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The Complexity of Reasoning for Fragments of Autoepistemic Logic

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Autoepistemic logic extends propositional logic by the modal operator L. A formula φ that is preceded by an L is said to be "believed". The logic was introduced by Moore in 1985 for modeling an ideally rational agent's behavior and reasoning about his own beliefs. In this paper we analyze all Boolean fragments of autoepistemic logic with respect to the computational complexity of the three most common decision problems expansion existence, brave reasoning and cautious reasoning. As a second contribution we classify the computational complexity of checking that a given set of formulae characterizes a stable expansion and that of counting the number of stable expansions of a given knowledge base. We improve the best known Δ_2^p-upper bound on the former problem to completeness for the second level of the Boolean hierarchy. To the best of our knowledge, this is the first paper analyzing counting problem for autoepistemic logic.

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1. INTRODUCTION

Non-monotonic logics are among the most important calculi in the area of knowledge representation and reasoning. Autoepistemic logic, introduced 1985 by Moore [Moore 1985], is one of the most prominent non-monotonic logics. It was originally created to overcome difficulties present in the non-monotonic modal logics proposed by McDermott and Doyle [McDermott and Doyle 1980], but was also shown to offer a unified approach to other models of non-monotonic reasoning: it is known to embed McCarthy's circumscription [McCarthy 1980] and Reiter's default logic [Reiter 1980] under a certain types of translation, and can be used to define the semantics of logic programs; see, *e.g.*, [Marek and Truszczynski 1991; Lifschitz and Schwarz 1993].

Autoepistemic logic extends classical logic with a unary modal operator L expressing the beliefs of an ideally rational agent. The sentence $L\varphi$ means that the agent can derive φ based on its knowledge. To formally capture the set of beliefs of an agent, the

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notion of *stable expansions* was introduced. Stable expansions are defined as the fixed points of an operator deriving the logical consequences of the agent's knowledge and belief. A given knowledge base may admit no or several such stable expansions. Hence, the following questions naturally arise. Given a knowledge base Σ, does Σ admit a stable expansion at all? And given a knowledge base Σ and a formula φ, is φ contained in at least one (resp. all) stable expansion of Σ?

While all these problems are undecidable for first-order autoepistemic logic, they are situated at the second level of the polynomial hierarchy in the propositional case [Gottlob 1992]; and thus harder to solve than the classical satisfiability or implication problem unless the polynomial hierarchy collapses below its second level. This increase in complexity raises the question for the sources of the hardness on the one hand, and for tractable restrictions on the other hand.

Results. In this paper, we study the computational complexity of the three decision problems mentioned above for fragments of autoepistemic logic, given by restricting the propositional part, *i.e.*, by restricting the set of allowed Boolean connectives. We bound the complexity of all three reasoning tasks for all finite sets of allowed Boolean functions. This approach has first been taken by Lewis, who showed that the satisfiability problem for (pure) propositional logic is NP-complete if the negation of the implication $(x \nrightarrow y)$ can be composed from the set of available Boolean connectives, and is polynomial-time solvable in all other cases. Since then, this approach has been applied to a wide range of problems including equivalence and implication problems [Reith 2003; Beyersdorff et al. 2009a], satisfiability and model checking in modal and temporal logics [Bauland et al. 2006; Bauland et al. 2009; Bauland et al. 2009; Meier et al. 2008], default logic [Beyersdorff et al. 2009b], circumscription [Thomas 2009], abduction [Creignou et al. 2010], and argumentation [Creignou et al. 2010].

Our contributions are theoretical in nature and aim to understand the sources of hardness as well as to provide an understanding which connectives take the role of $(x \nrightarrow y)$ in the context of autoepistemic logic. Furthermore our results exhibit new fragments of lower complexity which might lead to better algorithms for cases in which the set of Boolean connectives can be restricted. Our results might be of interest for knowledge representation and reasoning, however we cannot judge this at the moment.

Though at first sight, an infinite number of sets B of allowed propositional connectives have to be examined, we prove, making use of results from universal algebra, that essentially only seven cases can occur: (1) B can express all Boolean connectives, (2) B can express all monotone Boolean connectives, (3) B can express all linear connectives, (4) B is equivalent to $\{\vee\}$, (5) B is equivalent to $\{\wedge\}$, (6) B is equivalent to $\{\neg\}$, (7) B is empty. We first show, extending Gottlob's results, that the above problems are complete for a class from the second level of the polynomial hierarchy for the cases (1) and (2). In case (4) the complexity drops to completeness for a class from the first level of the hierarchy, whereas for (3) the problem becomes solvable in polynomial time while being hard for $\oplus\mathrm{L}$. Finally, for the cases from (5) to (7) it even drops down to solvability in logarithmic space.

Beyond the expansion existence and the brave and cautious reasoning problems, we also study the complexity of the problem to verify that a given set of formulae characterizes a stable expansion of a given knowledge base. This problem has first been studied by Niemelä in [Niemelä 1991], who showed that stable expansions can be characterized by the truth assignment made to the L-prefixed formulae in the given knowledge base. He showed that this problem in contained in Δ_2^{p}. We prove that this problem is actually contained in DP (*i.e.*, the second level of the Boolean hierarchy) in general and remains hard for DP for the cases (1) and (2). In case (3) the problem becomes $\oplus\mathrm{L}$-complete, and is decidable in logarithmic space in the cases from (4) to (7).

Thus, in summary the question for fragments of lower complexity that are yet expressive can be settled in a negative way: As the expressive power of the L-operator alone is enough to simulate negations, only very weak fragments of autoepistemic logic admit for efficient decision procedures. However, in the case for affine sets of Boolean connectives, our results indicate that the extension of systems of linear equations with a belief operator able to express that a variable takes the value 1 in all solutions does not increase the complexity of deciding the existence of solutions by much—the problem remains efficiently solvable. However, whether any such system can be transformed back to a traditional system of linear equation in polynomial-time remains open (cf. the complexity of the expansion existence problem).

Besides the decision variant, another natural question is concerned with the number of stable expansions. This refers to the so called *counting problem* for stable expansions. Recently, counting problems have gained quite a lot of attention in non-monotonic logics. For circumscription, the counting problem (that is, determining the number of minimal models of a propositional formula) has been studied in [Durand et al. 2005; Durand and Hermann 2008]. For propositional abduction, a different non-monotonic logic, some complexity results for the counting problem (that is, computing the number of so called "solutions" of a propositional abduction problem) were presented in [Hermann and Pichler 2007; Creignou et al. 2010]. Algorithms based on bounded treewidth have been proposed in [Jakl et al. 2008] for the counting problems in abduction and circumscription.

Here, we consider the complexity of the problem to count the number of stable expansions for a given knowledge base. To the best of our knowledge, this problem is addressed here for the first time. We show that in general it is $\#\cdot\mathrm{coNP}$-complete under parsimonious reductions, a class for which few natural problems are known to be complete under this strict type of reductions. More precisely, we show that the expansion counting problem is $\#\cdot\mathrm{coNP}$-complete in cases (1) and (2) from the above, drops to $\#\mathrm{P}$-completeness for the case (4), and is polynomial-time computable in cases (3) and (5) to (7).

Related Work. Similar complexity classifications have also been conducted for other non-monotonic logics, namely default logic [Beyersdorff et al. 2009b], circumscription [Thomas 2009]. Autoepistemic logic is known to embed circumscription [Niemelä 1993] and to be itself embeddable into default logic [Janhunen 1999] using modular polynomial-time transformations. Beyond, using a weaker notions of translations, a non-modular translation from default logic into autoepistemic logic exists [Gottlob 1995]. Consequently, the question arises whether results obtained in this paper are subsumed by results from [Beyersdorff et al. 2009b; Thomas 2009]. That this is not the case as follows from the fact that all of the three mentioned translation require the Boolean connectives $\{\rightarrow, \neg\}$, which alone suffice to simulate all Boolean connectives. Our results can rather be interpreted as complementary in the sense that differences in the computational complexity of fragments of these logics can be accounted for by the approach to model non-monotonicity in the respective logic.

For example, consider the set $B = \{\wedge, \vee\}$. From the results obtained in this article and in [Beyersdorff et al. 2009b], it follows that autoepistemic logic restricted to B-formulae cannot be translated into default logic restricted to B'-formulae unless $[B' \cup \{1\}] = \mathsf{BF}$. This indeed holds for all sets B such that $\mathsf{M} \subseteq [B \cup \{0, 1\}]$.

Although the $\#\cdot\mathrm{coNP}$-hardness of the circumscriptive model problem under subtractive reductions can be transferred to the expansion existence problem via the translation from [Niemelä 1993], membership in $\#\cdot\mathrm{coNP}$ and $\#\cdot\mathrm{coNP}$-hardness under parsimonious reductions cannot be obtained in this way.

Table I. A list of Boolean clones with definitions and bases.

Name	Definition	Base
BF	All Boolean functions	$\{\wedge, \neg\}$
M	$\{f : f \text{ is monotone}\}$	$\{\vee, \wedge, 0, 1\}$
L	$\{f : f \text{ is linear}\}$	$\{\oplus, 1\}$
V	$\{f : f \equiv c_0 \vee \bigvee_{i=1}^{n} c_i x_i \text{ where the } c_i\text{s are constant}\}$	$\{\vee, 0, 1\}$
E	$\{f : f \equiv c_0 \wedge \bigwedge_{i=1}^{n} c_i x_i \text{ where the } c_i\text{s are constant}\}$	$\{\wedge, 0, 1\}$
N	$\{f : f \text{ depends on at most one variable}\}$	$\{\neg, 0, 1\}$
I	$\{f : f \text{ is a projection or a constant}\}$	$\{\mathrm{id}, 0, 1\}$

Organization of the Article. Sections 2 and 3 contain preliminaries and the formal definition of autoepistemic logic. In Section 4 we classify the complexity of the decision problems mentioned above for all finite sets of allowed Boolean functions. Section 5 contains the classification of the problem to count the number of stable expansions. The interrelationship of the fragments of autoepistemic logic with those of default logic and circumscription resulting from the results in this paper are then treated in Section 6. Finally, Section 7 concludes with a discussion of the results.

2. PRELIMINARIES

We use standard notions of complexity theory. For decision problem, the arising complexity degrees encompass the classes L, P, NP, coNP, Σ_2^{p}, and Π_2^{p}. For more background information, the reader is referred to [Papadimitriou 1994]. We furthermore require the class DP defined as $\{A \cap B \mid A \in \mathrm{NP}, B \in \mathrm{coNP}\}$ and the class $\oplus\mathrm{L}$ defined as the class of languages L such that there exists a nondeterministic logspace Turing machine that exhibits an odd number of accepting paths on input x iff $x \in L$ for all x [Buntrock et al. 1992]. It is known that $\mathrm{L} \subseteq \oplus\mathrm{L} \subseteq \mathrm{P}$. Regarding hardness proofs of decision problems, we consider *logspace many-one reductions*, defined as follows: a language A is logspace many-one reducible to some language B (written $A \leq_{\mathrm{m}}^{\log} B$) if there exists a logspace-computable function f such that $x \in A$ iff $f(x) \in B$.

In the context of counting problems, denote by FP the set of all functions computable in polynomial time, and for an arbitrary complexity class $\mathcal{C}$, define $\#\cdot\mathcal{C}$ as the class the functions f for which there exists a set $A \in \mathcal{C}$ (the *witness set* for f) such that there exists a polynomial p such that for all $(x, y) \in A$, $|y| \leq \big|p(x)\big|$, and $f(x) = \big|\{y \mid (x,y) \in A\}\big|$, see [Hemaspaandra and Vollmer 1995]. In particular, we make use of the classes $\#\mathrm{P} = \#\cdot\mathrm{P}$ and $\#\cdot\mathrm{coNP}$. To obtain hardness results for counting problems, we will employ *parsimonious reductions* defined as follows: A counting function f parsimoniously reduces to function h if there is a function $g \in \mathrm{FP}$ such that for all x, $f(x) = h\big(g(x)\big)$. Note the analogy to the simple m-reductions for decision problems defined above.

We moreover assume familiarity with propositional logic. As we are going to consider problems parameterized by the set of Boolean connectives, we require some algebraic tools to classify the complexity of the infinitely many arising problems. A *clone* is a set B of Boolean functions that is closed under superposition, *i.e.*, B contains all projections and is closed under arbitrary compositions (see [Pippenger 1997, Chapter 1] or [Böhler et al. 2003]). For a set B of Boolean functions, we denote by $[B]$ the smallest clone containing B and call B a *base* for $[B]$. Post classified the lattice of all clones and found a finite base for each clone [Post 1941]. A list of all Boolean clones together with a basis for each of them can be found, *e.g.*, in [Böhler et al. 2003]. In order to introduce the clones relevant to this paper, say that an n-ary Boolean function f is *monotone* if $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$ implies $f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n)$, and that f is *linear* if $f \equiv x_1 \oplus \cdots \oplus x_n \oplus c$ for a constant $c \in \{0, 1\}$ and variables $x_1, \ldots, x_n$. The clones relevant to this paper together with their bases are listed in Table I.

3. AUTOEPISTEMIC LOGIC

Autoepistemic logic extends propositional logic by a modal operator L stating that its argument is "believed". Syntactically, the set of autoepistemic formulae $\mathcal{L}_{ae}$ is defined via $\varphi ::= p \mid f(\varphi, \dots, \varphi) \mid L\varphi$, where f is a Boolean function and p is a proposition. The consequence relation $\models$ of the underlying propositional logic is extended to $\mathcal{L}_{ae}$ by simply treating $L\varphi$ as an atomic formula. An *(autoepistemic) B-formula* is an autoepistemic formula using only functions from a finite set B of Boolean functions as connectives. The set of all autoepistemic B-formulae will be denoted by $\mathcal{L}_{ae}(B)$.

Let B be any finite set of Boolean functions. For $\Sigma \subseteq \mathcal{L}_{ae}(B)$, we write $\mathrm{Th}(\Sigma)$ for the deductive closure of Σ, *i.e.*, $\mathrm{Th}(\Sigma) := \{\varphi \in \mathcal{L}_{ae} \mid \Sigma \models \varphi\}$, and $\neg\Sigma$ for $\{\neg\varphi \mid \varphi \in \Sigma\}$. For $\varphi \in \mathcal{L}_{ae}(B)$, let $\mathrm{SF}(\varphi)$ be the set of its subformulae and let $\mathrm{SF}^L(\varphi) := \{L\psi \mid L\psi \in \mathrm{SF}(\varphi)\}$ be the set of its L-prefixed subformulae. The above notions are canonically extended to sets of formulae.

The key notion in autoepistemic logic are stable sets of beliefs grounded on the given premises (the knowledge of the agent). These sets, called *stable expansions*, are defined as the fixed points of the consequences of knowledge and belief.

Definition 3.1. Let $\Sigma \subseteq \mathcal{L}_{ae}(B)$. A set $\Delta \subseteq \mathcal{L}_{ae}$ is a *stable expansion* of Σ if it satisfies the condition $\Delta = \mathrm{Th}(\Sigma \cup L(\Delta) \cup \neg L(\bar{\Delta}))$, where $L(\Delta) := \{L\varphi \mid \varphi \in \Delta\}$ and $\neg L(\bar{\Delta}) := \{\neg L\varphi \mid \varphi \notin \Delta\}$, and $L(\Delta), \neg L(\bar{\Delta}) \subseteq \mathcal{L}_{ae}(B)$.

Example 3.2. Consider the following set Σ_{car} of autoepistemic formulae formalizing knowledge about cars.

$$\Sigma_{car} := \{car, threeWheeler \to rickshaw, car \land \neg L\,threeWheeler \to fourWheeler,\\ L\,rickshaw \to threeWheeler\}$$

The set Σ_{car} has two stable expansions: one being a superset of $\{\neg L\,threeWheeler, \neg L\,rickshaw\}$, and one being a superset of $\{L\,threeWheeler, L\,rickshaw\}$.

Indeed, if $L\,rickshaw$ is contained in a stable expansion Δ, then *threeWheeler* is derivable from the formulae in Σ_{car}, and by definition of stable expansions, $L\,threeWheeler \in \Delta$. On the other hand, if $L\,rickshaw$ is not contained in Δ, then we cannot derive *threeWheeler* from $\mathrm{Th}(\Sigma \cup L(\Delta) \cup \neg L(\bar{\Delta}))$, which implies $\neg L\,threeWheeler \in \neg L(\bar{\Delta}) \subseteq \Delta$. Thus, any stable expansion Δ has to satisfy $L\,threeWheeler \in \Delta$ iff $L\,rickshaw \in \Delta$.

To see that the given sets characterize stable expansions, observe that $\{L\,rickshaw, L\,threeWheeler\} \subseteq \Delta$ implies that $rickshaw, threeWheeler \in \mathrm{Th}(\Sigma \cup L(\Delta) \cup \neg L(\bar{\Delta})) = \Delta$, whereas $\{\neg L\,rickshaw, \neg L\,threeWheeler\} \subseteq \Delta$ implies that neither *rickshaw* nor *threeWheeler* can be derived from $\Sigma \cup L(\Delta) \cup \neg L(\bar{\Delta})$. Thus both sets can be extended to yield a stable expansion.

The three main decision problems in the context of autoepistemic logic are deciding whether a given set of premises has a stable expansion, and deciding whether a given formula in contained in at least one (resp. all) stable expansion. As we are to study the complexity of these problems for finite restricted sets B of Boolean functions, we formally define the *expansion existence problem* as

Problem: $\mathrm{EXP}(B)$
Input: A finite set $\Sigma \subseteq \mathcal{L}_{ae}(B)$
Output: Does Σ have a stable expansion?

and the *brave (resp. cautious) reasoning problems* as

Problem: $\mathrm{MEM_b}(B)$ (resp. $\mathrm{MEM_c}(B)$)
Input: A finite set $\Sigma \subseteq \mathcal{L}_{ae}(B)$, a formula $\varphi \in \mathcal{L}_{ae}(B)$
Output: Is φ contained in some (resp. any) stable expansion of Σ?

Coming back to Example 3.2, it is clear that for any finite set of Boolean functions B with $\{\wedge, \neg, \rightarrow\} \subseteq [B]$, we have $\Sigma_{car} \in \mathrm{EXP}(B)$, $(\Sigma_{car}, \mathit{fourWheeler}) \in \mathrm{MEM}_{\mathrm{b}}(B)$, but $(\Sigma_{car}, \mathit{fourWheeler}) \notin \mathrm{MEM}_{\mathrm{c}}(B)$.

A central tool for the study of the computational complexity of the above problems is the following finitary characterization of stable expansions given by Niemelä [Niemelä 1991].

Definition 3.3. For a set $\Sigma \subseteq \mathcal{L}_{ae}$, a set $\Lambda \subseteq \mathrm{SF}^L(\Sigma) \cup \neg\mathrm{SF}^L(\Sigma)$ is Σ-*full* if for each $L\varphi \in \mathrm{SF}^L(\Sigma)$,

(1) $\Sigma \cup \Lambda \models \varphi$ iff $L\varphi \in \Lambda$,
(2) $\Sigma \cup \Lambda \not\models \varphi$ iff $\neg L\varphi \in \Lambda$.

LEMMA 3.4 [NIEMELÄ 1991]. *Let* $\Sigma \subseteq \mathcal{L}_{ae}$.

(*1*) *Let* Λ *be a* Σ*-full set, then for every* $L\varphi \in \mathrm{SF}^L(\Sigma)$ *either* $L\varphi \in \Lambda$ *or* $\neg L\varphi \in \Lambda$.
(*2*) *The stable expansions of* Σ *and* Σ*-full sets are in one-to-one correspondence.*

It is not hard to see that any stable expansion contains a full set. As an intuitive explanation for the converse direction, observe that any set Λ such that for all $L\varphi \in \mathrm{SF}^L(\Sigma)$ either $L\varphi \in \Lambda$ or $\neg L\varphi \in \Lambda$ fixes the set of beliefs occurring in Σ. Thus, full sets can be interpreted as minimal justified sets of beliefs. Using an hierarchic construction, it can now be shown that these suffice determine all derived beliefs.

Using the finitary characterization of stable expansions as full sets, one can also define the *expansion checking problem* as the problem to decide, given two sets Σ and Λ of autoepistemic formulae, whether Λ is Σ-full:

Problem: $\mathrm{FULL}(B)$
Input: A finite set $\Sigma \subseteq \mathcal{L}_{ae}(B)$ and a finite set $\Lambda \subseteq \mathrm{SF}^L(\Sigma) \cup \neg\mathrm{SF}^L(\Sigma)$
Output: Is Λ a Σ-full set?

Notice also that Lemma 3.4 actually yields a characterization of stable expansions that is polynomial in the size of the given set Σ. To make this characterization more precise, say that a formula is *quasi-atomic* if it is atomic or else begins with an L. Further denote by $\mathrm{SF}^q(\varphi)$ the set of all maximal quasi-atomic subformulae of φ (in the sense that a quasi-atomic subformula is maximal if it is not a subformula of another quasi-atomic subformula of φ). Write $\mathrm{SE}(\Lambda)$ for the stable expansion corresponding to Λ and say that Λ is its *kernel*.

Definition 3.5. Let $\Sigma \subseteq \mathcal{L}_{ae}$ and let $\varphi \in \mathcal{L}_{ae}$. We define the consequence relation $\models_L$ recursively as

$$\Sigma \models_L \varphi \iff \Sigma \cup \mathrm{SB}(\varphi) \models \varphi,$$

where $\mathrm{SB}(\varphi) := \{L\chi \in \mathrm{SF}^q(\varphi) \mid \Sigma \models_L \chi\} \cup \{\neg L\chi \mid L\chi \in \mathrm{SF}^q(\varphi), \Sigma \not\models_L \chi\}$.

The point in defining the consequence relation $\models_L$ is that, once a Σ-full set has been determined, it describes membership in the stable expansion corresponding to Λ for arbitrary $\mathcal{L}_{ae}$-formulae φ.

LEMMA 3.6 [NIEMELÄ 1991]. *Let* $\Sigma \subseteq \mathcal{L}_{ae}$*, let* Λ *be a* Σ*-full set and* $\varphi \in \mathcal{L}_{ae}$*. It holds that* $\Sigma \cup \Lambda \models_L \varphi$ *iff* $\varphi \in \mathrm{SE}(\Lambda)$.

To illustrate the concept of a kernel, recall Σ_{car} from Example 3.2. The kernel of the stable expansion containing $L\mathit{rickshaw}$ is $\Lambda_1 := \{L\mathit{rickshaw}, L\mathit{threeWheeler}\}$; the kernel of the stable expansion containing $\neg L\mathit{rickshaw}$ is $\Lambda_2 := \{\neg L\mathit{rickshaw}, \neg L\mathit{threeWheeler}\}$. Clearly, $\Sigma_{car} \cup \Lambda_2 \models_L \mathit{fourWheeler}$.

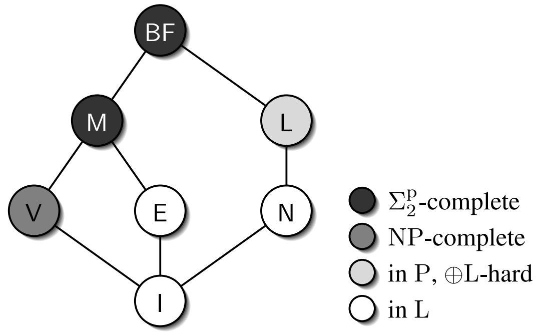

Fig. 1. Relevant clones and their inclusion structure; the shading indicates the complexity of $\mathrm{EXP}(B)$.

4. COMPLEXITY RESULTS

The complexity of the before defined decision problems for autoepistemic logic has already been investigated by Niemelä [Niemelä 1991] and Gottlob [Gottlob 1992]. Niemelä [Niemelä 1991] proved that in order to show that a set Λ is Σ-full, we may simply check whether it satisfies the conditions given in Definition 3.3. Thus the problem of verifying full sets Turing-reduces to the propositional implication problem, which yields membership in Δ_2^{p}. Consequently, the problem of deciding whether Σ has a stable expansion is nondeterministically Turing-reducible to the propositional implication problem and therefore contained in Σ_2^{p} (Lemma 3.4). According to Definition 3.5 and Lemma 3.6 the problem of deciding whether there exists a stable expansion Σ containing a given formula ϕ can be solved with a polynomial number of calls to an NP-oracle by a nondeterministic Turing reduction as follows. Guess a subset Λ; check that Λ is Σ-full and check that $\phi \in \mathrm{SE}(\Lambda)$. Therefore, the brave reasoning problem is in Σ_2^{p}, whereas the cautious reasoning problem is in Π_2^{p}. For the expansion existence, the brave reasoning and the cautious reasoning problem, corresponding hardness results were obtained by Gottlob [Gottlob 1992]. More precisely he obtained completeness results for the special case $B = \{\wedge, \vee, \neg\}$.

We investigate here the complexity of these problems for every B. Observe that the upper bounds, *i.e.*, membership in Σ_2^{p} (resp. Δ_2^{p} and Π_2^{p}) still hold for any B. In order to classify the complexity for the infinitely many cases of B we will make use of Post's lattice as follows: Suppose that $B \subseteq [B']$ for some finite sets B, B' of Boolean functions. Then every function in B can be expressed as a composition of functions from B'; in other words: for every $f \in B$ there is a propositional formula ϕ_f over connectives from B' defining f, and every $\mathcal{L}_{ae}(B)$-formula can be transformed into an equivalent $\mathcal{L}_{ae}(B')$-formula. If moreover in the formulae ϕ_f every free variable appears only once (in this case we say that ϕ_f is a *small* formula for f; and in the proofs below we will see that we can always construct such small formulae), then the transformation of a $\mathcal{L}_{ae}(B)$-formula ψ into an equivalent $\mathcal{L}_{ae}(B')$-formula ψ' is efficient in the sense that the length of ψ' can be bounded by a polynomial in the length of ψ: replacing all occurrences of $f \in B$ with $\phi_f \in \mathcal{L}_{ae}(B')$ yields a formula whose length is bounded by $|\psi| \cdot \max_{f\in B} |\phi_f|$. Thus, the upper bound for the complexity of $\mathrm{EXP}(B')$ yields an upper bound for the complexity of $\mathrm{EXP}(B)$, and a lower bound for the complexity of $\mathrm{EXP}(B)$ yields a lower bound for the complexity of $\mathrm{EXP}(B')$. If $[B] = [B']$ then $\mathrm{EXP}(B)$ and $\mathrm{EXP}(B')$ are of the same complexity (w.r.t. logspace reductions). Thus, the complexity of $\mathrm{EXP}(B)$ is determined by the clone $[B]$. This already brings some structure into the infinitely many problems $\mathrm{EXP}(B')$.

We next note that we may w.l.o.g. assume the availability of the Boolean constants.

LEMMA 4.1. *Let $\mathcal{P}$ be any of the problems* EXP, FULL*,* $\mathrm{MEM_c}$*, or* $\mathrm{MEM_b}$*. Then* $\mathcal{P}(B) \equiv_{\mathrm{m}}^{\log} \mathcal{P}(B \cup \{0,1\})$ *for all finite sets B of Boolean functions.*

PROOF. For the nontrivial direction, let $\Sigma \in \mathcal{L}_{ae}$. We map Σ to $\Sigma' := \Sigma[1/t, 0/Lf] \cup \{t\}$, where t and f are fresh propositions. Then the stable expansions of Σ' and Σ are in one-to-one correspondence, as any expansion of Σ' includes t and $\neg Lf$:

Since f occurs in the scope of an L-operator only, it cannot be derived from Σ' unless Σ' is inconsistent. Thus, by definition of stable expansions and Lemma 3.6, any stable expansion has to contain $\neg Lf$. □

As a consequence of Lemma 4.1, it suffices to consider the clones of the form $[B \cup \{0,1\}]$ (as can be seen immediately from the list of clones given in [Böhler et al. 2003]). These are the seven clones I, N, V, E, L, M, and BF (see Fig. 1). All other cases will have the same complexity of these, by the explanations above. Before we start proving our classification, we note one further observation:

LEMMA 4.2. *For every set $\Sigma \subseteq \mathcal{L}_{ae}$, the inconsistent set $\mathcal{L}_{ae}$ is a stable expansion of Σ iff $\Sigma \cup \mathrm{SF}^L(\Sigma)$ is inconsistent.*

PROOF. Suppose that $\mathcal{L}_{ae}$ is a stable expansion of Σ and let Λ denote its kernel. Then $\Sigma \cup \Lambda \models_L 0$ by virtue of Lemma 3.6. As $\Sigma \cup \Lambda \models_L 0$ iff $\Sigma \cup \Lambda \models 0$, we obtain $\Lambda = \mathrm{SF}^L(\Sigma)$ (notice that $\{L\chi \mid L\chi \in \mathrm{SF}^q(0)\} = \emptyset$, cf. Definition 3.5). In conclusion, $\Sigma \cup \mathrm{SF}^L(\Sigma)$ must be inconsistent. Conversely suppose that $\Sigma \cup \mathrm{SF}^L(\Sigma)$ is inconsistent. Then so is $\mathrm{Th}(\Sigma \cup L(\mathcal{L}_{ae}))$. Consequently, any stable expansion must contain all autoepistemic formulae. □

4.1. Expansion Existence

THEOREM 4.3. *Let B be a finite set of Boolean functions.*

— *If* $[B \cup \{0,1\}]$ *is* BF *or* M *then* $\mathrm{EXP}(B)$ *is* Σ_2^{p}*-complete.*
— *If* $[B \cup \{0,1\}]$ *is* V *then* $\mathrm{EXP}(B)$ *is* NP*-complete.*
— *If* $[B \cup \{0,1\}]$ *is* L *then* $\mathrm{EXP}(B)$ *is* $\oplus$L*-hard and contained in* P.
— *If* $[B \cup \{0,1\}]$ *is* E *or* N *or* I *then* $\mathrm{EXP}(B)$ *is in* L *(solvable in logarithmic space).*

The proof of this theorem requires several propositions.

LEMMA 4.4. *Let B be a finite set of Boolean functions such that* $\mathsf{M} \subseteq [B]$*. Then* $\mathrm{EXP}(B)$ *is* Σ_2^{p}*-complete.*

PROOF. Let B be a finite set of Boolean functions as required. We have to prove Σ_2^{p}-hardness. Let $\varphi := \exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \psi$ be a quantified Boolean formula in disjunctive normal form. In [Gottlob 1992], Gottlob shows that φ is valid iff the set $\Sigma := \{L\psi, Lx_1 \leftrightarrow x_1, \ldots, Lx_n \leftrightarrow x_n\}$ has a stable expansion. The idea of our proof is to modify the given reduction to only use monotone connectives, thus showing that $\mathrm{EXP}(B)$ is Σ_2^{p}-hard for every finite set $\mathsf{M} \subseteq [B]$. More precisely, we define

$$\psi' := \psi[\neg x_1/x_1', \ldots, \neg x_n/x_n', \neg y_1/y_1', \ldots, \neg y_m/y_m']$$

and let

$$\Sigma' := \{L\psi'\} \cup \{x_i \vee Lx_i', Lx_i \vee x_i' \mid 1 \le i \le n\} \cup \{y_j \vee y_j' \mid 1 \le j \le m\}.$$

Clearly, $\Sigma' \subseteq \mathcal{L}_{ae}(\{\wedge, \vee\})$. Moreover, for every $1 \le i \le n$, we have that any consistent stable expansion of Σ' contains either Lx_i or Lx_i' but not both: assume that Λ is a Σ'-full set such that $Lx_i \in \Lambda$ and $Lx_i' \in \Lambda$. Then, by definition of Σ', $\Sigma' \cup \Lambda \not\models x_i$ and $\Sigma' \cup \Lambda \not\models x_i'$, although $Lx_i, Lx_i' \in \Lambda$; a contradiction to Λ being Σ'-full. Otherwise, if Λ were a Σ'-full set such that $\neg Lx_i \in \Lambda$ and $\neg Lx_i' \in \Lambda$, then $\Sigma' \cup \Lambda \models x_i$ and $\Sigma' \cup \Lambda \models x_i'$, a contradiction to Λ being Σ'-full, because $Lx_i, Lx'_i \notin \Lambda$. In conclusion, any Σ'-full set and equivalently any consistent stable expansion of Σ' contains either Lx_i or Lx'_i but not both.

We show that Σ' has a stable expansion if and only if φ is valid. First suppose that Σ' has a stable expansion Δ. Let Λ denote its kernel. As $\Sigma' \cup \mathrm{SF}^L(\Sigma')$ is consistent, we obtain that $\Delta \neq \mathcal{L}_{ae}$ from Lemma 4.2. By the argument above, either $Lx_i \in \Delta$ or $Lx'_i \in \Delta$, but not both. Moreover, $L\psi' \in \Delta$, whence ψ' must be derivable from $\Sigma' \cup \Lambda$ by Definition 3.3. Note that this implies that ψ' is satisfied by all assignments setting either y_i or y'_i to 1; in particular, by all assignments that assign a complementary value to y_i and y'_i for every i. Define a truth assignment $\sigma\colon \{x_i \mid 1 \leq i \leq n\} \to \{0,1\}$ from Λ such that $\sigma(x_i) := 1$ if $Lx_i \in \Lambda$, and $\sigma(x_i) := 0$ otherwise. It follows that $\sigma \models \forall y_1 \cdots \forall y_m \psi$, thus φ is valid.

Now suppose that φ is valid. Then there exists an assignment $\sigma\colon \{x_i \mid 1 \leq i \leq n\} \to \{0,1\}$ such that any extension of σ to $y_1, \ldots, y_m$ satisfies ψ. Let $\Lambda := \{Lx_i, \neg Lx'_i \mid \sigma(x_i) = 1\} \cup \{\neg Lx_i, Lx'_i \mid \sigma(x_i) = 0\} \cup \{L\psi'\}$. We claim that Λ is Σ'-full. If $Lx_i \in \Lambda$, then $\neg Lx'_i \in \Lambda$; hence $\{Lx'_i \vee x_i, \neg Lx'_i\}$ implies x_i. Conversely, if $\Sigma' \cup \Lambda \models x_i$ then $\neg Lx'_i$ has to be in Λ, because x_i occurs in $L\psi'$ and the clause $Lx'_i \vee x_i$ only. From this, we obtain $Lx_i \in \Lambda$. Therefore, $\Sigma' \cup \Lambda \models x_i$ if and only if $Lx_i \in \Lambda$. From the definition of Λ now follows that $\Sigma' \cup \Lambda \not\models x_i$ if and only if $\neg Lx_i \in \Lambda$. The same holds for x'_i for each i. Due to the construction of Λ, the fact that the clause $y_i \vee y'_i$ enforces y'_i to be assigned a value equal to or bigger than the one assigned to $\neg y_i$, the definition of ψ' and its monotonicity, we also have $\Sigma' \cup \Lambda \models \psi'$. Hence, following Definition 3.3, Λ is a Σ'-full set and, by Lemma 3.4, Σ' has a stable expansion.

Finally, note that in any finite set of Boolean functions B such that $\mathsf{M} \subseteq [B]$, conjunction and disjunction can be defined by small formulae, *i.e.*, there exist formulae $\phi_\wedge \equiv x \wedge y$ and $\phi_\vee \equiv x \vee y$ such that x and y occur exactly once in these formulae, see [Schnoor 2010]. □

We cannot transfer the above result to $\mathrm{EXP}(B)$ for $[B] = \mathsf{V}$, because we may not assume ψ to be in conjunctive normal form. But, using a similar idea, we can show that the problem is NP-complete.

LEMMA 4.5. *Let B be a finite set of Boolean functions such that $[B \cup \{0,1\}] = \mathsf{V}$. Then $\mathrm{EXP}(B)$ is NP-complete.*

PROOF. We first show that $\mathrm{EXP}(B)$ is efficiently verifiable, thus proving membership in NP. Given a set $\Sigma \subseteq \mathcal{L}_{ae}$ and a candidate Λ for a Σ-full set, substitute $L\varphi$ by the Boolean value assigned to by Λ and call the resulting set Σ'. Note that Σ' is still equivalent to a set of disjunctions. Therefore the conditions $\Sigma' \models \varphi$ if $L\varphi \in \Lambda$ and $\Sigma' \not\models \varphi$ if $\neg L\varphi \in \Lambda$ can be verified in polynomial time, for $\mathrm{IMP}(B) \in \mathrm{P}$ [Beyersdorff et al. 2009a].

To show NP-hardness, we reduce 3SAT to $\mathrm{EXP}(B)$ as follows. Let $\varphi := \bigwedge_{1 \leq i \leq n} c_i$ with clauses $c_i = \ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$, $1 \leq i \leq n$, be given and let $x_1, \ldots, x_m$ enumerate the propositions occurring in φ. From φ we construct the set

$$\Sigma := \{Lc'_i \mid 1 \leq i \leq n\} \cup \{x_i \vee Lx'_i, Lx_i \vee x'_i \mid 1 \leq i \leq m\},$$

where $c'_i = c_i[\neg x_1/x'_1, \ldots, \neg x_m/x'_m]$ for $1 \leq i \leq n$. Analogously to Lemma 4.4, we obtain that for any stable expansion Δ of Σ either $x_i \in \Delta$ or $x'_i \in \Delta$, but not both. First, suppose that Δ is a stable expansion of Σ. It is easily observed that $\Sigma \cup \mathrm{SF}^L(\Sigma)$ is consistent, therefore $\Delta \neq \mathcal{L}_{ae}$. Let Λ be the kernel of Δ. As $\Delta \neq \mathcal{L}_{ae}$ and $Lc'_i \in \Sigma$ for all $1 \leq i \leq n$, Definition 3.3 implies that $\Sigma \cup \Lambda \models_L c'_i$ and hence $\Sigma \cup \Lambda \models c_i$ for all $1 \leq i \leq n$. From this it follows that φ is satisfied by the assignment σ setting $\sigma(x_i) = 1$ iff $Lx_i \in \Delta$.

Conversely, suppose that φ is satisfied by the assignment σ. Define the set $\Lambda := \{Lx_i, \neg Lx'_i \mid \sigma(x_i) = 1\} \cup \{Lx'_i, \neg Lx_i \mid \sigma(x_i) = 0\} \cup \{Lc'_i \mid 1 \leq i \leq n\}$. As $\sigma \models c_i$ for any $1 \leq i \leq n$, we obtain that $\Sigma \cup \Lambda \models c'_i$. Concluding, Λ is a Σ-full set. □

Next, we turn to the case $[B \cup \{0,1\}] = \mathsf{L}$. Say that an L-prefixed formula is *L-atomic* if it is of the form $L\varphi$ for some atomic formula φ.

LEMMA 4.6. *Let $\Sigma \subseteq \mathcal{L}_{ae}(\{\oplus, 1\})$. If $\mathrm{SF}^L(\Sigma)$ contains only L-atomic formulae, then one can decide in polynomial time whether Σ has a stable expansion.*

PROOF. The idea is to use Gaussian elimination twice. Let Σ be as required and suppose that Σ consists of m formulas. Then the set Σ can be seen as a system of linear equations and thus written as $A\mathbf{x} = B\mathbf{y} \oplus C$, where $\mathbf{x} = (x_1, \dots, x_n)^{\mathrm{T}}$, $\mathbf{y} = (Lx_1, \dots, Lx_n)^{\mathrm{T}}$, A and B are Boolean matrices having m rows, and C is a Boolean vector.

By applying Gaussian elimination to A we obtain an equivalent system $A'\mathbf{x} = B'\mathbf{y} \oplus C'$ with an upper triangular matrix A'. Let r denote the number of free variables in $A'\mathbf{x}$ and suppose w.l.o.g. that these are $x_1, \dots, x_r$. By subsequently eliminating the variables $x_{r+1}, \dots, x_n$, we arrive at a system T equivalent to Σ of the form:

$$\begin{aligned}\{x_i = f_i(x_1, \dots, x_r) \oplus g_i(Lx_1, \dots, Lx_n) \oplus c_i \mid r < i \leq n\} \cup \\ \{0 = g_i(Lx_1, \dots, Lx_n) \oplus c_i \mid n < i \leq m + r\},\end{aligned}$$

where for each i the functions f_i and g_i are linear, and c_i is the constant 0 or 1 (recall that all L-prefixed formulae in Σ are L-atomic). The number of equations in T is still m.

Observe that $\Sigma \cup \mathrm{SF}^L(\Sigma)$ is inconsistent iff $T[Lx_1/1, \dots Lx_n/1]$ has no solution. In this case Σ has $\mathcal{L}_{ae}$ as a stable expansion. Let us now show how to construct a Σ-full set Λ such that $\mathrm{SE}(\Lambda) \neq \mathcal{L}_{ae}$.

Since the variables $x_1, \dots, x_r$ are free, they cannot be derived from $\Sigma \cup \Lambda$ whatever Λ is. The same occurs for every $i \geq r + 1$ such that $f_i(x_1, \dots, x_r)$ is not a constant function. Suppose this is the case for $r + 1 \leq i \leq s$. Then any Σ-full set has to contain $\neg Lx_j$ for $1 \leq j \leq s$. Let T' be the system obtained by considering all remaining equations while replacing Lx_i with 0 for each $1 \leq i \leq s$. For each equation in T', the function f_i (if present) is a constant function ε_i. Therefore T' consists of the following equations:

$$\begin{aligned}\{x_i = \varepsilon_i \oplus g'_i(Lx_{s+1}, \dots, Lx_n) + c_i \mid s < i \leq n\} \cup \\ \{0 = g'_i(Lx_{s+1}, \dots, Lx_n) \oplus c_i \mid n < i \leq m + r\}\end{aligned}$$

with $g'_i(Lx_{s+1}, \dots, Lx_n) := g_i(0, \dots, 0, Lx_{s+1}, \dots, Lx_n)$ for $s < i \leq m + r$. Thus, for every $\Lambda \subseteq \mathrm{SF}^L(\Sigma) \cup \neg\mathrm{SF}^L(\Sigma)$ such that $\{\neg Lx_1, \dots, \neg Lx_s\} \subseteq \Lambda$ and every i, $\Sigma \cup \Lambda \models x_i$ (resp., $\Sigma \cup \Lambda \not\models x_i$) if and only if $T' \cup \Lambda \models x_i$ (resp., $T' \cup \Lambda \not\models x_i$).

We claim that the solution of the system $T'[x_{s+1}/Lx_{s+1}, \dots, x_n/Lx_n]$ are in one-to-one correspondence with the Σ-full sets corresponding to the consistent stable expansions of Σ. From this, we are able conclude, as Σ has a stable expansion iff $T[Lx_1/1, \dots Lx_n/1]$ has no solution (in this case $\mathcal{L}_{ae}$ is a stable expansion) or $T'[x_{s+1}/Lx_{s+1}, \dots, x_n/Lx_n]$ has at least one solution.

Before actually proving the claim, let us illustrate the described procedure. Consider the set $\Sigma := \{x_1 \oplus x_3 \oplus Lx_1 \oplus Lx_2 \oplus 1,\ x_1 \oplus x_2 \oplus x_3 \oplus Lx_2 \oplus Lx_3,\ x_2 \oplus x_3 \oplus Lx_1\}$. From Σ we obtain the following system $A\mathbf{x} = B\mathbf{y} \oplus C$ of linear equations (assuming all formulae in Σ need to be fulfilled, and thus all equations in the corresponding system equal 1):

$$\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \cdot \mathbf{x} = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 0\end{pmatrix} \cdot \mathbf{y} \oplus \begin{pmatrix}0\\ 1\\ 1\end{pmatrix}.$$

After performing Gaussian elimination on A, we arrive at

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \mathbf{x} = \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix} \cdot \mathbf{y} \oplus \begin{pmatrix}0\\ 1\\ 0\end{pmatrix},$$

which yields the following system T:

$$\{x_1 = x_3 \oplus Lx_1 \oplus Lx_2,\ x_2 = Lx_1 \oplus Lx_3 \oplus 1,\ x_3 = Lx_3\}.$$

As $T[Lx_1/1, Lx_2/1, Lx_3/1]$ is consistent, $\mathcal{L}_{ae}$ is no stable expansion of Σ. Further, in T the function f_1 is not constant. Therefore the system T' is given by the second and third equation with Lx_1 replaced by 0:

$$\{x_2 = Lx_1 \oplus Lx_3 \oplus 1,\ x_3 = Lx_3\}.$$

The system $T'[x_2/Lx_2, x_3/Lx_3]$ now has four solutions, namely $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, and $(1,1,1)$. Concluding, Σ possesses four stable expansions.

CLAIM 1. *Let I and J form a partition of $\{s+1,\ldots,n\}$. Then $(Lx_{s+1},\ldots,Lx_n)$ with $Lx_i = 0$ if $i \in I$ and $Lx_j = 1$ if $j \in J$ is a solution of the system $T'[x_{s+1}/Lx_{s+1},\ldots,x_n/Lx_n]$ if and only if $\Lambda = \{\neg Lx_1,\ldots,\neg Lx_s\} \cup \{\neg Lx_i \mid i \in I\} \cup \{Lx_j \mid j \in J\}$ is a Σ-full set.*

To prove the claim, let $\Lambda = \{\neg Lx_1,\ldots,\neg Lx_s\} \cup \{\neg Lx_i \mid i \in I\} \cup \{Lx_j \mid j \in J\}$ be a Σ-full set. Observe that $\Sigma \cup \Lambda$ is consistent and that either $T' \cup \Lambda \models x_i$ or $T' \cup \Lambda \models \neg x_i$, for each i. Denote by λ the truth assignment induced by Λ on $\mathrm{SF}^L(\Sigma)$. Then, for every $i > s$, $Lx_i \in \Lambda$ iff $\lambda(Lx_i) = 1$ iff $T' \cup \Lambda \models x_i$ iff $\varepsilon_i \oplus g'_i\big(\lambda(Lx_{s+1}),\ldots,\lambda(Lx_n)\big) \oplus c_i = 1$; and $\neg Lx_i \in \Lambda$ iff $\lambda(Lx_i) = 0$ iff $T' \cup \Lambda \models \neg x_i$ iff $\varepsilon_i \oplus g'_i\big(\lambda(Lx_{s+1}),\ldots,\lambda(Lx_n)\big) \oplus c_i = 0$. This means that for every i, we have $\varepsilon_i \oplus g'_i\big(\lambda(Lx_{s+1}),\ldots,\lambda(Lx_n)\big) \oplus c_i = \lambda(Lx_i)$. Therefore λ is a solution of the system $\{Lx_i = \varepsilon_i \oplus g_i(0,\ldots,0, Lx_{s+1},\ldots,Lx_n) \oplus c_i \mid s < i \leq n\}$, and hence of the system $T'[x_{s+1}/Lx_{s+1},\ldots,x_n/Lx_n]$.

Conversely, suppose that λ is a solution of $T'[x_{s+1}/Lx_{s+1},\ldots,x_n/Lx_n]$. In particular, λ satisfies $\lambda(Lx_i) = \{\varepsilon_i \oplus g'_i\big(\lambda(Lx_{s+1}),\ldots,\lambda(Lx_n)\big) \oplus c_i \mid s+1 \leq i \leq n\}$. Set $\Lambda := \{\neg Lx_1,\ldots,\neg Lx_s\} \cup \{\neg Lx_i \mid s+1 \leq i \leq n, \lambda(x_i) = 0\} \cup \{Lx_i \mid s+1 \leq i \leq n, \lambda(x_i) = 1\}$. Then $T' \cup \Lambda$ is equivalent to $\{Lx_i = \varepsilon_i \oplus g'_i\big(\lambda(Lx_{s+1}),\ldots,\lambda(Lx_n)\big) \oplus c_i \mid s < i \leq n\} \cup \{0 = g'_i\big(\lambda(Lx_{s+1}),\ldots,\lambda(Lx_n)\big) \oplus c_i \mid n < i \leq m+r\}$. Therefore $T' \cup \Lambda \models x_i$ iff $\lambda(Lx_i) = 1$ and $T' \cup \Lambda \models \neg x_i$ iff $\lambda(Lx_i) = 0$. Hence, Λ is a Σ-full set. This proves the claim. □

Note that solving this last system by Gaussian elimination also gives the total number of possible Σ-full sets: the number of consistent stable expansions is equal to the number of solutions of $T'[x_{s+1}/Lx_{s+1},\ldots,x_n/Lx_n]$; (while testing for the inconsistent stable expansion can also be accomplished in polynomial-time as seen at the beginning of the proof).

LEMMA 4.7. *Let B be a finite set of Boolean functions such that $[B \cup \{0,1\}] = \mathsf{L}$. Then $\mathrm{EXP}(B)$ is $\oplus\mathrm{L}$-hard and contained in P.*

PROOF. Let B be as required and Σ be a set of autoepistemic B-formulae. Then Σ can be written in polynomial time as a set $\big\{c_k \oplus \bigoplus_{i \in I_k} x_i \,\big|\, k \in \mathbb{N}, c_k \in \{0,1\}\big\}$, where k is the number of equations in Σ (see, *e.g.*, [Beyersdorff et al. 2009a]).

We transform this set to Σ' as follows: considering the formulae in $\mathrm{SF}^L(\Sigma)$ ordered by their length, introduce a fresh variable y_φ for every non-atomic formula φ such that $L\varphi \in \mathrm{SF}^L(\Sigma)$; add the equations $y_\varphi \leftrightarrow \varphi$; and replace all occurrences of $L\varphi$ by Ly_φ. We claim that the Σ-full sets and the Σ'-full sets are in one-to-one correspondence. This establishes the upper bound, because Σ' satisfies the conditions of Lemma 4.6.

To prove the claim, we give an inductive argument on the number of non-L-atomic formulae in Σ: Suppose that Σ contains m non-L-atomic formulae, and that the claim is satisfied for all $m' < m$. Further let $\Lambda \subseteq \mathrm{SF}^L(\Sigma) \cup \neg\mathrm{SF}^L(\Sigma)$. Then Σ contains a non-L-atomic subformula $L\varphi \in \mathrm{SF}^L(\Sigma)$ such that all formulae in $\mathrm{SF}^L(\varphi)$ are L-atomic.

Define

$$\Sigma_\varphi := \Sigma[L\varphi/Ly_\varphi] \cup \{\varphi \oplus y_\varphi \oplus 1\},$$
$$\Lambda_\varphi := \begin{cases} (\Lambda \setminus \{L\varphi\}) \cup \{Ly_\varphi\}, & \text{if } L\varphi \in \Lambda, \\ (\Lambda \setminus \{\neg L\varphi\}) \cup \{\neg Ly_\varphi\}, & \text{otherwise.} \end{cases}$$

That is, Σ_φ differs from Σ in that we substituted $L\varphi$ with Ly_φ, and added the formula $\varphi \oplus y_\varphi \oplus 1$. Observe that $\Sigma \cup \Lambda \models \varphi$ iff $\Sigma_\varphi \cup \Lambda_\varphi \models y_\varphi$. Therefore, since $L\varphi \in \Lambda$ iff $Ly_\varphi \in \Lambda_\varphi$ and $\neg L\varphi \in \Lambda$ iff $\neg Ly_\varphi \in \Lambda_\varphi$, it holds that Λ is Σ-full iff Λ_φ is Σ_φ-full.

By induction hypothesis, there now exists a set Σ'_φ of autoepistemic formulae such that all formulae in $\mathrm{SF}^L(\Sigma'_\varphi)$ are L-atomic, $\varphi \oplus y_\varphi \oplus 1 \in \mathrm{SF}^L(\Sigma'_\varphi)$, and the Σ_φ-full sets and Σ'_φ-full sets are in one-to-one correspondence. Let $\Lambda'_\varphi \subseteq \mathrm{SF}^L(\Sigma'_\varphi) \cup \neg\mathrm{SF}^L(\Sigma'_\varphi)$ and define

$$\Sigma' := \Sigma'_\varphi[Ly_\varphi/L\varphi] \setminus \{\varphi \oplus y_\varphi \oplus 1\},$$
$$\Lambda' := \begin{cases} (\Lambda'_\varphi \setminus \{Ly_\varphi\}) \cup \{L\varphi\}, & \text{if } Ly_\varphi \in \Lambda'_\varphi, \\ (\Lambda'_\varphi \setminus \{\neg Ly_\varphi\}) \cup \{\neg L\varphi\}, & \text{otherwise.} \end{cases}$$

Then, by arguments analogous to the above, Λ'_φ is Σ'_φ-full if and only if Λ' is Σ'-full. Hence, the stable expansions of Σ and Σ_φ, of Σ_φ and Σ'_φ, and of Σ'_φ and Σ are in one-to-one correspondence. This proves the claim.

It hence remains to establish $\oplus$L-hardness. We give a reduction from $\mathrm{IMP}(B)$ for $[B \cup \{0,1\}] = \mathsf{L}$, *i.e.*, the problem to decide whether $\Gamma \models \psi$ for a given set Γ of B-formulae and a given B-formula ψ. Since $\mathrm{IMP}(B)$ is $\oplus$L-complete in this case, the proposition follows. For an instance (Γ, ψ) of $\mathrm{IMP}(B)$, let $\Sigma := \Gamma \cup \{L\psi\}$. Indeed, if $\Gamma \models \psi$, then $\Lambda := \{L\psi\}$ is Σ-full; and if $\Lambda := \{L\psi\}$ is Σ-full, then $\Gamma \models \psi$. Thus, $\mathrm{IMP}(B) \leq^{\log}_{\mathrm{m}} \mathrm{EXP}(B)$ via the mapping $(\Gamma, \psi) \mapsto \Sigma$. □

LEMMA 4.8. *Let B be a finite set of Boolean functions such that $[B] \subseteq \mathsf{N}$ or $[B] \subseteq \mathsf{E}$. Then $\mathrm{EXP}(B)$ is solvable in* L. *It moreover holds that, for every set $\Sigma \subseteq \mathcal{L}_{ae}(B)$, there is at most one consistent stable expansion.*

PROOF. Let B be a finite set of Boolean functions such that $[B] \subseteq \mathsf{N}$ and let $\Sigma \subseteq \mathcal{L}_{ae}(B)$ be given. For Σ to have a consistent stable expansion, φ has to be in Σ for all $L\varphi \in \Sigma$, while φ must not be in Σ for all $\neg L\varphi \in \Sigma$ or $L\neg\varphi \in \Sigma$. This can be verified in space logarithmic in the size of Σ. As $\Sigma \equiv \bigwedge \Sigma$, the result for $[B] \subseteq \mathsf{E}$ follows from the above. □

The proof of Theorem 4.3 now immediately follows from Lemmas 4.4–4.8. Note that by Lemma 4.1 and the discussion following that lemma, this covers all cases and, hence, Theorem 4.3 gives a complete classification.

From this theorem and its proof one can easily settle the complexity of the existence of a consistent stable expansion as well as the complexity of the brave and cautious reasoning.

COROLLARY 4.9. *For all finite sets B of Boolean functions, the complexity of the problem to decide whether a set of autoepistemic B-formulae has a consistent stable expansion is the same as for the problem to decide the existence of a stable expansion.*

PROOF. The corollary follows immediately from the proof of Theorem 4.3. Indeed, in each hardness proof (see Lemmas 4.4, 4.5 and 4.7) we have shown that the set of B-premises constructed in that proof, Σ or Σ', does not admit $\mathcal{L}_{ae}$ as a stable expansion. Therefore, Σ or Σ' has a stable expansion iff it has a consistent stable expansion. This proves all the hardness results. As for the upper bounds, Lemmas 4.4 and 4.5 are easily seen to extend to the existence of a consistent stable expansion. For the tractable cases $[B] \subseteq \mathsf{E}$ and $[B] \subseteq \mathsf{N}$, one can decide the existence of a consistent stable expansion in logarithmic space. This follows from the proof of Lemma 4.8. Finally, for $[B] \subseteq \mathsf{L}$, observe that the proof of Lemma 4.7 actually allows to compute full sets corresponding to consistent stable expansions in polynomial time. □

4.2. Expansion Checking

THEOREM 4.10. *Let B be a finite set of Boolean functions.*

— *If* $[B \cup \{0,1\}]$ *is* BF *or* M *then* $\mathrm{FULL}(B)$ *is* DP*-complete.*
— *If* $[B \cup \{0,1\}]$ *is* L *then* $\mathrm{FULL}(B)$ *is* $\oplus\mathrm{L}$*-complete.*
— *If* $[B \cup \{0,1\}]$ *is* V *or* E *or* N *or* I *then* $\mathrm{FULL}(B)$ *is in* L *(solvable in logarithmic space).*

The proof of Theorem 4.10 will be established from the following three lemmas.

LEMMA 4.11. *Let B be a finite set of Boolean functions such that $\mathsf{M} \subseteq [B]$. Then $\mathrm{FULL}(B)$ is DP-complete.*

PROOF. Let B be a finite set of Boolean functions as required. To prove membership in DP, we give a simple reduction to the canonical DP-complete problem $\mathrm{SAT\text{-}UNSAT}$. Given an instance (Σ, Λ) of $\mathrm{FULL}(B)$, we first construct the sets

$$S_1 := \{\Sigma' \cup \{\neg\varphi\} \mid \neg L\varphi \in \Lambda\},$$
$$S_2 := \{\Sigma' \cup \{\neg\varphi\} \mid L\varphi \in \Lambda\},$$

where Σ' is obtained from Σ by substituting $L\varphi$ with 1 if $L\varphi \in \Lambda$, and $L\varphi$ with 0 otherwise. Observe that elements of S_1 and S_2 are sets of propositional formulae. It now holds that $(\Sigma, \Lambda) \in \mathrm{FULL}(B)$ iff all sets of formulae in S_1 are satisfiable while all sets of formulae in S_2 are unsatisfiable. Thus, $(\Sigma, \Lambda) \mapsto (S_1, S_2)$ gives a polynomial-time many-one reduction from $\mathrm{FULL}(B)$ to $\mathrm{SAT\text{-}UNSAT} \in \mathrm{DP}$. The claim follows from the closure of DP under such reductions.

To prove DP-hardness, we give a reduction from the DP-complete problem $\mathrm{CRITSAT}$ [Papadimitriou and Wolfe 1988], which is the problem of deciding whether a given formula in conjunctive normal form is unsatisfiable but removing any of its clauses makes it satisfiable.

The reduction will use the following mapping: Let φ be a given propositional formula in conjunctive normal form over propositions $x_1, \ldots, x_n$. Write ψ for the negation normal form of $\neg\varphi$. Denote by ψ' the formula obtained from ψ by replacing all negative literals x_i by fresh propositions y_i. Then it holds that φ is unsatisfiable iff $\bigwedge_{i=1}^{n}(x_i \vee y_i) \models \psi'$. Notice that these formulae are monotone. Define g to be the mapping from φ to the B-representation of $L\psi'$. By the arguments from the last paragraph in the proof of Lemma 4.4, we may w.l.o.g. assume that g can be computed in space logarithmic in the length of its argument.

Thus, given an instance $\varphi \equiv \bigwedge_{1 \leq i \leq m} c_i$ of $\mathrm{CRITSAT}$ over propositions $x_1, \ldots, x_n$, we map φ to the following sets $\Sigma \subseteq \mathcal{L}_{ae}(\{\wedge, \vee\})$ and $\Lambda \subseteq \mathrm{SF}^L(\Sigma) \cup \neg\mathrm{SF}^L(\Sigma)$:

$$\Sigma := \Big\{ \bigwedge_{i=1}^{n}(x_i \vee y_i), g(\varphi) \Big\} \cup \Big\{ g(\varphi_{-i}) \vee p_i, Lp_i \,\Big|\, 1 \leq i \leq m \Big\},$$
$$\Lambda := \{g(\varphi)\} \cup \{\neg g(\varphi_{-i}), Lp_i \mid 1 \leq i \leq m\},$$

where φ_{-j} denotes the formula $\bigwedge_{1 \leq i \leq m, i \neq j} c_i$, $1 \leq i \leq m$ (recall that g maps its argument to an L-prefixed formula).

Observe that if $\varphi \in$ CRITSAT, then φ is unsatisfiable while all formulae φ_{-i} are satisfiable. As a result, any Σ-full set has to contain $g(\varphi)$ and $\neg g(\varphi_{-i})$. Thus, we have $\Sigma \cup \Lambda' \models p_i$ for all Σ-full sets Λ'. In particular, Λ is Σ-full.

On the other hand, if Λ is Σ-full then, by definition of $g(\varphi)$, φ has to be unsatisfiable while φ_{-i} is satisfiable for all $1 \leq i \leq m$. Hence, $\varphi \in$ CRITSAT. □

LEMMA 4.12. *Let B be a finite set of Boolean functions such that $[B] = \mathsf{L}$. Then $\mathrm{FULL}(B)$ is $\oplus\mathrm{L}$-complete.*

PROOF. To prove membership in $\oplus$L, we will use the result that $\mathrm{L}^{\oplus\mathrm{L}} = \oplus\mathrm{L}$ [Hertrampf et al. 2000]. Given an instance (Σ, Λ) of $\mathrm{FULL}(B)$, we construct sets of affine formulae equations corresponding to the set of formulae in S_1 and S_2 from the proof of Lemma 4.11:

$$T_1 := \{\Sigma' \cup \{\varphi \oplus 1\} \mid L\varphi \in \Lambda\},$$
$$T_2 := \{\Sigma' \cup \{\varphi \oplus 1\} \mid \neg L\varphi \in \Lambda\},$$

where Σ' is obtained from Σ by substituting $L\varphi$ with 1 if $L\varphi \in \Lambda$, and $L\varphi$ with 0 otherwise. Analogous to the above proof, we have that $(\Sigma, \Lambda) \in \mathrm{FULL}(B)$ iff each system of linear equations from T_1 has no solutions and each system of linear equations from T_2 has a solution, which can be checked for each system of linear equations with a call to a $\oplus$L-oracle. As each system of linear equations in T_1 or T_2 can be constructed from (Σ, Λ) in logarithmic space (see, *e.g.*, [Beyersdorff et al. 2009a]), we obtain membership in $\mathrm{L}^{\oplus\mathrm{L}} = \oplus\mathrm{L}$.

Hardness for $\oplus$L is apparent from the proof of Lemma 4.7, as the given reduction $(\Gamma, \psi) \mapsto \Sigma$ can be easily adapted to map instances of $\mathrm{IMP}(B)$ to $\mathrm{FULL}(B)$ by setting $\Lambda := \{L\psi\}$. □

LEMMA 4.13. *Let B be a finite set of Boolean functions such that $[B] \subseteq \mathsf{V}$, $[B] \subseteq \mathsf{E}$ or $[B] \subseteq \mathsf{N}$. Then $\mathrm{FULL}(B)$ is solvable in* L.

PROOF. To see that $\mathrm{FULL}(B) \in \mathrm{L}$ for all finite sets of Boolean functions such that $[B] \subseteq \mathsf{V}$ or $[B] \subseteq \mathsf{E}$ or $[B] \subseteq \mathsf{N}$, recall that $(\Sigma, \Lambda) \in \mathrm{FULL}(B)$ iff $\Sigma' \models \varphi$ for all $L\varphi \in \Lambda$ and $\Sigma' \not\models \varphi$ for all $\neg L\varphi \in \Lambda$, where Σ' is obtained from Σ by substituting $L\varphi$ with 1 if $L\varphi \in \Lambda$, and $L\varphi$ with 0 otherwise. As implication for all considered types of B-formulae can be decided in logarithmic space [Beyersdorff et al. 2009a], we obtain that $\mathrm{FULL}(B) \in \mathrm{L}$. □

4.3. Brave and Cautious Reasoning

THEOREM 4.14. *Let B be a finite set of Boolean functions.*

- *If $[B \cup \{0,1\}]$ is BF or M then $\mathrm{MEM_b}(B)$ is Σ_2^{p}-complete, whereas $\mathrm{MEM_c}(B)$ is Π_2^{p}-complete.*
- *If $[B \cup \{0,1\}]$ is V then $\mathrm{MEM_b}(B)$ is NP-complete, whereas $\mathrm{MEM_c}(B)$ is coNP-complete.*
- *If $[B \cup \{0,1\}]$ is L then $\mathrm{MEM_b}(B)$ and $\mathrm{MEM_c}(B)$ are $\oplus$L-hard and in P.*
- *If $[B \cup \{0,1\}]$ is E or N or I then $\mathrm{MEM_b}(B)$ and $\mathrm{MEM_c}(B)$ are in L.*

To prove Theorem 4.14, we require two lemmas that provide upper bounds on the complexity of $\mathrm{MEM_b}(B)$ and $\mathrm{MEM_c}(B)$ via reduction to the expansion existence problem.

LEMMA 4.15. *Let B be a finite set of Boolean functions such that $[B] = \mathsf{L}$. Then $\mathrm{MEM_b}(B) \leq_{\mathrm{m}}^{\log} \mathrm{EXP}(B)$.*

PROOF. Let B be a finite set of Boolean functions such that $[B] = \mathsf{L}$. Given $\Sigma \subseteq \mathcal{L}_{ae}(B)$ and $\varphi \in \mathcal{L}_{ae}(B)$, map the pair (Σ, φ) to $\Sigma' := \Sigma \cup \{L\varphi \oplus p \oplus 1, Lp\}$, where p is a fresh proposition. We claim that φ is contained in a stable expansion of Σ iff $\Sigma' \in \mathrm{EXP}(B)$.

First suppose that φ is contained in a stable expansion Δ of Σ and let Λ denote its kernel. We claim that $\Lambda' := \Lambda \cup \{L\varphi, Lp\}$ is Σ'-full:

— $\Sigma' \cup \Lambda' \models \varphi$, because $\Sigma \cup \Lambda \models_L \varphi$;
— $\Sigma' \cup \Lambda' \models p$, because $\Sigma \cup \{L\varphi, L\varphi \oplus p \oplus 1\} \models p$;
— for all $L\psi \in \Lambda$, we have $\Sigma' \cup \Lambda' \equiv \Sigma \cup \Lambda \cup \{L\varphi, L\varphi \oplus p \oplus 1, Lp\} \models_L \psi$; whereas for all $\neg L\psi \in \Lambda$, we still have $\Sigma' \cup \Lambda' \equiv \Sigma \cup \Lambda \cup \{L\varphi, L\varphi \oplus p \oplus 1, Lp\} \not\models_L \psi$.

Hence, Σ' has a stable expansion.

Conversely, suppose that φ is not bravely entailed. Hence Σ does not have $\mathcal{L}_{ae}$ as a stable expansion and $\neg L\varphi \in \Delta$ for all stable expansions Δ of Σ. Observe that $\Sigma' \cup \mathrm{SF}^L(\Sigma') = \Sigma \cup \mathrm{SF}^L(\Sigma) \cup \{L\varphi \oplus p \oplus 1, Lp\} \cup \{L\varphi, Lp\}$ is consistent, therefore $\mathcal{L}_{ae}$ is not a stable expansion of Σ'. Hence, assume that Δ' is a consistent stable expansion of Σ'. Then either $Lp \in \Delta'$ or $\neg Lp \in \Delta'$. In the former case, Δ' would also have to contain $L\varphi$, while φ can not be derived. A contradiction to Δ' being a stable expansion of Σ'. In the latter case, we have that $\mathrm{Th}(\Sigma' \cup L(\Delta') \cup \neg L(\overline{\Delta'})) \supset \{\neg Lp, Lp\}$. Thus $\mathrm{Th}(\Sigma' \cup L(\Delta') \cup \neg L(\overline{\Delta'})) = \mathcal{L}_{ae} \supset \Delta'$;a contradiction to Δ' being a stable expansion. We conclude that Σ' does not possess any stable expansions. □

LEMMA 4.16. *Let B be a finite set of Boolean functions such that $[B] = \mathsf{L}$. Then $\overline{\mathrm{MEM_c}}(B) \leq_{\mathrm{m}}^{\log} \mathrm{EXP}^*(B)$, where $\mathrm{EXP}^*(B)$ denotes the problem of deciding the existence of a consistent stable expansion.*

PROOF. The proof is similar to the proof of Lemma 4.15. Let B be a finite set of Boolean functions such that $[B] = \mathsf{L}$. Given $\Sigma \subseteq \mathcal{L}_{ae}(B)$ and $\varphi \in \mathcal{L}_{ae}(B)$, map the pair (Σ, φ) to $\Sigma' := \Sigma \cup \{L\varphi \oplus p, Lp\}$, where p is a fresh proposition. We claim that φ is contained in any stable expansion of Σ iff $\Sigma' \notin \mathrm{EXP}^*(B)$.

First suppose that there exists a stable expansion Δ of Σ that does not contain φ. Let Λ denote its kernel. Then, for the same arguments as above, $\Lambda' := \Lambda \cup \{\neg L\varphi, Lp\}$ is a Σ'-full set. Conversely, suppose that φ is contained in all stable expansions Δ of Σ. Let Δ' denote a consistent stable expansion of Σ'. If $Lp \in \Delta'$, then Δ' would also have to contain $\neg L\varphi$, while φ can be derived. A contradiction to Δ' being a stable expansion of Σ'. Otherwise, if $\neg Lp \in \Delta'$, then $\Sigma' \cup L(\Delta') \cup \neg L(\bar{\Delta}')$ is inconsistent—contradictory to Δ' being a consistent stable expansion. We conclude that Σ' does not possess any consistent stable expansion. □

PROOF PROOF OF THEOREM 4.14. According to Lemma 4.1 one can suppose w.l.o.g. that B contains the two constants 0 and 1. Since 1 belongs to all stable expansions, a set Σ of B-premises has a stable expansion iff 1 belongs to some stable expansion of Σ. Since 0 does not belong to any consistent stable expansion, a set Σ of B-premises has no consistent stable expansion iff 0 belongs to any stable expansion of Σ. Therefore, the lower bounds follow from Theorem 4.3 and Corollary 4.9.

As for the upper bounds, membership in Σ_2^{p} and Π_2^{p} in the general case follows from the discussion preceding Theorem 4.3.

For $[B] \subseteq \mathsf{V}$ the proof of Lemma 4.5 shows that, given $\Sigma \subseteq \mathcal{L}_{ae}(B)$, we can compute a Σ-full set Λ in NP and therefore get a membership result in NP for $\mathrm{MEM_b}(B)$, resp., in coNP for $\mathrm{MEM_c}(B)$. By Lemma 3.6, it remains to check whether $\Sigma \cup \Lambda \models_L \varphi$. To this end, we nondeterministically guess a set $T \subseteq \mathrm{SF}^q(\varphi) \cap \mathrm{SF}^L(\varphi)$, verify that $\Sigma \cup \Lambda \cup T \cup \{\neg L\chi \mid L\chi \in (\mathrm{SF}^q(\varphi) \cap \mathrm{SF}^L(\varphi)) \setminus T\} \models \varphi$, and recursively check that

— $\Sigma \cup \Lambda \models_L \chi$ for all $L\chi \in T$,
— $\Sigma \cup \Lambda \not\models_L \chi$ for all $L\chi \in (\mathrm{SF}^q(\varphi) \cap \mathrm{SF}^L(\varphi)) \setminus T$.

This recursion terminates after at most $|\varphi|$ steps as $|\mathrm{SF}^q(\varphi) \cap \mathrm{SF}^L(\varphi)| \leq |\mathrm{SF}(\varphi)| \leq |\varphi|$ and $\Sigma \cup \Lambda \models_L \chi$ iff $\Sigma \cup \Lambda \models \chi$ for all propositional formulae χ. The above hence constitutes a polynomial-time Turing reduction to the implication problem for propositional B-formulae. As implication testing for B-formulae is in P, we obtain that $\Sigma \cup \Lambda \models_L \varphi$ is polynomial-time decidable; thence $\mathrm{MEM_b}(B) \in \mathrm{NP}$ and $\mathrm{MEM_c}(B) \in \mathrm{coNP}$.

For $[B] \subseteq \mathsf{N}$ and $[B] \subseteq \mathsf{E}$, the proof of Lemma 4.8 shows that, given $\Sigma \subseteq \mathcal{L}_{ae}(B)$, computation of a Σ-full set Λ can be performed in L, while deciding $\Sigma \cup \Lambda \models_L \varphi$ reduces to testing whether $\Sigma \cup \Lambda \models \psi$ for the (unique) atomic subformula $\psi \in \mathrm{SF}(\varphi)$.

Finally, for $[B \cup \{0,1\}] = \mathsf{L}$, the claim follows from Lemmas 4.15, 4.16, and 4.7, and Corollary 4.9. □

5. COUNTING COMPLEXITY

Besides deciding existence of stable expansions or entailment of formulae, another natural question is concerned with the total number of stable expansions of a given autoepistemic theory. We define the counting problem for stable expansions as

Problem: $\#\mathrm{EXP}(B)$
Input: A finite set $\Sigma \subseteq \mathcal{L}_{ae}(B)$
Output: The number of stable expansions of Σ.

The complexity of this problem is classified by the following theorem.

THEOREM 5.1. *Let B be a finite set of Boolean functions.*

—*If $[B \cup \{0,1\}]$ is BF or M then $\#\mathrm{EXP}(B)$ is $\#\cdot\mathrm{coNP}$-complete.*
—*If $[B \cup \{0,1\}]$ is V then $\#\mathrm{EXP}(B)$ is $\#\mathrm{P}$-complete.*
—*If $[B \cup \{0,1\}] \subseteq \mathsf{L}$ or $[B \cup \{0,1\}] \subseteq \mathsf{E}$ then $\#\mathrm{EXP}(B)$ is in FP.*

PROOF. We first prove the lower bounds. It is easily observed that the reduction given in the proof of Lemma 4.1 is parsimonious. For the claimed lower bounds it hence suffices to prove the $\#\cdot\mathrm{coNP}$-hardness of $\#\mathrm{EXP}(B)$ for $[B] = \mathsf{M}$ and the $\#\mathrm{P}$-hardness of $\#\mathrm{EXP}(B)$ for $[B] = \mathsf{V}$. For the latter, notice that the reduction given in Lemma 4.5 is also a parsimonious reduction from $\#3\mathrm{SAT}$, which is $\#\mathrm{P}$-complete via parsimonious reductions [Valiant 1979]. For the former, notice that the proof of Lemma 4.4 establishes a parsimonious reduction from the problem $\#\Pi_1\mathrm{SAT}$, which is $\#\cdot\mathrm{coNP}$-complete via parsimonious reductions [Durand et al. 2005].

We are thus left to prove the upper bounds. Let B be a finite set of Boolean functions such that $[B] = \mathsf{BF}$. In the paragraph starting Section 4, it has been argued that the problem of deciding $\mathrm{EXP}(B)$ nondeterministically Turing-reduces to the propositional implication problem (see also [Niemelä 1991]): given $\Sigma \subseteq \mathcal{L}_{ae}(B)$, guess a subset $\Lambda^+ \subseteq \mathrm{SF}^L(\Sigma)$ and verify that $\Lambda := \Lambda^+ \cup \{\neg L\varphi \mid \varphi \in \mathrm{SF}^L(\Sigma), L\varphi \notin \Lambda^+\}$ is a Σ-full set using the conditions given in Definition 3.3. It is thus clear that $\#\mathrm{EXP}$ is contained in $\#\cdot\mathrm{P}^{\mathrm{NP}}$, as a Turing machine implementing the above algorithm can be build in a way such that there is a bijection between its computation paths and the possible sets Λ^+. The first claim now follows from $\#\cdot\mathrm{P}^{\mathrm{NP}} = \#\cdot\mathrm{coNP}$ [Hemaspaandra and Vollmer 1995].

Next, let B be such that $[B] = \mathsf{V}$. Then there exists a nondeterministic Turing machine M such that the number of accepting paths of M on input $\Sigma \subseteq \mathcal{L}_{ae}(B)$ corresponds to the number of stable expansions of Σ (cf. the proof of Lemma 4.5). Hence, $\#\mathrm{EXP}(B) \in \#\mathrm{P}$.

Next, suppose that $[B] \subseteq \mathsf{L}$ and let Σ denote the given autoepistemic theory. Let T' denote the system of linear equations obtained from Σ in the proofs of Lemma 4.6. Then the number of consistent stable expansions of Σ is equal to the number of solutions of the system $T'[x_{s+1}/Lx_{s+1}, \ldots, x_n/Lx_n]$, which can be computed in polynomial time by Gaussian elimination. Moreover, $\mathcal{L}_{ae}$ is a stable expansion of Σ iff $\Sigma \cup \mathrm{SF}^L(\Sigma)$ is inconsistent, which is polynomial-time decidable. Hence, $\#\mathrm{EXP}(B) \in \mathrm{FP}$.

Finally, the case $[B] \subseteq \mathsf{E}$ follows from the fact that for any $\Sigma \subseteq \mathcal{L}_{ae}(B)$ an equivalent representation $\Sigma' \in \mathcal{L}_{ae}(\mathsf{I})$ can be computed efficiently. □

6. INTERRELATIONSHIP WITH OTHER NON-MONOTONIC LOGICS

Having classified the complexity of the expansion existence and the brave and cautious reasoning problems, this section studies the arising implications for translations between autoepistemic logic and Reiter's default logic as well as McCarthy's circumscription. In the interest of space, we will restrict our study to fragments being able to simulate the monotone functions. A complete treatment of the intertranslatability of fragments of autoepistemic logic, default logic and circumscription is given in [Thomas 2010].

Where B is a finite set of Boolean functions, denote by B-autoepistemic logic, B-default logic, and B-circumscription the respective logic restricted to formulae using Boolean functions from B (for a formal definition of default logic and circumscription, see [Reiter 1980; Lifschitz 1985]). For any non-monotonic logical theory T, write $T \models_{\mathrm{c}} \varphi$, if φ is cautiously entailed by T, where a formula φ is cautiously entailed in autoepistemic logic (resp. default logic, circumscription) if φ is contained in all stable expansions (resp. contained in all stable extensions, satisfied in all circumscriptive models) of the given knowledge base. We define a *translation* f to be a polynomial-time computable function mapping theories from one non-monotonic logic to another such that the set of cautiously entailed formulae is invariant under f (*i.e.*, $T \models_{\mathrm{c}} \varphi$ iff $f(T) \models_{\mathrm{c}} \varphi$). Hence, our notion of translation is quite weak in that it subsumes the notions studied in [Gottlob 1995; Janhunen 1999].

Janhunen showed in [Janhunen 1999] that there exists a polynomial-time computable faithful translation from autoepistemic logic to default logic. Theorem 4.3 and results from [Beyersdorff et al. 2009b] imply that for fragments of autoepistemic logic able to express $\wedge$ and $\vee$ this holds only if in default logic we can simulate *all* Boolean functions.

THEOREM 6.1. *Let B and B' be finite sets of Boolean functions such that $\mathsf{M} \subseteq [B \cup \{0,1\}]$ and $[B' \cup \{1\}] \neq \mathsf{BF}$. Then there exists no translation from B-autoepistemic logic to B'-default logic unless the polynomial-time hierarchy collapses to Δ_2^{p}.*

PROOF. Let B and B' be finite sets of Boolean functions such that $[B' \cup \{1\}] \neq \mathsf{BF}$. If $\mathsf{M} \subseteq [B \cup \{0,1\}]$ then the consistent expansion existence problem for B-autoepistemic logic is Σ_2^{p}-complete, whereas the cautious reasoning problem for B'-default logic is solvable in Δ_2^{p}. If there exists a polynomial-time computable translation f preserving the set of cautiously entailed formulae then we have $\Sigma_2^{\mathrm{p}} = \Delta_2^{\mathrm{p}}$, as a given set Σ admits no consistent stable expansion iff 0 is contained in all stable expansions of Σ iff 0 is contained in all stable extensions of $f(\Sigma)$. □

Thus, considering only clones that contain the constant 1, Janhunen's translation is optimal with respect to the required Boolean functions.

As for the reverse direction, there exists a translation from default logic to autoepistemic logic [Gottlob 1995]. Indeed, under our weak notion of translations, there exists a translation from monotone default logic to monotone autoepistemic logic.

THEOREM 6.2. *Let B and B' be finite sets of Boolean functions such that $\mathsf{M} \subseteq [B \cup \{0,1\}]$ and $[B] \subseteq [B' \cup \{0,1\}]$. Then there exists a translation from B-default logic to B'-autoepistemic logic.*

PROOF. Let B and B' be finite sets of Boolean functions as in the statement of the theorem. We split the proof into two cases. If $[B \cup \{0,1\}] = \mathsf{BF}$, then using Lemma 4.1 we can easily adapt the translation given in [Gottlob 1995] to map the given B-default theory to a set of autoepistemic B'-formulae.

If, otherwise, $[B \cup \{0,1\}] = \mathsf{M}$, then the given B-default theory (W, D) possesses at most one stable extension. Define

$$\Sigma := W \cup \left\{ L\alpha \vee p_\alpha, Lp_\alpha \vee \gamma \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D \text{ and } \beta \text{ is satisfiable} \right\}$$

for fresh, mutually different propositions p_α. We define the translation function f to be the mapping $(W, D) \mapsto \Sigma'$, where Σ' denotes the B'-representation of Σ.

To see that f is polynomial-time computable, recall that the consistency of a set of monotone formulae is decidable in polynomial time and that, for $\mathsf{M} \subseteq [B' \cup \{0,1\}]$, B efficiently implements $\wedge$ and $\vee$, see [Schnoor 2010].

To prove correctness of the translation, suppose that (W, D) has the stable extension E. Write $\mathrm{GD}(E)$ for the defaults in D that are applicable in E. From the definition of extension (systematically; cf. [Reiter 1980]), it follows that there exists an ordering $\delta_1, \ldots, \delta_n$ of the defaults in $\mathrm{GD}(E)$ such that for all $0 \leq i \leq n$ and

$$\Lambda_i := \left\{ L\alpha_j, \neg Lp_{\alpha_j} \;\middle|\; \delta_j = \frac{\alpha_j : \beta_j}{\gamma_j} \text{ and } j \leq i \right\}$$

we obtain $\Sigma \cup \Lambda_{i-1} \models \alpha_i$ and $\Sigma \cup \Lambda_{i-1} \cup \{L\alpha_i\} \not\models p_{\alpha_i}$. As the p_{α_i}'s do not occur in any formula except $L\alpha_i \vee p_{\alpha_i}$ and $Lp_{\alpha_i} \vee \gamma$, this eventually leads to $\Sigma \cup \Lambda_n \models \alpha$ if and only if $L\alpha \in \Lambda_n$ and, $\Sigma \cup \Lambda_n \cup \{L\alpha\} \not\models p_\alpha$ if and only if $\neg Lp_\alpha \in \Lambda_n$, for all premises α in $\mathrm{GD}(E)$. As $W \cup \mathrm{GD}(E) \not\models \alpha$ for all $\frac{\alpha:\beta}{\gamma} \in D \setminus \mathrm{GD}(E)$, setting

$$\Lambda := \Lambda_n \cup \left\{ \neg L\alpha, Lp_\alpha \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D \setminus \mathrm{GD}(E) \right\}.$$

we obtain $\Sigma \cup \Lambda \models \alpha$ if and only if $L\alpha \in \Lambda$ and, $\Sigma \cup \Lambda \cup \{L\alpha\} \not\models p_\alpha$ if and only if $\neg Lp_\alpha \in \Lambda$. Thus, Λ is a Σ-full set.

Finally, suppose that (W, D) does not have a stable extension. Then there has to exist an applicable default $\frac{\alpha:\beta}{\gamma} \in D$, whose conclusion is equivalent to 0. By construction, Σ then contains a formula that is equivalent to Lp_α. Since the only other occurrence of p_α in Σ is in $L\alpha \vee p_\alpha$, $L\alpha$ has to be 0. However, the applicability of $\frac{\alpha:\beta}{\gamma}$ implies that α can be derived. Therefore, any Σ-full set has to contain $L\alpha$. Hence, no consistent stable expansion of Σ may exist. □

As for the relation to circumscription, let $\leq_{(P,Q,Z)}$ denote the preorder on assignments defined by $\sigma \leq_{(P,Q,Z)} \sigma'$ if $\sigma(q) = \sigma'(q)$ for all $q \in Q$ and $\sigma(p) \leq \sigma'(p)$ for all $p \in P$. Niemelä [Niemelä 1993] showed that, given a set Γ of propositional formulae and a partition of the propositions occurring in Γ into sets (P, Q, Z), the minimal models of Γ with respect to $\leq_{(P,Q,Z)}$ are in one-to-one correspondence with the stable expansions of

$$\Sigma := \Gamma \cup \{\neg Lx \rightarrow \neg x \mid x \in P \cup Q\} \cup \{\neg L \neg x \rightarrow x \mid x \in Q\}$$

and coincide up to the propositional language. This nicely contrasts with the fact that we cannot map any proper fragment of circumscription to monotone autoepistemic logic:

THEOREM 6.3. *Let B and B' be finite sets of Boolean functions such that $[B'] \subseteq \mathsf{M}$. Then there exists no translation from B-circumscription to B'-autoepistemic logic.*

PROOF. Let B and B' be finite sets of Boolean functions such that $[B'] \subseteq \mathsf{M}$. Then $[B' \cup \{0,1\}] \subseteq \mathsf{M}$. Then $\Gamma := \{y\}$ and $(P, Q, Z) := (\{x\}, \emptyset, \{y\})$ cautiously entail $\neg x$, as all $\leq_{(P,Q,Z)}$-minimal models of Γ set x to 0. On the other hand, any consistent stable expansion of a set of autoepistemic B'-formulae is satisfied by the assignment setting all propositions to 1. Consequently, $\big(\{y\}, (\{x\}, \emptyset, \{y\})\big)$ cannot be translated into a set of autoepistemic B'-formulae. □

For the converse direction, it was shown that in the general case no translation in the sense of Niemelä [Niemelä 1993] or Janhunen [Janhunen 1999] can exist. Our results show that, unless the polynomial-time hierarchy collapses to NP, this also holds for monotone autoepistemic logic.

THEOREM 6.4. *Let B and B' be finite sets of Boolean functions such that $\mathsf{M} \subseteq [B \cup \{0,1\}]$. Then there exists no translation from B-autoepistemic logic to B'-circumscription unless the polynomial-time hierarchy collapses to* NP.

PROOF. Let B and B' be as in the statement of the theorem. Consequently, $\mathrm{EXP}^*(B)$, the problem to decide whether a given set of autoepistemic B-formulae has a consistent stable expansion, is Σ_2^{p}-complete by Corollary 4.9. Assume that there exists a translation f from B-autoepistemic logic to B'-circumscription. Then $\mathrm{EXP}'(B) \leq_{\mathrm{m}}^{\log} \mathrm{SAT}(B')$ via $\Sigma \in \mathrm{EXP}'(B) \iff \Sigma \not\models_{\mathrm{c}} 0 \iff \Gamma \not\models_{\mathrm{c}} 0 \iff \Gamma \in \mathrm{SAT}(B')$, where Γ and (P,Q,Z) denote the image of Σ under the translation f. Consequently, $\mathrm{NP} = \Sigma_2^{\mathrm{p}}$. □

7. CONCLUSION

In this paper we followed the approach of Lewis to build formulae from a given finite set B of allowed Boolean functions [Lewis 1979] and studied the complexity of the expansion existence, the brave (resp. cautious) reasoning problem, the expansion checking problem, and the counting problem for stable expansions involving B-formulae.

We showed that for all sets of allowed Boolean functions, the computational complexity of the expansion existence and reasoning problems is divided into four presumably different levels (see Figure 1 and Table II): all three problems remain complete for classes of the second level of the polynomial hierarchy as long as the connectives $\wedge$ and $\vee$ can be expressed; if, otherwise, only disjunctions can be expressed the complexity drops to completeness for the first level of the polynomial hierarchy; in all remaining cases, the problems become tractable (either contained in L or contained in P and $\oplus\mathrm{L}$-hard). We obtained a non-trivial polynomial-time upper bound for the case of not-unary affine functions. Note however that the exact complexity of the problems in this case remains open. This clone has also remained unclassified in a number of previous related works on different modal and non-monotonic logics [Bauland et al. 2006; Thomas 2009].

In addition these problems, we also classified the complexity of the expansion checking problem as being DP-complete as long as the connectives $\wedge$ and $\vee$ can be expressed; in all other cases, the problem becomes tractable (with this case splitting into $\oplus\mathrm{L}$-completeness and membership in L).

As for the problem of counting the number of stable expansions, its computational complexity is trichotomic: $\#\cdot\mathrm{coNP}$-complete, $\#\mathrm{P}$-complete, or contained in FP. We think it is important to note that for our classification of counting problems the conceptually simple parsimonious reductions are sufficient, while for related classifications in the literature less restrictive (and more complicated) reductions such as subtractive or complementive reductions had to be used (see, *e.g.*, [Durand et al. 2005; Durand and Hermann 2008; Bauland et al. 2010] and some of the results of [Hermann and Pichler 2007]). Parsimonious reductions are not only the conceptually simplest ones since they are direct analogues of the usual many-one reductions among languages. They also form the strongest (strictest) type of reduction with a number of good properties, *e.g.*, all relevant counting classes are closed under parsimonious reductions (and not under the other mentioned types of reductions). Thus, one of the contributions of our paper is a natural counting problem complete in the class $\#\cdot\mathrm{coNP}$ under the simplest type of reductions.

Finally, we examined the interrelationship of autoepistemic logic with the non-monotonic logics default logic and circumscription. Our results imply that using a

Table II. Overview of complexity results

Problem	$[B \cup \{0,1\}]$ is			
	BF or M	V	L	N or E or I
$\mathrm{EXP}(B)$	Σ_2^p-complete	NP-complete	in P, $\oplus$L-hard	in L
$\mathrm{MEM_b}(B)$	Σ_2^p-complete	NP-complete	in P, $\oplus$L-hard	in L
$\mathrm{MEM_c}(B)$	Π_2^p-complete	coNP-complete	in P, $\oplus$L-hard	in L
$\mathrm{FULL}(B)$	DP-complete	in L	$\oplus$L-complete	in L

comparatively weak notion of translations monotone default logic can be translated to monotone autoepistemic logic, while a translation in the converse direction would imply a collapse of the polynomial-time hierarchy. Similarly, we were able to show that the translation from circumscription to autoepistemic logic given by Niemelä [Niemelä 1993] does not extend to the monotone fragments; this also holds for translations from autoepistemic logic to circumscription, for which the absence of translations was already shown in [Niemelä 1993], although for a stricter type of translations.

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