROG' of APT-programs, inf(PROG') exists and is in  $PROG_{B_{\mathcal{L}},t_{max}}$ .

Let  $PROG' = \{\mathcal{K}_1, \ldots, \mathcal{K}_i, \ldots, \mathcal{K}_n\}$ . We create  $\mathcal{K}' \equiv inf(PROG')$  as follows. Consider all  $\phi$  such that  $\phi : [\ell_i, u_i]$  appears in each  $\mathcal{K}_i$ . Add  $\phi : [\min(\ell_i), \max(u_i)]$  to  $\mathcal{K}'$ . Next, consider all  $F, G, \Delta t$  s.t.  $F \stackrel{efr}{\simeq} G : [\Delta t, \ell_i, u_i]$  appears in all  $\mathcal{K}_i$ . Add  $F, G, \Delta t$  s.t.  $F \stackrel{efr}{\simeq} G : [\Delta t, \min(\ell_i), \max(u_i)]$  to  $\mathcal{K}'$ . Clearly, for each element in  $\mathcal{K}'$ , there is an element in every  $\mathcal{K}_i$  with the same or tighter probability bounds. It is also obvious tha  $\mathcal{K}' \in PROG_{B_{\mathcal{L}}, t_{max}}$ . Assume that there is a  $\mathcal{K}''$  (not equivalent to  $\mathcal{K}'$ ) that is below each  $\mathcal{K}_i$  but above  $\mathcal{K}'$ . Then, for all elements in  $\mathcal{K}'$ , there must be a corresponding element (with tighter probability bounds) in  $\mathcal{K}''$  s.t. the probability bounds is looser than any  $\mathcal{K}_i$ . However, by the construction, this is clearly not possible unless  $\mathcal{K}' \equiv \mathcal{K}''$ , so we have a contradiction.

CLAIM 2: For a set PROG' of APT-programs, sup(PROG') exists and is in  $PROG_{B_{\mathcal{L}},t_{max}}$ .

Let  $PROG' = \{\mathcal{K}_1, \ldots, \mathcal{K}_i, \ldots, \mathcal{K}_n\}$ . Let  $\mathcal{K}' = \bigcup_i \{\mathcal{K}_i\}$ . Clearly, by the definition of  $\sqsubseteq$ , this is a least upper bound of PROG'. We must show that  $\mathcal{K}'$  is in  $PROG_{B_{\mathcal{L}},t_{max}}$ . We have two cases. (1) If  $\mathcal{K}'$  is inconsistent, then it is equivalent to  $\top$  and in  $PROG_{B_{\mathcal{L}},t_{max}}$ . (2) If  $\mathcal{K}'$  is consistent, then it is also in  $PROG_{B_{\mathcal{L}},t_{max}}$ .  $\Box$ 

# B.12 Proof of Lemma 4.22

 $\mathcal{K} \sqsubseteq \Gamma(\mathcal{K}).$ 

PROOF. Follows directly from the definition of  $\Gamma$  - all rules and ptf's in  $\mathcal{K}$  are in  $\Gamma(\mathcal{K})$  with equivalent or tighter probability bounds. All IC's in  $\mathcal{K}$  remain in  $\Gamma(\mathcal{K})$ .  $\Box$ 

# B.13 Proof of Lemma 4.23

 $\Gamma$  is monotonic.

PROOF. Given  $\mathcal{K}_1 \sqsubseteq \mathcal{K}_2$ , we must show  $\Gamma(\mathcal{K}_1) \sqsubseteq \Gamma(\mathcal{K}_2)$ . Suppose, BWOC, there exists  $\phi : [\ell, u] \in \Gamma(\mathcal{K}_1)$  (see note <sup>11</sup>) s.t. there does not exist  $\phi : [\ell', u'] \in \Gamma(\mathcal{K}_2)$  where  $[\ell', u'] \subseteq [\ell, u]$ . Therefore, there must exist a set of ptf's and/or rules (call this set  $\mathcal{K}'_1$ ) in  $\mathcal{K}_1$  s.t. for each element in  $\mathcal{K}'_1$ , there does not exist a an element in  $\mathcal{K}_2$  s.t. the probability bounds are tighter. However, as  $\mathcal{K}_1 \sqsubseteq \mathcal{K}_2$ , this cannot be possible, and we have a contradiction.  $\Box$ 

# B.14 Proof of Theorem 4.24

 $\Gamma$  has a least fixed point.

PROOF. Follows directly from Lemma 4.22 and Lemma 4.23.  $\Box$ 

# B.15 Proof of Lemma 4.25

If APT-logic program  $\mathcal{K}$  entails rule  $F \stackrel{efr}{\leadsto} G : [\Delta t, \ell, u]$  or  $\phi : [\ell, u]$  such that one of the following is true:

• 
$$\ell > u$$

•  $\ell < 0 \text{ or } \ell > 1$ 

• 
$$u < 0$$
 or  $u > 1$ 

Then  $\mathcal{K}$  is **inconsistent** - i.e. there exists no interpretation I such that  $I \models \mathcal{K}$ .

**PROOF.** Following directly from the definitions of satisfaciton and entailment, if  $\mathcal{K}$  entails such a rule or ptf, there can be no satisfying interpretation.  $\Box$ 

#### B.16 Proof of Theorem 4.26

For APT-logic program  $\mathcal{K}$ , if there exists natural number *i* such that  $\Gamma(\mathcal{K}) \uparrow i$  that contains rule  $F \stackrel{efr}{\simeq} G : [\Delta t, \ell, u]$  or  $\phi : [\ell, u]$  such that one of the following is true:

- $\bullet \ \ell > u$
- $\bullet \ \ell < 0 \ {\rm or} \ \ell > 1$
- u < 0 or u > 1

# Then $\mathcal{K}$ is **inconsistent**.

PROOF. We know by Propositions 4.16 that any number of applications of  $\Gamma$  result in an APT-program entailed by  $\mathcal{K}$ . Therefore, all of the elements of that program must be entailed by  $\mathcal{K}$ . By Lemma 4.25, the statement follows.  $\Box$ 

<sup>&</sup>lt;sup>11</sup>Resp.  $F \stackrel{efr}{\sim} G : [\Delta t, \ell, u] \in \Gamma(\mathcal{K}_1)$ , we note that the proof can easily be mirrored for rules, we only show with ptfs here.

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#### B.17 Proof of Proposition 4.27

If there does not exist at least one thread that satisfies all integrity constraints in an APT-logic program, then that program is inconsistent.

PROOF. For an APT-logic program to be consistent, then there must exist a satisfying interpretation such that the sum of the probabilities assigned to all threads is 1. However, if there is no thread that satisfies all integrity constraints, then the sum of the probabilities of all threads in a satisfying interpretation is 0 - a contradiction.  $\Box$ 

# B.18 Proof of Proposition 4.29

If  $\mathsf{lo}_i > \left\lceil \frac{(\mathsf{blk}_i - 1) \cdot t_{max}}{\mathsf{blk}_i} \right\rceil$  then there does not exist a partial thread for ground atom  $A_i$  such that the single block-size and occurrence IC associated with  $A_i$  hold. Follows directly from the following Proposition:

PROPOSITION B.1. For atom  $a_i$ , block size  $blk_i$  and  $t_{max}$ , if more than  $\left| \frac{(blk_i-1)\cdot t_{max}}{blk_i} \right|$ worlds must be true, then all partial threads will have a block of size  $blk_i$ .

PROOF. CLAIM 1: If we require less than (or equal)  $\left\lceil \frac{(\mathsf{blk}_i-1) \cdot t_{max}}{\mathsf{blk}_i} \right\rceil$  worlds to satisfy the atom, there exists at least one partial thread that does not contain a block.

Simply consider  $\mathsf{blk}_i - 2$  sub-sequences of  $\mathsf{blk}_i - 1$  worlds, and one sub-sequence of  $\leq \mathsf{blk}_i - 1$  worlds satisfying atom  $a_i$  - each separated by a world that does not satisfy  $a_i$ . Obviously, this partial thread does not contain a block.

CLAIM 2: If we require more than  $\left\lceil \frac{(\mathsf{blk}_i-1)\cdot t_{max}}{\mathsf{blk}_i} \right\rceil$  worlds to satisfy the atom, there can be no sequence of two consecutive worlds that do not satisfy  $a_i$ , or there exists a block.

This follows from the pigeon hole principle - if two consecutive worlds satisfy  $\neg a_i$ , then there must exists a sequence of at least  $\mathsf{blk}_i$  worlds that satisfy  $a_i$ .

PROOF OF PROPOSITION: Suppose we have a partial thread with  $\left\lceil \frac{(\mathsf{blk}_i-1)\cdot t_{max}}{\mathsf{blk}_i} \right\rceil$  worlds satisfying the atom, and require one additional world to satisfy  $a_i$ . By claim 2, this world must be between two sub-sequences, as there are no more than two non-satisfying worlds, hence the statement of the proposition follows.  $\Box$ 

#### B.19 Proof of Propositon 4.30

For ground atom  $A_i$  with (with associated ICs), if  $\mathsf{up}_i > \left\lceil \frac{(\mathsf{blk}_i - 1) \cdot t_{max}}{\mathsf{blk}_i} \right\rceil$  we know that for numbers of worlds satisfying  $A_i$  cannot be in the range  $\left[ \left\lceil \frac{(\mathsf{blk}_i - 1) \cdot t_{max}}{\mathsf{blk}_i} \right\rceil, up_i \right]$ .

PROOF. As, in this case,  $\mathsf{up}_i > \left\lceil \frac{(\mathsf{blk}_i - 1) \cdot t_{max}}{\mathsf{blk}_i} \right\rceil$ , lowering the value of  $\mathsf{up}_i$  will not cause an inconsistency unless Proposition 4.29 applies. We note that by Proposition B.1, we cannot have threads with more than this amount of worlds satisfying  $a_i$ .  $\Box$ 

B.20 Proof of Proposition 4.31

**ThEX** can be solved in O(1).

Proof. As the check in Proposition 4.29 can be performed in O(1) time, the statement follows.  $\hfill\square$ 

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# C. PROOFS FOR SECTION 5

#### C.1 Proof of Lemma 5.2

Given non-ground formulas  $F_{ng}, G_{ng}$ , time  $\Delta t$ , and non-ground time formula  $\phi_{ng}$ . Let  $(\alpha_{in}, \beta_{in}) = EFR\_IN(F_{ng}, G_{ng}, \Delta t, \phi_{ng})$  and  $[\alpha_{out}, \beta_{out}] = EFR\_OUT(F_{ng}, G_{ng}, \Delta t, \phi_{ng})$ . Then the following holds true:

- (1) If  $Th \models \phi_{ng}$ , then for all ground instances F, G of  $F_{ng}, G_{ng}$  we have  $efr(F, G, \Delta t, Th) \in$  $|\alpha_{out}, \beta_{out}|$
- (2) If  $Th \models \phi_{ng}$ , then there exists ground instances F, G of  $F_{ng}, G_{ng}$  we have  $efr(F, G, \Delta t, Th) \ge \alpha_{in}$
- (3) If  $Th \models \phi_{nq}$ , then there exists ground instances F, G of  $F_{nq}, G_{nq}$  we have  $efr(F, G, \Delta t, Th) \leq \beta_{in}$

PROOF. CLAIM 1: Part 1 is true..

Suppose, BWOC, there is some thread,  $Th \models \phi_{nq}$  s.t. there are ground instances F, G of  $F_{nq}, G_{nq}$  s.t.  $efr(F, G, \Delta t, Th) \notin [\alpha_{out}, \beta_{out}]$ . This directly contradicts Definition 5.1.

CLAIM 2: Part 2 is true.

This directly contradicts Definition 5.1.

CLAIM 3: Part 3 is true.

This directly contradicts Definition 5.1.  $\Box$ 

#### C.2 Proof of Theorem 5.3

Given non-ground APT-program  $\mathcal{K}^{(ng)}$  that contains the following:

Non-ground rule: 
$$F_{ng} \stackrel{egr}{\sim} G_{ng} : [\Delta t, \ell, u]$$
  
Non-ground ptf:  $\phi_{ng} : [1, 1]$ 

Let  $(\alpha_{in}, \beta_{in}) = EFR\_IN(F_{ng}, G_{ng}, \Delta t, \phi_{ng})$ . If we are given  $\alpha_{in} \leq \alpha_{in}$  and  $\beta_{in}^+ \geq \alpha_{in}$  $\beta_{in}$ , then,  $\mathcal{K}^{(ng)}$  is not consistent if one (or both) of the following is true:

- (1)  $\alpha_{in}^- > u$
- (2)  $\beta_{in}^+ < \ell$

PROOF. CLAIM 1: If  $\alpha_{in}^- > u$ , then  $\mathcal{K}^{(ng)}$  is not consistent. Suppose, BWOC that  $\alpha_{in}^- > u$  and  $\mathcal{K}^{(ng)}$  is consistent. Then, by Lemma 5.2 there exists ground instances F, G of  $F_{ng}, G_{ng}$  s.t.  $EFR(F, G, \Delta t, gnd(\phi_{ng})) \subseteq [\alpha_{in}^{-}, 1].$ Therefore, by Theorem 4.6,  $\mathcal{K}^{(ng)}$  entails  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \alpha_{in}^{-}, 1]$ . However, as  $\mathcal{K}^{(ng)}$ includes  $F_{ng} \overset{efr}{\sim} G_{ng}$  :  $[\Delta t, \ell, u]$ , then  $\mathcal{K}^{(ng)}$  also entails  $F \overset{efr}{\sim} G$  :  $[\Delta t, \ell, u]$ . As  $[\alpha_{in}^{-},1] \cap [\ell,u] = \emptyset$ , we know that  $\mathcal{K}^{(ng)}$  cannot be consistent (by Lemma 4.25) – a contradiction.

CLAIM 2: If  $\beta_{in}^+ < \ell$ , then  $\mathcal{K}^{(ng)}$  is not consistent. Suppose, BWOC, that  $\beta_{in}^+ < \ell$  and  $\mathcal{K}^{(ng)}$  is consistent. Then, by Lemma 5.2 there exists ground instances F, G of  $F_{ng}, G_{ng}$  s.t.  $EFR(F, G, \Delta t, gnd(\phi_{ng})) \subseteq [0, \beta_{in}^+]$ . Therefore, by Theorem 4.6,  $\mathcal{K}^{(ng)}$  entails  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, 0, \beta_{in}^+]$ . However, as  $\mathcal{K}^{(ng)}$ includes  $F_{ng} \overset{efr}{\sim} G_{ng}$  :  $[\Delta t, \ell, u]$ , then  $\mathcal{K}^{(ng)}$  also entails  $F \overset{efr}{\sim} G$  :  $[\Delta t, \ell, u]$ . As ACM Transactions on Computational Logic, Vol. V, No. N, April 2010.

 $[0, \beta_{in}^+] \cap [\ell, u] = \emptyset$ , we know that  $\mathcal{K}^{(ng)}$  cannot be consistent (by Lemma 4.25) – a contradiction.

C.3 Proof of Corollary 5.4

Given non-ground APT-program  $\mathcal{K}^{(ng)}$  that contains the following:

Non-ground rule:  $F_{ng} \stackrel{efr}{\sim} G_{ng} : [\Delta t, \ell, u]$ Non-ground ptf:  $\phi_{ng} : [\ell', u']$ 

Let  $(\alpha_{in}, \beta_{in}) = EFR IN(F_{ng}, G_{ng}, \Delta t, \phi_{ng})$ . If we are given  $\alpha_{in}^- \leq \alpha_{in}$  and  $\beta_{in}^+ \geq \beta_{in}$ , then,  $\mathcal{K}^{(ng)}$  is not consistent if  $\alpha_{in}^- \cdot \ell' > u$ .

PROOF. Suppose, BWOC,  $\alpha_{in}^- \cdot \ell' > u$  and  $\mathcal{K}^{(ng)}$  is consistent. Then, by Lemma 5.2 there exists ground instances F, G of  $F_{ng}, G_{ng}$  s.t.  $EFR(F, G, \Delta t, gnd(\phi_{ng})) \subseteq [\alpha_{in}^-, 1]$ . Therefore, by Corollary 4.7,  $\mathcal{K}^{(ng)}$  entails  $F \stackrel{efr}{\longrightarrow} G : [\Delta t, \alpha_{in}^- \cdot \ell', 1]$ . However, as  $\mathcal{K}^{(ng)}$  includes  $F_{ng} \stackrel{efr}{\longrightarrow} G_{ng} : [\Delta t, \ell, u]$ , then  $\mathcal{K}^{(ng)}$  also entails  $F \stackrel{efr}{\longrightarrow} G : [\Delta t, \ell, u]$ . As  $[\alpha_{in}^- \cdot \ell', 1] \cap [\ell, u] = \emptyset$ , we know that  $\mathcal{K}^{(ng)}$  cannot be consistent (by Lemma 4.25) – a contradiction.  $\Box$ 

# C.4 Proof of Proposition 5.5

If the list returned by NG-INCONSIST-CHK contains any elements, then  $\mathcal{K}^{(ng)}$  is not consistent.

**PROOF.** Follows directly from Theorem 5.3 and Corollary 5.4.  $\Box$ 

# C.5 Proof of Proposition 5.6

NG-INCONSIST-CHK performs  $O(|\mathcal{K}^{(ng)}|^2)$  comparisons.

PROOF. The algorithm consists of two nested loops. The outer loop considers all ptf's in the program – requiring  $O(|\mathcal{K}^{(ng)}|)$  time, while the inner loop considers all rules in the program – also requiring  $O(|\mathcal{K}^{(ng)}|)$  time. The statement follows.  $\Box$ 

#### C.6 Proof of Lemma 5.8

 $\mathcal{K} \subseteq \Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}) \text{ wrt } \langle PROG_{B_{\mathcal{L}}, t_{max}}, \sqsubseteq \rangle$ 

PROOF. Follows directly from Definition 5.7.  $\Box$ 

#### C.7 Proof of Lemma 5.9

 $\Lambda_{\mathcal{K}^{(ng)}}$  is monotonic.

PROOF. Given  $\mathcal{K}_1 \sqsubseteq \mathcal{K}_2$  (both ground), we must show  $\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}_1) \sqsubseteq \Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}_2)$ . Suppose, BWOC, there is an element (rule, ptf, or IC) of  $\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}_1)$  that either has a tighter probability bound than a corresponding element in  $\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}_2)$  or not in  $\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}_2)$ . However, this is a contradiction as all elements in  $\mathcal{K}_1$  are in  $\mathcal{K}_2$  – or in  $\mathcal{K}_2$  with a tighter probability bound. Therefore, such an element would be in  $\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}_2)$  – a contradiction.  $\Box$ 

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C.8 Proof of Theorem 5.11

 $\Lambda_{\mathcal{K}^{(ng)}}$  has a least fixed point.

PROOF. Follows directly from Lemma 5.8 and Lemma 5.9.  $\Box$ 

# C.9 Proof of Lemma 5.12

Given non-ground program  $\mathcal{K}^{(ng)}$ , and ground program  $\mathcal{K}$ ,  $lfp(\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K})) \subseteq ground(\mathcal{K}^{(ng)}) \cup \mathcal{K}$ .

PROOF. Suppose, BWOC, that  $lfp(\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K})) \not\subseteq ground(\mathcal{K}^{(ng)}) \cup \mathcal{K}$ . Then, there must exist a ground rule, ptf, or IC in element in  $lfp(\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}))$  that is not in  $ground(\mathcal{K}^{(ng)}) \cup \mathcal{K}$ . However, all elements in  $lfp(\Lambda_{\mathcal{K}^{(ng)}}(\mathcal{K}))$  are either elements of  $\mathcal{K}$  or ground instances of elements in  $\mathcal{K}^{(ng)}$  – hence a contradiction.  $\Box$ 

# C.10 Proof of Theorem 5.13

Definition C.1 Tightening. For APT-rule  $F \stackrel{efr}{\leadsto} G : [\Delta t, \ell, u]$  or ptf  $\phi : [\ell, u]$ , for any  $[\ell', u'] \subseteq [\ell, u]$ ,

(1)  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell', u']$  is a **tightening** of  $F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell, u]$ 

(2)  $\phi : [\ell, u]$  is a **tightening** of  $\phi : [\ell', u']$ 

Definition C.2 Update. Given ground APT-program  $\mathcal{K}$ , ground rule  $r = F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell_1, u_1]$ , and ground ptf  $p = \phi : [\ell_2, u_2]$ , any tightening to the bounds of r or p causes by an application of the operator  $\Gamma$  is an **update**.

Definition C.3 Update Widget. Given ground APT-program  $\mathcal{K}$ , ground rule  $r = F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell_1, u_1]$ , and ground ptf  $p = \phi : [\ell_2, u_2]$ , ground atomic time formula A : t, we define the following **update widgets**.

- (1) Let the ground rule  $r' = F \stackrel{efr}{\rightsquigarrow} G : [\Delta t, \ell', u']$  be a tightening of r where  $\ell' = \mathsf{l\_bnd}(F, G, \Delta t, \mathcal{K})$  or  $u' = \mathsf{u\_bnd}(F, G, \Delta t, \mathcal{K})$ . Then an **update widget** consists of a graph of a vertex  $v_{r'}$  for r' (called a **top vertex**) and set V of vertices one vertex for each ground rule and ptf in  $\mathcal{K}$  that led to the tightening (as per Definition 4.13) (called **bottom vertices**) and directed edges from all elements in V to  $v_{r'}$ .
- (2) Let the ground ptf  $p' = \phi : [\ell', u']$  be a tightening of  $\phi : [\ell_2, u_2]$  where  $\ell' \in \{\mathbf{l\_bnd}(\phi, \mathcal{K}), 1 \mathbf{u\_bnd}(\neg \phi, \mathcal{K})\}$  or  $u' \in \{\mathbf{u\_bnd}(\phi, \mathcal{K}), 1 \mathbf{l\_bnd}(\neg \phi, \mathcal{K})\}$ . Then an **update widget** consists of a graph of a vertex  $v_{p'}$  for p' (called a **top vertex**) and set V of vertices - one vertex for each ground rule and ptf in  $\mathcal{K}$  that led to the tightening (as per Definition 4.13) (called **bottom vertices**) and directed edges from all elements in V to  $v_{p'}$ .
- (3) If K entails A : t : [0,0] due to the presence of ptf's and IC's (as per Propositions 4.10-4.11), then Then an update widget consists of a graph of a vertex v<sub>A:t:[0,0]</sub> for A : t : [0,0] (called a top vertex) and set V of vertices one for each IC and ptf in K that led to the entailment of A : t : [0,0] (called bottom vertices) and directed edges from all elements in V to v<sub>r'</sub>.

Definition C.4 Deduction Tree. A series of update widgets with the top vertices of all but one widgets are the bottom vertices for another widget is called a **deduction tree**. A vertex that is not a bottom vertex for any widget in the tree is a **root** and a vertex that is not top vertex for any widget in the tree is a **leaf**. For a given deduction tree, T, let leaf(T) be the set of ptf's or rules corresponding with leaf nodes in the tree.

Definition C.5 Corresponding Deduction Tree. Given ground APT-program  $\mathcal{K}$ , for ground ptf  $p = \phi$  :  $[\ell_2, u_2]$ , s.t.  $p \in lfp(\Gamma(\mathcal{K}))$ , then the **corresponding deduction tree** is a deduction tree, rooted in a node representing p s.t. for each update performed by  $\Gamma$ , there is a corresponding update widget in the tree. For program  $\mathcal{K}$  and ptf p, let  $T_{\mathcal{K},p}$  be the corresponding deduction tree.

LEMMA C.6. If  $\phi : [\ell, u] \in lfp(\Gamma(\mathcal{K} \cup \{\phi : [0, 1]\})$  then there exists  $\phi : [\ell', u'] \in lfp(\Gamma(leaf(T_{\mathcal{K},\phi:[\ell,u]}) \cup \{\phi : [0, 1]\})$  s.t.  $[\ell', u'] \subseteq [\ell, u]$ .

PROOF. Suppose, BWOC, that  $[\ell', u'] \not\subseteq [\ell, u]$ . Then, there must exist an update performed by  $\Gamma$  that uses some ptf or rule other  $\in \mathcal{K}$  s.t. other  $\notin leaf(T_{\mathcal{K},\phi:[\ell,u]})$ . However, by the Definition C.5 this is not possible as  $T_{\mathcal{K},\phi:[\ell,u]}$  accounts for all updates performed by  $\Gamma$ .  $\Box$ 

# Theorem 5.13

Given non-ground program  $\mathcal{K}^{(ng)}$ 

$$\phi: [\ell, u] \in lfp(\Gamma(lfp(\Lambda_{\mathcal{K}^{(ng)}}(\{\phi: [0, 1]\}))))$$

iff

$$\phi: [\ell, u] \in lfp(\Gamma(ground(\mathcal{K}^{(ng)}) \cup \{\phi: [0, 1]\}))$$

PROOF. CLAIM 1: If  $\phi : [\ell, u] \in lfp(\Gamma(lfp(\Lambda_{\mathcal{K}^{(ng)}}(\{\phi : [0, 1]\}))))$  then for some  $[\ell', u'] \subseteq [\ell, u], \phi : [\ell', u'] \in lfp(\Gamma(ground(\mathcal{K}^{(ng)}) \cup \{\phi : [0, 1]\})).$ By Lemma 5.12, we know that  $lfp(\Lambda_{\mathcal{K}^{(ng)}}(\{\phi : [0, 1]\})) \subseteq ground(\mathcal{K}^{(ng)}) \cup \{\phi : [0, 1]\})$ , so the claim follows.

CLAIM 2: If  $\phi : [\ell, u] \in lfp(\Gamma(ground(\mathcal{K}^{(ng)}) \cup \{\phi : [0, 1]\}))$  then for some  $[\ell', u'] \subseteq [\ell, u], \phi : [\ell, u] \in lfp(\Gamma(lfp(\Lambda_{\mathcal{K}^{(ng)}}(\{\phi : [0, 1]\})))).$ By Definition 5.7 and Definition C.5,  $leaf(T_{\mathcal{K},\phi:[\ell,u]}) \cup \{\phi : [0, 1]\} \subseteq lfp(\Lambda_{\mathcal{K}^{(ng)}}(\{\phi : [0, 1]\}))$ . Hence, we can apply Lemma C.6 and the claim follows. The statement of the theorem follows directly from claims 1-2.  $\Box$ 

# D. SUPPLEMENTAL INFORMATION FOR SECTION 6

# D.1 Proof of Proposition 6.1

OC-EXTRACT runs in time  $O((n - t_{max}) \cdot t_{max})$ .

PROOF. This follows directly from the two for loops in the algorithm - the first iterating  $(n - t_{max})$  time and a nested loop iterating  $t_{max}$  times.  $\Box$ 

#### D.2 Proof of Proposition 6.2

There are no historical threads such that atom  $a_i$  is satisfied by less than  $lo_i$  or more than  $up_i$  worlds when  $lo_i, up_i$  are produced by OC-EXTRACT.

PROOF. Suppose, by way of contradiction, that there exists a historical thread that does not meet the constraints. As we examine all possible historical threads in OC-EXTRACT and take the minimum and maximum number of times  $a_i$  is satisfied over all these threads, we have a contradiction.  $\Box$ 

# D.3 Proof of Proposition 6.3

BLOCK-EXTRACT runs in time O(n).

PROOF. Follows directly from the for loop in the algorithm - which iterates n times.  $\Box$ 

# D.4 Proof of Proposition 6.4

Given  $blk_i$  as returned by BLOCK-EXTRACT, there is no sequence of  $blk_i$  or more consecutive historical worlds that satisfy atom  $a_i$ .

PROOF. Suppose there is a sequence of at least  $blk_i$  or more. However, the algorithm maintains the variable *best* which is the greatest number of consecutive time points in the historical data where  $a_i$  is true – this is a contradiction.  $\Box$ 

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