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Tableau Calculi for Logic Programs under Answer Set Semantics

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ACM Transactions on Computational Logic, Vol. V, No. N, Month 20YY, Pages 1–40.

We present proofs of results by sections. Proofs of Theorem 3.1 from Section 3 and Theorem 4.6 from Section 4 are postponed to Appendix A.2, where they can be derived as consequences of more general results.

A.1 Proofs of Results from Section 4

To begin with, we show Proposition 4.1 and 4.2 on correspondences between tableau rules and logic programming operators as well as *smodels*' propagation.

PROPOSITION 4.1. *Let Π be a normal program and $\mathbf{A}$ an assignment. Then, we have that*

(*1*) $\mathbb{T}_\Pi(\mathbf{A}) = \big(D_{\{FTA\}}(\Pi, D_{\{FTB\}}(\Pi, \mathbf{A}))\big)^{\boldsymbol{T}}$;

(*2*) $\mathbb{N}_\Pi(\mathbf{A}) = \big(D_{\{FFA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))\big)^{\boldsymbol{F}}$;

(*3*) $\mathbb{U}_\Pi(\mathbf{A}) = \big(D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))\big)^{\boldsymbol{F}}$.

PROOF. We separately consider the items of the statement:

(1) We have that $p \in \mathbb{T}_\Pi(\mathbf{A})$ iff $p = head(r)$ for some $r \in \Pi$ such that $body(r)^+ \subseteq \mathbf{A}^{\boldsymbol{T}}$ and $body(r)^- \subseteq \mathbf{A}^{\boldsymbol{F}}$ iff $p = head(r)$ for some $r \in \Pi$ such that $\boldsymbol{T}body(r) \in D_{\{FTB\}}(\Pi, \mathbf{A})$, so that $p \in \big(D_{\{FTA\}}(\Pi, D_{\{FTB\}}(\Pi, \mathbf{A}))\big)^{\boldsymbol{T}}$.

(2) We have that $p \in \mathbb{N}_\Pi(\mathbf{A})$ iff $p \in atom(\Pi)$ such that $head(r) \neq p$ or $(body(r)^+ \cap \mathbf{A}^{\boldsymbol{F}}) \cup (body(r)^- \cap \mathbf{A}^{\boldsymbol{T}}) \neq \emptyset$ for every $r \in \Pi$ iff $p \in atom(\Pi)$ such that $\boldsymbol{F}B \in D_{\{FFB\}}(\Pi, \mathbf{A})$ for every $B \in body(p)$, so that $p \in \big(D_{\{FFA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))\big)^{\boldsymbol{F}}$.

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(3) We have that $p \in \mathbb{U}_\Pi(\mathbf{A})$ iff $p \in U$ for some $U \subseteq atom(\Pi)$ such that $(B^+ \cap \mathbf{A}^{\boldsymbol{F}}) \cup (B^- \cap \mathbf{A}^{\boldsymbol{T}}) \neq \emptyset$ for every $B \in EB_\Pi(U)$ iff $p \in U$ for some $U \subseteq atom(\Pi)$ such that $\boldsymbol{F}B \in D_{\{FFB\}}(\Pi, \mathbf{A})$ for every $B \in EB_\Pi(U)$, so that $p \in \left(D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))\right)^{\boldsymbol{F}}$.

We have thus shown that all items of the statement hold. $\square$

PROPOSITION 4.2. *Let* Π *be a normal program and* $\mathbf{A}$ *an assignment. Then, we have that*

(1) $D_{\{FI\}}(\Pi, \mathbf{A}) = D_{\{FTA\}}(\Pi, D_{\{FTB\}}(\Pi, \mathbf{A}))$*;*
(2) $D_{\{ARC\}}(\Pi, \mathbf{A}) = D_{\{FFA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))$*;*
(3) $D_{\{CTH\}}(\Pi, \mathbf{A}) = D_{\{BTB\}}(\Pi, D_{\{BTA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}) \cup \{\boldsymbol{T}p \mid p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)\}))$*;*
(4) $D_{\{CFH\}}(\Pi, \mathbf{A}) = D_{\{BFB\}}(\Pi, D_{\{BFA\}}(\Pi, \mathbf{A}) \cup \{\boldsymbol{T}p \mid p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)\} \cup \{\boldsymbol{F}p \mid p \in \mathbf{A}^{\boldsymbol{F}} \cap atom(\Pi)\})$*;*
(5) $D_{\{AM\}}(\Pi, \mathbf{A}) = D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))$.

PROOF. We separately consider the items of the statement:

(1) We have that $\boldsymbol{T}p \in D_{\{FI\}}(\Pi, \mathbf{A})$ iff $p = head(r)$ for some $r \in \Pi$ such that $body(r)^+ \subseteq \mathbf{A}^{\boldsymbol{T}}$ and $body(r)^- \subseteq \mathbf{A}^{\boldsymbol{F}}$ iff $p = head(r)$ for some $r \in \Pi$ such that $\boldsymbol{T}body(r) \in D_{\{FTB\}}(\Pi, \mathbf{A})$, so that $\boldsymbol{T}p \in D_{\{FTA\}}(\Pi, D_{\{FTB\}}(\Pi, \mathbf{A}))$.
(2) We have that $\boldsymbol{F}p \in D_{\{ARC\}}(\Pi, \mathbf{A})$ iff $p \in atom(\Pi)$ such that $(B^+ \cap \mathbf{A}^{\boldsymbol{F}}) \cup (B^- \cap \mathbf{A}^{\boldsymbol{T}}) \neq \emptyset$ for every $B \in body(p)$ iff $p \in atom(\Pi)$ such that $\boldsymbol{F}B \in D_{\{FFB\}}(\Pi, \mathbf{A})$ for every $B \in body(p)$, so that $\boldsymbol{F}p \in D_{\{FFA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))$.
(3) We have that $\boldsymbol{t}l \in D_{\{CTH\}}(\Pi, \mathbf{A})$ iff $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ and $l \in body(r)$ for some $r \in \Pi$ such that $(B^+ \cap \mathbf{A}^{\boldsymbol{F}}) \cup (B^- \cap \mathbf{A}^{\boldsymbol{T}}) \neq \emptyset$ for every $B \in body(p) \setminus \{body(r)\}$ iff $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ and $l \in body(r)$ for some $r \in \Pi$ such that $\{\boldsymbol{F}B \mid B \in body(p) \setminus \{body(r)\}\} \subseteq D_{\{FFB\}}(\Pi, \mathbf{A})$, so that $\boldsymbol{T}body(r) \in D_{\{BTA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}) \cup \{\boldsymbol{T}p \mid p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)\})$ and $\{\boldsymbol{t}l \mid l \in body(r)\} \subseteq D_{\{BTB\}}(\Pi, D_{\{BTA\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}) \cup \{\boldsymbol{T}p \mid p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)\}))$.
(4) We have that $\boldsymbol{f}l \in D_{\{CFH\}}(\Pi, \mathbf{A})$ iff $l \in body(r)$ for some $r \in \Pi$ such that $\boldsymbol{F}head(r) \in \mathbf{A}$ and $\boldsymbol{t}l' \in \mathbf{A}$ for every $l' \in body(r) \setminus \{l\}$ iff $\boldsymbol{F}body(r) \in D_{\{BFA\}}(\Pi, \mathbf{A})$ and $\{\boldsymbol{t}l' \mid l' \in body(r) \setminus \{l\}\} \subseteq \{\boldsymbol{T}p \mid p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)\} \cup \{\boldsymbol{F}p \mid p \in \mathbf{A}^{\boldsymbol{F}} \cap atom(\Pi)\}$ for some $r \in \Pi$ and $l \in body(r)$, so that $\boldsymbol{f}l \in D_{\{BFB\}}(\Pi, D_{\{BFA\}}(\Pi, \mathbf{A}) \cup \{\boldsymbol{T}p \mid p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)\} \cup \{\boldsymbol{F}p \mid p \in \mathbf{A}^{\boldsymbol{F}} \cap atom(\Pi)\})$.
(5) We have that $\boldsymbol{F}p \in D_{\{AM\}}(\Pi, \mathbf{A})$ iff $p \in U$ for some $U \subseteq atom(\Pi)$ such that $(B^+ \cap \mathbf{A}^{\boldsymbol{F}}) \cup (B^- \cap \mathbf{A}^{\boldsymbol{T}}) \neq \emptyset$ for every $B \in EB_\Pi(U)$ iff $p \in U$ for some $U \subseteq atom(\Pi)$ such that $\boldsymbol{F}B \in D_{\{FFB\}}(\Pi, \mathbf{A})$ for every $B \in EB_\Pi(U)$, so that $\boldsymbol{F}p \in D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi, D_{\{FFB\}}(\Pi, \mathbf{A}))$.

We have thus shown that all items of the statement hold. $\square$

In view of Proposition 4.2, we derive the following relationship between tableau calculi using the deterministic tableau rules in Figure 1 or 3, respectively.

COROLLARY 4.3. *Let* Π *be a normal program and* $\mathbf{A}$ *an assignment. Then, we have that* $D^*_{\{FI,ARC,CTH,CFH,AM\}}(\Pi, \mathbf{A}) \subseteq D^*_{\mathcal{T}_{smodels}}(\Pi, \mathbf{A})$.

PROOF. This result follows immediately from Proposition 4.2, since any entry deducible by some of the tableau rules in $\{FI, ARC, CTH, CFH, AM\}$ can likewise be deduced by iterated applications of the tableau rules *(a)–(h)* and $WFN[2^{atom(\Pi)}]$ in Figure 1, which are the deterministic tableau rules contained in $\mathcal{T}_{smodels}$. $\square$

Next, we show the one-to-one correspondence between models of $\mathit{Comp}(\Pi)$ and non-contradictory complete branches in tableaux of $\mathcal{T}_{comp}$, stated in Theorem 4.4. To this end, we first provide Lemma A.1, linking models of $\mathit{Comp}(\Pi)$ to non-contradictory complete branches.

LEMMA A.1. *Let Π be a normal program and $X \subseteq \mathit{atom}(\Pi) \cup \mathit{body}(\Pi)$.*
Then, we have that $(X \cap \mathit{atom}(\Pi)) \cup \{p_B \mid B \in X \cap \mathit{body}(\Pi)\}$ is a model of $\mathit{Comp}(\Pi)$ iff $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}) \subseteq \{\boldsymbol{T}v \mid v \in X\} \cup \{\boldsymbol{F}v \mid v \in (\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)) \setminus X\}$ for every assignment $\mathbf{A} \subseteq \{\boldsymbol{T}v \mid v \in X\} \cup \{\boldsymbol{F}v \mid v \in (\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)) \setminus X\}$.

PROOF. Let $M = (X \cap \mathit{atom}(\Pi)) \cup \{p_B \mid B \in X \cap \mathit{body}(\Pi)\}$ and $\mathbf{A}' = \{\boldsymbol{T}v \mid v \in X\} \cup \{\boldsymbol{F}v \mid v \in (\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)) \setminus X\}$ in the following consideration of the implications of the statement.

($\Rightarrow$) Assume that $\mathbf{A} \subseteq \mathbf{A}'$ but $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}'$. Then, some of the following cases applies:

(1) If $D_{\{FTB,BFB\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}'$, for some $B = \{p_1, \ldots, p_m, \mathit{not}\ p_{m+1}, \ldots, \mathit{not}\ p_n\} \in \mathit{body}(\Pi)$, we have that $\{\boldsymbol{F}B, \boldsymbol{T}p_1, \ldots, \boldsymbol{T}p_m, \boldsymbol{F}p_{m+1}, \ldots, \boldsymbol{F}p_n\} \subseteq \mathbf{A}'$, so that $p_B \notin M$ and $\{p_1, \ldots, p_m, p_{m+1}, \ldots, p_n\} \cap M = \{p_1, \ldots, p_m\}$. Since $\mathit{Comp}(\Pi)$ includes $\big(p_B \leftrightarrow (p_1 \wedge \cdots \wedge p_m \wedge \neg p_{m+1} \wedge \cdots \wedge \neg p_n)\big)$, this shows that M is not a model of $\mathit{Comp}(\Pi)$.

(2) If $D_{\{FFB,BTB\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}'$, for some $B \in \mathit{body}(\Pi)$ and $l \in B$, we have that $\{\boldsymbol{T}B, \boldsymbol{f}l\} \subseteq \mathbf{A}'$, so that $p_B \in M$ and $B^+ \not\subseteq M$ or $B^- \cap M \neq \emptyset$. Since $\mathit{Comp}(\Pi)$ includes $\big(p_B \leftrightarrow (\bigwedge_{p \in B^+} p \wedge \bigwedge_{q \in B^-} \neg q)\big)$, this shows that M is not a model of $\mathit{Comp}(\Pi)$.

(3) If $D_{\{FTA,BFA\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}'$, for some $p \in \mathit{atom}(\Pi)$ and $B \in \mathit{body}(p)$, we have that $\{\boldsymbol{F}p, \boldsymbol{T}B\} \subseteq \mathbf{A}'$, so that $p \notin M$ and $p_B \in M$. Since $\mathit{Comp}(\Pi)$ includes $\big(p \leftrightarrow (\bigvee_{B \in \mathit{body}(p)} p_B)\big)$, this shows that M is not a model of $\mathit{Comp}(\Pi)$.

(4) If $D_{\{FFA,BTA\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}'$, for some $p \in \mathit{atom}(\Pi)$ and $\mathit{body}(p) = \{B_1, \ldots, B_m\}$, we have that $\{\boldsymbol{T}p, \boldsymbol{F}B_1, \ldots, \boldsymbol{F}B_m\} \subseteq \mathbf{A}'$, so that $p \in M$ and $\{p_{B_1}, \ldots, p_{B_m}\} \cap M = \emptyset$. Since $\mathit{Comp}(\Pi)$ includes $\big(p \leftrightarrow (p_{B_1} \vee \cdots \vee p_{B_m})\big)$, this shows that M is not a model of $\mathit{Comp}(\Pi)$.

In each of the above cases, M is not a model of $\mathit{Comp}(\Pi)$, which in turn shows that, if M is a model of $\mathit{Comp}(\Pi)$, then $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}) \subseteq \mathbf{A}'$ for every assignment $\mathbf{A} \subseteq \mathbf{A}'$.

($\Leftarrow$) Assume that M is not a model of $\mathit{Comp}(\Pi)$. Then, some of the following cases applies:

(1) If $p_B \notin M$ and $\{p_1, \ldots, p_m, p_{m+1}, \ldots, p_n\} \cap M = \{p_1, \ldots, p_m\}$ for some $B = \{p_1, \ldots, p_m, \mathit{not}\ p_{m+1}, \ldots, \mathit{not}\ p_n\} \in \mathit{body}(\Pi)$, we have that $\{\boldsymbol{T}p_1, \ldots, \boldsymbol{T}p_m, \boldsymbol{F}p_{m+1}, \ldots, \boldsymbol{F}p_n\} \subseteq \mathbf{A}'$, so that $\boldsymbol{T}B \in D_{\{FTB\}}(\Pi, \mathbf{A}')$. Since $\boldsymbol{T}B \notin \mathbf{A}'$, this shows that $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \not\subseteq \mathbf{A}'$.

(2) If $p_B \in M$ and $\{p_1, \ldots, p_m, p_{m+1}, \ldots, p_n\} \cap M \neq \{p_1, \ldots, p_m\}$ for some $B = \{p_1, \ldots, p_m, \mathit{not}\ p_{m+1}, \ldots, \mathit{not}\ p_n\} \in \mathit{body}(\Pi)$, we have that $\{\boldsymbol{F}p_1, \ldots, \boldsymbol{F}p_m, \boldsymbol{T}p_{m+1}, \ldots, \boldsymbol{T}p_n\} \cap \mathbf{A}' \neq \emptyset$, so that $\boldsymbol{F}B \in D_{\{FFB\}}(\Pi, \mathbf{A}')$. Since $\boldsymbol{F}B \notin \mathbf{A}'$, this shows that $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \not\subseteq \mathbf{A}'$.

(3) If $p \notin M$ and $p_B \in M$ for some $p \in \mathit{atom}(\Pi)$ and $B \in \mathit{body}(p)$, we have that $\boldsymbol{T}B \in \mathbf{A}'$, so that $\boldsymbol{T}p \in D_{\{FTA\}}(\Pi, \mathbf{A}')$. Since $\boldsymbol{T}p \notin \mathbf{A}'$, this shows that $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \not\subseteq \mathbf{A}'$.

(4) If $p \in M$ and $\{p_{B_1}, \ldots, p_{B_m}\} \cap M = \emptyset$ for some $p \in \mathit{atom}(\Pi)$ and $\mathit{body}(p) = \{B_1, \ldots, B_m\}$, we have that $\{\boldsymbol{F}B_1, \ldots, \boldsymbol{F}B_m\} \subseteq \mathbf{A}'$, so that $\boldsymbol{F}p \in D_{\{FFA\}}(\Pi, \mathbf{A}')$. Since $\boldsymbol{F}p \notin \mathbf{A}'$, this shows that $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \not\subseteq \mathbf{A}'$.

In each of the above cases, $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \not\subseteq \mathbf{A}'$, which in turn shows that, if $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}) \subseteq \mathbf{A}'$ for every assignment $\mathbf{A} \subseteq \mathbf{A}'$, then M is a model of $\mathit{Comp}(\Pi)$. $\square$

THEOREM 4.4. *Let Π be a normal program.*
Then, we have that the following holds for tableau calculus $\mathcal{T}_{comp}$:

(*1*) *Every incomplete tableau for Π and $\emptyset$ can be extended to a complete tableau for Π and $\emptyset$.*
(*2*) *$Comp(\Pi)$ has a model X iff every complete tableau for Π and $\emptyset$ has a unique non-contradictory branch $(\Pi, \mathbf{A})$ such that $(\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)) \cup \{p_B \mid B \in \mathbf{A}^{\boldsymbol{T}} \cap body(\Pi)\} = X$.*
(*3*) *$Comp(\Pi)$ has no model iff every complete tableau for Π and $\emptyset$ is a refutation.*

PROOF. We separately consider the items of the statement:

(1) By applying *Cut*$[atom(\Pi) \cup body(\Pi)]$, an incomplete branch in a tableau for Π and $\emptyset$ can be extended to a subtableau such that, for every branch $(\Pi, \mathbf{A})$ in it, we have that $atom(\Pi) \cup body(\Pi) \subseteq \mathbf{A}^{\boldsymbol{T}} \cup \mathbf{A}^{\boldsymbol{F}}$. Furthermore, if $(\Pi, \mathbf{A})$ is not complete, then $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}$, so that the application of some of the tableau rules *(a)–(h)* in $\mathcal{T}_{comp}$ yields a contradictory and thus complete branch.
(2) ($\Rightarrow$) Assume that $X \subseteq atom(\Pi) \cup \{p_B \mid B \in body(\Pi)\}$ is a model of $Comp(\Pi)$, and consider the following assignment:

$$\begin{aligned} \mathbf{A} \;=\; & \{\boldsymbol{T}p \mid p \in X \cap atom(\Pi)\} \cup \{\boldsymbol{F}p \mid p \in atom(\Pi) \setminus X\} \\ \cup\; & \{\boldsymbol{T}B \mid B \in body(\Pi), p_B \in X\} \cup \{\boldsymbol{F}B \mid B \in body(\Pi), p_B \notin X\} \end{aligned}$$

Then, by Lemma A.1, $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \subseteq \mathbf{A}$ for every assignment $\mathbf{A}' \subseteq \mathbf{A}$. Since either $\mathbf{A}' \cup \{\boldsymbol{T}v\} \subseteq \mathbf{A}$ or $\mathbf{A}' \cup \{\boldsymbol{F}v\} \subseteq \mathbf{A}$ for any application of *Cut*$[atom(\Pi) \cup body(\Pi)]$ on a branch $(\Pi, \mathbf{A}')$ such that $\mathbf{A}' \subseteq \mathbf{A}$, we have that the assignment in exactly one of the resulting branches is contained in $\mathbf{A}$. Along with $\emptyset \subseteq \mathbf{A}$, it follows that every complete tableau for Π and $\emptyset$ has a unique non-contradictory branch $(\Pi, \mathbf{A})$ such that $(\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)) \cup \{p_B \mid B \in \mathbf{A}^{\boldsymbol{T}} \cap body(\Pi)\} = X$.
($\Leftarrow$) Assume that $(\Pi, \mathbf{A})$ is a non-contradictory complete branch, that is, $\mathbf{A}^{\boldsymbol{T}} \cup \mathbf{A}^{\boldsymbol{F}} = atom(\Pi) \cup body(\Pi)$ and $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}) \subseteq \mathbf{A}$. Then, by Lemma A.1 (along with the fact that $D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A}') \subseteq D_{\{(a)\text{–}(h)\}}(\Pi, \mathbf{A})$ for every $\mathbf{A}' \subseteq \mathbf{A}$), we have that $X = (\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)) \cup \{p_B \mid B \in \mathbf{A}^{\boldsymbol{T}} \cap body(\Pi)\}$ is a model of $Comp(\Pi)$.
(3) From the second item, if $Comp(\Pi)$ has a model, then every complete tableau for Π and $\emptyset$ has a non-contradictory branch; by the first item, there is some complete tableau for Π and $\emptyset$, so that some complete tableau for Π and $\emptyset$ is not a refutation. Conversely, if some complete tableau for Π and $\emptyset$ is not a refutation, it has a non-contradictory branch $(\Pi, \mathbf{A})$, and $(\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)) \cup \{p_B \mid B \in \mathbf{A}^{\boldsymbol{T}} \cap body(\Pi)\}$ is a model of $Comp(\Pi)$, as shown in the proof of the second item.

We have thus shown that all items of the statement hold. $\square$

For proving Proposition 4.5, stating that tableau rule *WFN*$[2^{atom(\Pi)}]$ is as powerful as the iterated application of more restrictive tableau rules *FFA* and *WFN*$[loop(\Pi)]$ (along with *FFB*), we first show as an auxiliary result that *WFN*$[loop(\Pi)]$ is applicable wrt a fixpoint of *FFB* and *FFA* if *WFN*$[2^{atom(\Pi)}]$ is.

LEMMA A.2. *Let Π be a normal program and $\mathbf{A}$ an assignment.*
Then, we have that $D_{\{FFB, WFN[2^{atom(\Pi)}]\}}(\Pi, \mathbf{A}) \subseteq \mathbf{A}$ iff $D_{\{FFB, FFA, WFN[loop(\Pi)]\}}(\Pi, \mathbf{A}) \subseteq \mathbf{A}$.

PROOF. ($\Rightarrow$) Assume that $D_{\{FFB, FFA, WFN[loop(\Pi)]\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}$. Then, $D_{\{FFB\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}$ or $D_{\{FFA, WFN[loop(\Pi)]\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}$. If $D_{\{FFB\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}$, it is clear that

$D_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$. Otherwise, if $D_{\{FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$, there is some $p \in atom(\Pi) \setminus \mathbf{A}^{\boldsymbol{F}}$ such that $EB_\Pi(\{p\}) \subseteq body(p) \subseteq \mathbf{A}^{\boldsymbol{F}}$ or $p \in U$ for an $U \in loop(\Pi)$ satisfying $EB_\Pi(U) \subseteq \mathbf{A}^{\boldsymbol{F}}$. Given that $\{\{p\} \mid p \in atom(\Pi)\} \cup loop(\Pi) \subseteq 2^{atom(\Pi)}$, we conclude that there is some $p \in atom(\Pi) \setminus \mathbf{A}^{\boldsymbol{F}}$ such that $\boldsymbol{F}p \in D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A})$, so that $D_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$.

($\Leftarrow$) Assume that $D_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$. Then, $D_{\{FFB\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$ or $D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$. If $D_{\{FFB\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$, it is clear that $D_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$. Otherwise, if $D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$, there is some $U \subseteq atom(\Pi)$ such that $U \not\subseteq \mathbf{A}^{\boldsymbol{F}}$ and $EB_\Pi(U) \subseteq \mathbf{A}^{\boldsymbol{F}}$. Since $D_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$ if $D_{\{FFB,FFA\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$, assume that $D_{\{FFB,FFA\}}(\Pi,\mathbf{A}) \subseteq \mathbf{A}$. Then, for every $B \in EB_\Pi(U \setminus \mathbf{A}^{\boldsymbol{F}}) \setminus EB_\Pi(U)$, the fact that $B^+ \cap (U \cap \mathbf{A}^{\boldsymbol{F}}) \neq \emptyset$ implies $B \in \mathbf{A}^{\boldsymbol{F}}$. Along with $EB_\Pi(U) \subseteq \mathbf{A}^{\boldsymbol{F}}$, we conclude that $EB_\Pi(U \setminus \mathbf{A}^{\boldsymbol{F}}) \subseteq \mathbf{A}^{\boldsymbol{F}}$. Moreover, since $U \setminus \mathbf{A}^{\boldsymbol{F}}$ is finite, there is some strongly connected component of the subgraph of the dependency graph of Π induced by $U \setminus \mathbf{A}^{\boldsymbol{F}}$, given by $(U \setminus \mathbf{A}^{\boldsymbol{F}}, \{(head(r), p) \mid r \in \Pi, head(r) \in U \setminus \mathbf{A}^{\boldsymbol{F}}, p \in body(r)^+ \cap (U \setminus \mathbf{A}^{\boldsymbol{F}})\})$, such that its vertices L do not reach atoms in $(U \setminus \mathbf{A}^{\boldsymbol{F}}) \setminus L$.[2] The latter means that $B^+ \cap ((U \setminus \mathbf{A}^{\boldsymbol{F}}) \setminus L) = \emptyset$ holds for every $p \in L$ and $B \in body(p)$, so that $EB_\Pi(L) \subseteq EB_\Pi(U \setminus \mathbf{A}^{\boldsymbol{F}}) \subseteq \mathbf{A}^{\boldsymbol{F}}$. Since $L \cap \mathbf{A}^{\boldsymbol{F}} = \emptyset$, for every $p \in L$, $D_{\{FFA\}}(\Pi,\mathbf{A}) \subseteq \mathbf{A}$ implies $body(p) \not\subseteq \mathbf{A}^{\boldsymbol{F}}$, while $EB_\Pi(L) \subseteq \mathbf{A}^{\boldsymbol{F}}$ yields $body(p) \cap EB_\Pi(L) \subseteq \mathbf{A}^{\boldsymbol{F}}$; that is, there is some $B \in body(p) \setminus \mathbf{A}^{\boldsymbol{F}}$, and $B^+ \cap L \neq \emptyset$ holds for each $B \in body(p) \setminus \mathbf{A}^{\boldsymbol{F}}$. Along with the fact that $U \setminus \mathbf{A}^{\boldsymbol{F}}$ is non-empty, we conclude that the strongly connected component of L (contained in the subgraph of the dependency graph of Π induced by $U \setminus \mathbf{A}^{\boldsymbol{F}}$) includes some edge, so that $L \in loop(\Pi)$. We have thus shown that $EB_\Pi(L) \subseteq \mathbf{A}^{\boldsymbol{F}}$ holds for some $L \in loop(\Pi)$ such that $L \not\subseteq \mathbf{A}^{\boldsymbol{F}}$, so that $D_{\{WFN[loop(\Pi)]\}}(\Pi,\mathbf{A}) \subseteq D_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A}) \not\subseteq \mathbf{A}$. $\square$

PROPOSITION 4.5. *Let Π be a normal program and $\mathbf{A}$ an assignment. Then, we have that $D^*_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) = D^*_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A})$.*

PROOF. By Lemma A.2, we have that $D^*_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A})$ is closed under $\{FFB, FFA, WFN[loop(\Pi)]\}$ and that $D^*_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A})$ is closed under $\{FFB, WFN[2^{atom(\Pi)}]\}$. Along with the fact that $D^*_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A})$ and $D^*_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A})$ are the unique smallest branches that extend $(\Pi,\mathbf{A})$ and are closed under $\{FFB, WFN[2^{atom(\Pi)}]\}$ or $\{FFB, FFA, WFN[loop(\Pi)]\}$, respectively, we conclude that $D^*_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A}) \subseteq D^*_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A})$ and that $D^*_{\{FFB,WFN[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \subseteq D^*_{\{FFB,FFA,WFN[loop(\Pi)]\}}(\Pi,\mathbf{A})$. $\square$

We have thus proven the formal results presented in Section 4, except for Theorem 4.6, whose proof is provided at the end of Appendix A.2.

A.2 Proofs of Results from Section 5

For proving the soundness and completeness of our generic tableau method relative to the language constructs considered in Section 5, we first provide some lemmas in Appendix A.2.1 and A.2.2. After demonstrating the main soundness and completeness result in Appendix A.2.3, the correspondences shown in Appendix A.2.4 between generic tableau rules and the basic ones

[2] Note that the "condensation" of $(U \setminus \mathbf{A}^{\boldsymbol{F}}, \{(head(r), p) \mid r \in \Pi, head(r) \in U \setminus \mathbf{A}^{\boldsymbol{F}}, p \in body(r)^+ \cap (U \setminus \mathbf{A}^{\boldsymbol{F}})\})$, obtained by contracting each strongly connected component to a single vertex, is a directed acyclic graph (cf. [Purdom 1970]).

for normal programs, as introduced in Section 3, allow us to derive Theorem 3.1 and 4.6 as consequences of more general results.

A.2.1 *Lemmas on Soundness.* The first two lemmas provide properties of non-contradictory complete branches that hold in view of the generic tableau rules in Figure 4.

LEMMA A.3. *Let Π be a disjunctive program and $\mathcal{T}$ a tableau calculus such that $\{I{\uparrow}, I{\downarrow}\} \cap \mathcal{T} \neq \emptyset$.*

Then, for every non-contradictory complete branch $(\Pi, \mathbf{A})$ and every $(\alpha \leftarrow \beta) \in \Pi$, we have that $\boldsymbol{t}\beta \notin \mathbf{A}$ or $\boldsymbol{f}\alpha \notin \mathbf{A}$.

PROOF. Consider any $(\alpha \leftarrow \beta) \in \Pi$ and any branch $(\Pi, \mathbf{A})$ such that $\boldsymbol{t}\beta \in \mathbf{A}$ and $\boldsymbol{f}\alpha \in \mathbf{A}$. Then, we have that $\boldsymbol{t}\alpha \in D_{\{I\uparrow\}}(\Pi, \mathbf{A})$ and $\boldsymbol{f}\beta \in D_{\{I\downarrow\}}(\Pi, \mathbf{A})$. Since $\{I{\uparrow}, I{\downarrow}\} \cap \mathcal{T} \neq \emptyset$, this shows that $(\Pi, \mathbf{A})$ cannot be (extended to) a non-contradictory complete branch. □

LEMMA A.4. *Let Π be a disjunctive program and $\mathcal{T}$ a tableau calculus such that $U{\uparrow} \in \mathcal{T}$.*

Then, for every non-contradictory complete branch $(\Pi, \mathbf{A})$ and every $S \subseteq atom(\Pi)$, we have that $sup_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$ or $\mathbf{A}^{\boldsymbol{T}} \cap S = \emptyset$.

PROOF. Consider any $S \subseteq atom(\Pi)$ and any branch $(\Pi, \mathbf{A})$ such that $sup_{\mathbf{A}}(\Pi, S, S) = \emptyset$ and $\mathbf{A}^{\boldsymbol{T}} \cap S \neq \emptyset$. Then, there is some $p \in \mathbf{A}^{\boldsymbol{T}} \cap S$ such that $\boldsymbol{F}p \in D_{\{U\uparrow\}}(\Pi, \mathbf{A})$. Since $U{\uparrow} \in \mathcal{T}$, this shows that $(\Pi, \mathbf{A})$ cannot be (extended to) a non-contradictory complete branch. □

For non-contradictory complete branches $(\Pi, \mathbf{A})$, the next lemmas show that the truth value of a variable $v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi)$ matches the valuation of $\tau[v]$ wrt $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, provided the inclusion of appropriate tableau rules, presented in Figure 5, 7, and 8, respectively, in a calculus.

LEMMA A.5. *Let Π be a disjunctive program and $\mathbf{A}$ a total assignment.*

Then, for every $p \in atom(\Pi)$, we have that

(*1*) $\boldsymbol{t}p \in \mathbf{A}$ *iff* $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[p]$*;*

(*2*) $\boldsymbol{t}not\ p \in \mathbf{A}$ *iff* $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[not\ p]$*;*

(*3*) $\boldsymbol{f}p \in \mathbf{A}$ *iff* $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[p]$*;*

(*4*) $\boldsymbol{f}not\ p \in \mathbf{A}$ *iff* $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[not\ p]$.

PROOF. We have that $\tau[p] = p$ and $\tau[not\ p] = \neg\tau[p] = \neg p$, and the following holds:

(1) $\boldsymbol{t}p \in \mathbf{A}$ iff $\boldsymbol{T}p \in \mathbf{A}$ iff $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models p$;

(2) $\boldsymbol{t}not\ p \in \mathbf{A}$ iff $\boldsymbol{F}p \in \mathbf{A}$ iff $p \notin \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \neg p$;

(3) $\boldsymbol{f}p \in \mathbf{A}$ iff $\boldsymbol{F}p \in \mathbf{A}$ iff $p \notin \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models p$;

(4) $\boldsymbol{f}not\ p \in \mathbf{A}$ iff $\boldsymbol{T}p \in \mathbf{A}$ iff $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \neg p$.

We have thus shown that all items of the statement hold. □

LEMMA A.6. *Let Π be a disjunctive program and $\mathcal{T}$ a tableau calculus such that $\{TLU{\uparrow}, FL{\uparrow}, FU{\uparrow}\} \subseteq \mathcal{T}$.*

Then, for every non-contradictory complete branch $(\Pi, \mathbf{A})$ and every $v \in card(\Pi)$, we have that $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$.

PROOF. Consider any $v = j\{l_1, \dots, l_n\}k \in card(\Pi)$ and any non-contradictory complete branch $(\Pi, \mathbf{A})$. For every $l \in \{l_1, \dots, l_n\}$, we have that $l \in atom(\Pi)$ or $l = not\ p$ for some $p \in atom(\Pi)$. By Lemma A.5, $\boldsymbol{t}l \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[l]$, and $\boldsymbol{f}l \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[l]$. We further consider the cases that $\boldsymbol{T}v \in \mathbf{A}$ and $\boldsymbol{F}v \in \mathbf{A}$, respectively:

(1) If $\boldsymbol{T}v \in \mathbf{A}$, then $\boldsymbol{F}v \notin D_{\{FL\uparrow,FU\uparrow\}}(\Pi,\mathbf{A})$. That is, $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\}| \leq n-j$ and $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}| \leq k$. In view of $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}| + |\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\}| = n$, $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\}| \leq n-j$ yields $j \leq |\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}|$. We have thus shown that $j \leq |\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}| \leq k$. Hence, for any $L \subseteq \{l_1,\dots,l_n\}$ such that $|L| < j$, it holds that $\{l \in \{l_1,\dots,l_n\} \setminus L \mid \boldsymbol{t}l \in \mathbf{A}\} \neq \emptyset$, so that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models (\bigvee_{l\in\{l_1,\dots,l_n\}\setminus L}\tau[l])$. Moreover, for any $L \subseteq \{l_1,\dots,l_n\}$ such that $k < |L|$, it holds that $\{l \in L \mid \boldsymbol{f}l \in \mathbf{A}\} \neq \emptyset$, so that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models (\bigwedge_{l\in L}\tau[l])$. Combining the cases for $|L| < j$ and $k < |L|$ yields that

$$\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \bigwedge_{L\subseteq\{l_1,\dots,l_n\},|L|<j \text{ or } k<|L|}\big((\bigwedge_{l\in L}\tau[l]) \rightarrow (\bigvee_{l\in\{l_1,\dots,l_n\}\setminus L}\tau[l])\big).$$

(2) If $\boldsymbol{F}v \in \mathbf{A}$, then $\boldsymbol{T}v \notin D_{\{TLU\uparrow\}}(\Pi,\mathbf{A})$. That is, $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}| < j$ or $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\}| < n-k$. In view of $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}| + |\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\}| = n$, $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\}| < n-k$ yields $k < |\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}|$. For $L' = \{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}$, we have thus shown that $|L'| < j$ or $k < |L'|$. Since $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \big((\bigwedge_{l\in L'}\tau[l]) \rightarrow (\bigvee_{l\in\{l_1,\dots,l_n\}\setminus L'}\tau[l])\big)$, we conclude that

$$\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \bigwedge_{L\subseteq\{l_1,\dots,l_n\},|L|<j \text{ or } k<|L|}\big((\bigwedge_{l\in L}\tau[l]) \rightarrow (\bigvee_{l\in\{l_1,\dots,l_n\}\setminus L}\tau[l])\big).$$

We have thus shown that $\boldsymbol{T}v \in \mathbf{A}$ and $\boldsymbol{F}v \in \mathbf{A}$ imply $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$ and $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[v]$, respectively. That is, $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$. □

LEMMA A.7. *Let Π be a disjunctive program and $\mathcal{T}$ a tableau calculus such that $\{TC\uparrow, FC\uparrow\} \subseteq \mathcal{T}$.*
If $card(\Pi) = \emptyset$ or $\{TLU\uparrow, FL\uparrow, FU\uparrow\} \subseteq \mathcal{T}$, then for every non-contradictory complete branch $(\Pi,\mathbf{A})$ and every $v \in conj(\Pi)$, we have that $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$.

PROOF. Consider any $v = \{l_1,\dots,l_n\} \in conj(\Pi)$ and any non-contradictory complete branch $(\Pi,\mathbf{A})$, and assume that $card(\Pi) = \emptyset$ or $\{TLU\uparrow, FL\uparrow, FU\uparrow\} \subseteq \mathcal{T}$. For every $l \in \{l_1,\dots,l_n\}$, we have that $l \in atom(\Pi) \cup card(\Pi)$ or $l = not\ \pi$ and $\tau[l] = \neg\tau[\pi]$ for some $\pi \in atom(\Pi) \cup card(\Pi)$. By Lemma A.5 and A.6, $\boldsymbol{t}l \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[l]$, and $\boldsymbol{f}l \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[l]$. We further consider the cases that $\boldsymbol{T}v \in \mathbf{A}$ and $\boldsymbol{F}v \in \mathbf{A}$, respectively:

(1) If $\boldsymbol{T}v \in \mathbf{A}$, then $\boldsymbol{F}v \notin D_{\{FC\uparrow\}}(\Pi,\mathbf{A})$. That is, $\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\} = \emptyset$ and $\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\} = \{l_1,\dots,l_n\}$, so that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models (\tau[l_1] \wedge \dots \wedge \tau[l_n])$.
(2) If $\boldsymbol{F}v \in \mathbf{A}$, then $\boldsymbol{T}v \notin D_{\{TC\uparrow\}}(\Pi,\mathbf{A})$. That is, $\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\} \neq \{l_1,\dots,l_n\}$ and $\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\} \neq \emptyset$, so that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models (\tau[l_1] \wedge \dots \wedge \tau[l_n])$.

We have thus shown that $\boldsymbol{T}v \in \mathbf{A}$ and $\boldsymbol{F}v \in \mathbf{A}$ imply $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$ and $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[v]$, respectively. That is, $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$. □

LEMMA A.8. *Let Π be a disjunctive program and $\mathcal{T}$ a tableau calculus such that $\{TD\uparrow, FD\uparrow\} \subseteq \mathcal{T}$.*
Then, for every non-contradictory complete branch $(\Pi,\mathbf{A})$ and every $v \in disj(\Pi)$, we have that $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$.

PROOF. Consider any $v = \{l_1;\dots;l_n\} \in disj(\Pi)$ and any non-contradictory complete branch $(\Pi,\mathbf{A})$. For every $l \in \{l_1,\dots,l_n\}$, we have that $l \in atom(\Pi)$ or $l = not\ p$ for some $p \in atom(\Pi)$. By Lemma A.5, $\boldsymbol{t}l \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[l]$, and $\boldsymbol{f}l \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \not\models \tau[l]$. We further consider the cases that $\boldsymbol{T}v \in \mathbf{A}$ and $\boldsymbol{F}v \in \mathbf{A}$, respectively:

(1) If $\boldsymbol{T}v \in \mathbf{A}$, then $\boldsymbol{F}v \notin D_{\{FD\uparrow\}}(\Pi, \mathbf{A})$. That is, $\{l \in \{l_1, \dots, l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\} \neq \{l_1, \dots, l_n\}$ and $\{l \in \{l_1, \dots, l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\} \neq \emptyset$, so that $\mathbf{A}^{\boldsymbol{T}} \cap \mathit{atom}(\Pi) \models (\tau[l_1] \vee \dots \vee \tau[l_n])$.

(2) If $\boldsymbol{F}v \in \mathbf{A}$, then $\boldsymbol{T}v \notin D_{\{TD\uparrow\}}(\Pi, \mathbf{A})$. That is, $\{l \in \{l_1, \dots, l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\} = \emptyset$ and $\{l \in \{l_1, \dots, l_n\} \mid \boldsymbol{f}l \in \mathbf{A}\} = \{l_1, \dots, l_n\}$, so that $\mathbf{A}^{\boldsymbol{T}} \cap \mathit{atom}(\Pi) \not\models (\tau[l_1] \vee \dots \vee \tau[l_n])$.

We have thus shown that $\boldsymbol{T}v \in \mathbf{A}$ and $\boldsymbol{F}v \in \mathbf{A}$ imply $\mathbf{A}^{\boldsymbol{T}} \cap \mathit{atom}(\Pi) \models \tau[v]$ and $\mathbf{A}^{\boldsymbol{T}} \cap \mathit{atom}(\Pi) \not\models \tau[v]$, respectively. That is, $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap \mathit{atom}(\Pi) \models \tau[v]$. □

A.2.2 *Lemmas on Completeness.* In order to abstract from the language constructs admitted in a program, the following definition formulates conditions under which we call $\overleftarrow{\mathit{sup}}$, $\overrightarrow{\mathit{sup}}$, min, and max, respectively, *well-behaved*. We then proceed by showing that these four concepts are well-behaved for disjunctive programs.

DEFINITION A.9. *Let α be a literal.*
Then, we define $\overleftarrow{\mathit{sup}}$, $\overrightarrow{\mathit{sup}}$, min, and max, respectively, as well-behaved for α if, for every $S \subseteq \mathcal{P}$ and every assignment $\mathbf{A}$, we have that

(1) *if $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha, S)$ holds, then $\overleftarrow{\mathit{sup}}_{\mathbf{A}'}(\alpha, S)$ holds for every $\mathbf{A}' \subseteq \mathbf{A}$;*

(2) *if $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\alpha, S)$ holds, then $\overrightarrow{\mathit{sup}}_{\mathbf{A}'}(\alpha, S')$ holds for every $\mathbf{A}' \subseteq \mathbf{A}$ and every $S' \subseteq S$;*

(3) *if $\ell \in \mathit{min}_{\mathbf{A}}(\alpha, S)$, then $\overleftarrow{\mathit{sup}}_{\mathbf{A} \cup \{\overline{\ell}\}}(\alpha, S)$ does not hold;*

(4) *if $\ell \in \mathit{max}_{\mathbf{A}}(\alpha, S)$, then $\overrightarrow{\mathit{sup}}_{\mathbf{A} \cup \{\overline{\ell}\}}(\alpha, S)$ does not hold.*

LEMMA A.10. *Let α be a disjunctive literal and β a cardinality literal or a possibly negated conjunction of cardinality literals.*
Then, we have that $\overleftarrow{\mathit{sup}}$ and min are well-behaved for α and that $\overrightarrow{\mathit{sup}}$ and max are well-behaved for β.

PROOF. Let $S \subseteq \mathcal{P}$ and $\mathbf{A}$ an arbitrary assignment.
We first consider the possible cases such that $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha, S)$ holds:

(1) If $\alpha \in S$, we have that $\overleftarrow{\mathit{sup}}_{\mathbf{A}'}(\alpha, S)$ holds for all assignments $\mathbf{A}'$.

(2) If $\alpha = j\{l_1, \dots, l_n\}k \in \mathit{card}(\mathcal{P})$, then $\{l_1, \dots, l_n\} \cap S \neq \emptyset$ and $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\}| < k$. Since for all $\mathbf{A}' \subseteq \mathbf{A}$, we have that $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}'\}| \leq |\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\}| < k$, we conclude that $\overleftarrow{\mathit{sup}}_{\mathbf{A}'}(\alpha, S)$ holds.

(3) If $\alpha = \{l_1; \dots; l_n\} \in \mathit{disj}(\mathcal{P})$, then $\{l_1, \dots, l_n\} \cap S \neq \emptyset$ and $\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\} = \emptyset$. Since for all $\mathbf{A}' \subseteq \mathbf{A}$, we have that $\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}'\} \subseteq \{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\} = \emptyset$, we conclude that $\overleftarrow{\mathit{sup}}_{\mathbf{A}'}(\alpha, S)$ holds.

We next consider the possible cases such that $\ell \in \mathit{min}_{\mathbf{A}}(\alpha, S)$:

(1) If $\alpha = j\{l_1, \dots, l_n\}k \in \mathit{card}(\mathcal{P})$ and $\ell \in \mathit{min}_{\mathbf{A}}(\alpha, S) = \{\boldsymbol{f}l \mid l \in \{l_1, \dots, l_n\} \setminus S, \boldsymbol{t}l \notin \mathbf{A}\}$, then $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\}| = k - 1$. That is, $\overline{\ell} = \boldsymbol{t}l \notin \mathbf{A}$ for some $l \in \{l_1, \dots, l_n\} \setminus S$, so that $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A} \cup \{\overline{\ell}\}\}| = |\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\}| + 1 = k$, which means that $\overleftarrow{\mathit{sup}}_{\mathbf{A} \cup \{\overline{\ell}\}}(\alpha, S)$ does not hold.

(2) If $\alpha = \{l_1; \dots; l_n\} \in \mathit{disj}(\mathcal{P})$ and $\ell \in \mathit{min}_{\mathbf{A}}(\alpha, S) = \{\boldsymbol{f}l \mid l \in \{l_1, \dots, l_n\} \setminus S\}$, then $\overline{\ell} = \boldsymbol{t}l$ for some $l \in \{l_1, \dots, l_n\} \setminus S$. We conclude that $\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A} \cup \{\overline{\ell}\}\} \neq \emptyset$, which means that $\overleftarrow{\mathit{sup}}_{\mathbf{A} \cup \{\overline{\ell}\}}(\alpha, S)$ does not hold.

We now come to the possible cases such that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta, S)$ holds:

(1) If $\beta = not\ v$, where $v \in \mathcal{P} \cup card(\mathcal{P}) \cup conj(\mathcal{P})$, we have that $\overrightarrow{sup}_{\mathbf{A}'}(\beta, S')$ holds for every assignment $\mathbf{A}'$ and every $S' \subseteq \mathcal{P}$.

(2) If $\beta \in \mathcal{P} \setminus S$, then $\beta \in \mathcal{P} \setminus S'$ for every $S' \subseteq S$, so that $\overrightarrow{sup}_{\mathbf{A}'}(\beta, S')$ holds for every assignment $\mathbf{A}'$ and every $S' \subseteq S$.

(3) If $\beta = j\{l_1, \dots, l_n\}k \in card(\mathcal{P})$, then $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\}| \geq j$. Since for every $\mathbf{A}' \subseteq \mathbf{A}$ and every $S' \subseteq S$, we have that $|\{l \in \{l_1, \dots, l_n\} \setminus S' \mid \boldsymbol{f}l \notin \mathbf{A}'\}| \geq |\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\}| \geq j$, we conclude that $\overrightarrow{sup}_{\mathbf{A}'}(\beta, S')$ holds.

(4) If $\beta = \{l_1, \dots, l_n\} \in conj(\mathcal{P})$, then $\overrightarrow{sup}_{\mathbf{A}}(l, S)$ holds for every $l \in \{l_1, \dots, l_n\}$. Furthermore, since one of the first three cases applies to each $l \in \{l_1, \dots, l_n\}$, we have that $\overrightarrow{sup}_{\mathbf{A}'}(l, S')$ holds for every $\mathbf{A}' \subseteq \mathbf{A}$ and every $S' \subseteq S$, so that $\overrightarrow{sup}_{\mathbf{A}'}(\beta, S')$ holds as well.

Finally, we consider the possible cases such that $\ell \in max_{\mathbf{A}}(\beta, S)$:

(1) If $\beta = j\{l_1, \dots, l_n\}k \in card(\mathcal{P})$ and $\ell \in max_{\mathbf{A}}(\beta, S) = \{\boldsymbol{t}l \mid l \in \{l_1, \dots, l_n\} \setminus S, \boldsymbol{f}l \notin \mathbf{A}\}$, then $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\}| = j$. That is, $\overline{\ell} = \boldsymbol{f}l \notin \mathbf{A}$ for some $l \in \{l_1, \dots, l_n\} \setminus S$, so that $|\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{f}l \notin \mathbf{A} \cup \{\overline{\ell}\}\}| = |\{l \in \{l_1, \dots, l_n\} \setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\}| - 1 = j - 1$, which means that $\overrightarrow{sup}_{\mathbf{A} \cup \{\overline{\ell}\}}(\beta, S)$ does not hold.

(2) If $\beta = \{l_1, \dots, l_n\} \in conj(\mathcal{P})$ and $\ell \in max_{\mathbf{A}}(\beta, S) = \bigcup_{l \in \{l_1, \dots, l_n\}} max_{\mathbf{A}}(l, S)$, then $\ell \in max_{\mathbf{A}}(l, S)$ for some $l \in \{l_1, \dots, l_n\} \cap card(\mathcal{P})$. That is, the previous case applies to l, so that $\overrightarrow{sup}_{\mathbf{A} \cup \{\overline{\ell}\}}(l, S)$ and $\overrightarrow{sup}_{\mathbf{A} \cup \{\overline{\ell}\}}(\beta, S)$ do not hold.

We have thus, for $S \subseteq \mathcal{P}$ and an arbitrary assignment $\mathbf{A}$, considered all possible cases and shown that $\overleftarrow{sup}$ and min are well-behaved for α and that $\overrightarrow{sup}$ and max are well-behaved for β. $\square$

The concept of well-behavedness allows us to identify the property that $sup_{\mathbf{A}}(\Pi, S, T)$ is antimonotone wrt both $\mathbf{A}$ and T.

LEMMA A.11. *Let Π be a disjunctive program, $S \subseteq \mathcal{P}$, $T \subseteq \mathcal{P}$, and $\mathbf{A}$ an assignment. If $\overleftarrow{sup}$ and $\overrightarrow{sup}$ are well-behaved for all literals in $\{\alpha \mid (\alpha \leftarrow \beta) \in \Pi\}$ and $\{\beta \mid (\alpha \leftarrow \beta) \in \Pi\}$, respectively, then we have that $sup_{\mathbf{A}}(\Pi, S, T) \subseteq sup_{\mathbf{A}'}(\Pi, S, T')$ for every $\mathbf{A}' \subseteq \mathbf{A}$ and every $T' \subseteq T$.*

PROOF. Assume that $\overleftarrow{sup}$ and $\overrightarrow{sup}$ are well-behaved for all literals in $\{\alpha \mid (\alpha \leftarrow \beta) \in \Pi\}$ and $\{\beta \mid (\alpha \leftarrow \beta) \in \Pi\}$, respectively, and consider any $(\alpha \leftarrow \beta) \in sup_{\mathbf{A}}(\Pi, S, T) = \{(\alpha \leftarrow \beta) \in \Pi \mid \boldsymbol{f}\beta \notin \mathbf{A}, \overleftarrow{sup}_{\mathbf{A}}(\alpha, S), \overrightarrow{sup}_{\mathbf{A}}(\beta, T)\}$. In view of Definition A.9, for every $\mathbf{A}' \subseteq \mathbf{A}$ and every $T' \subseteq T$, we have that $\overleftarrow{sup}_{\mathbf{A}}(\alpha, S)$ and $\overrightarrow{sup}_{\mathbf{A}}(\beta, T)$ imply $\overleftarrow{sup}_{\mathbf{A}'}(\alpha, S)$ and $\overrightarrow{sup}_{\mathbf{A}'}(\beta, T')$, respectively, and $\boldsymbol{f}\beta \notin \mathbf{A}'$ follows immediately from $\boldsymbol{f}\beta \notin \mathbf{A}$. From this, we conclude that $(\alpha \leftarrow \beta) \in sup_{\mathbf{A}'}(\Pi, S, T') = \{(\alpha \leftarrow \beta) \in \Pi \mid \boldsymbol{f}\beta \notin \mathbf{A}', \overleftarrow{sup}_{\mathbf{A}'}(\alpha, S), \overrightarrow{sup}_{\mathbf{A}'}(\beta, T')\}$. $\square$

We are now ready to prove that, for a total assignment $\mathbf{A}$ such that the deterministic tableau rules in Figure 4 do not yield a contradiction, the entries of $\mathbf{A}$ are preserved when applying these tableau rules wrt any assignment contained in $\mathbf{A}$.

LEMMA A.12. *Let Π be a disjunctive program and $\mathbf{A}$ a total assignment such that $\boldsymbol{t}\beta \notin \mathbf{A}$ or $\boldsymbol{f}\alpha \notin \mathbf{A}$ for every $(\alpha \leftarrow \beta) \in \Pi$ and $sup_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$ or $\mathbf{A}^{\boldsymbol{T}} \cap S = \emptyset$ for every $S \subseteq atom(\Pi)$. If $\overleftarrow{sup}$ and min are well-behaved for all literals in $\{\alpha \mid (\alpha \leftarrow \beta) \in \Pi\}$ and if $\overrightarrow{sup}$ and max are well-behaved for all literals in $\{\beta \mid (\alpha \leftarrow \beta) \in \Pi\}$, then for every $\mathbf{A}' \subseteq \mathbf{A}$, we have that $D_{\{(a)\text{–}(f)\}}(\Pi, \mathbf{A}') \subseteq \mathbf{A}$.*

PROOF. Assume that $\overleftarrow{sup}$ and min are well-behaved for all literals in $\{\alpha \mid (\alpha \leftarrow \beta) \in \Pi\}$ and that $\overrightarrow{sup}$ and max are well-behaved for all literals in $\{\beta \mid (\alpha \leftarrow \beta) \in \Pi\}$, and consider any $\mathbf{A}' \subseteq \mathbf{A}$. We show that any entry deducible by $I\!\uparrow$, $I\!\downarrow$, $N\!\uparrow$, $N\!\downarrow$, $U\!\uparrow$, or $U\!\downarrow$ in $(\Pi, \mathbf{A}')$ belongs to $\mathbf{A}$:

($I\!\uparrow$) If $\boldsymbol{t}\alpha \in D_{\{I\uparrow\}}(\Pi, \mathbf{A}')$, we have that $\boldsymbol{t}\beta \in \mathbf{A}'$ for some $(\alpha \leftarrow \beta) \in \Pi$. Since $\boldsymbol{t}\beta \in \mathbf{A}$, it holds that $\boldsymbol{f}\alpha \notin \mathbf{A}$, which yields $\boldsymbol{t}\alpha \in \mathbf{A}$ because $\mathbf{A}$ is total.

($I\!\downarrow$) If $\boldsymbol{f}\beta \in D_{\{I\downarrow\}}(\Pi, \mathbf{A}')$, we have that $\boldsymbol{f}\alpha \in \mathbf{A}'$ for some $(\alpha \leftarrow \beta) \in \Pi$. Since $\boldsymbol{f}\alpha \in \mathbf{A}$, it holds that $\boldsymbol{t}\beta \notin \mathbf{A}$, which yields $\boldsymbol{f}\beta \in \mathbf{A}$ because $\mathbf{A}$ is total.

($N\!\uparrow$) If $\boldsymbol{F}p \in D_{\{N\uparrow\}}(\Pi, \mathbf{A}')$, we have that $p \in atom(\Pi)$ and $sup_{\mathbf{A}'}(\Pi, \{p\}, \emptyset) = \emptyset$. By Lemma A.11, we conclude that $sup_{\mathbf{A}}(\Pi, \{p\}, \{p\}) = \emptyset$. Thus, it holds that $\boldsymbol{T}p \notin \mathbf{A}$, which yields $\boldsymbol{F}p \in \mathbf{A}$ because $\mathbf{A}$ is total.

($N\!\downarrow$) If $\ell \in D_{\{N\downarrow\}}(\Pi, \mathbf{A}')$, we have that $\ell \in \{\boldsymbol{t}\beta\} \cup min_{\mathbf{A}'}(\alpha, \{p\}) \cup max_{\mathbf{A}'}(\beta, \emptyset)$ for some $p \in (\mathbf{A}')^{\boldsymbol{T}} \cap atom(\Pi)$ such that $sup_{\mathbf{A}'}(\Pi, \{p\}, \emptyset) = \{\alpha \leftarrow \beta\}$. Since $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, it holds that $sup_{\mathbf{A}}(\Pi, \{p\}, \{p\}) \neq \emptyset$. However, given that min and max are well-behaved for α and β, respectively, we also have that $(\alpha \leftarrow \beta) \notin sup_{\mathbf{A}' \cup \{\overline{\ell}\}}(\Pi, \{p\}, \emptyset)$. By Lemma A.11, we conclude that $sup_{\mathbf{A} \cup \{\overline{\ell}\}}(\Pi, \{p\}, \{p\}) \subseteq sup_{\mathbf{A}' \cup \{\overline{\ell}\}}(\Pi, \{p\}, \emptyset) \subseteq sup_{\mathbf{A}'}(\Pi, \{p\}, \emptyset) \setminus \{\alpha \leftarrow \beta\} = \emptyset$. That is, $sup_{\mathbf{A} \cup \{\overline{\ell}\}}(\Pi, \{p\}, \{p\}) = \emptyset \neq sup_{\mathbf{A}}(\Pi, \{p\}, \{p\})$, which yields $\overline{\ell} \notin \mathbf{A}$. Finally, since $\mathbf{A}$ is total, $\overline{\ell} \notin \mathbf{A}$ implies $\ell \in \mathbf{A}$.

($U\!\uparrow$) If $\boldsymbol{F}p \in D_{\{U\uparrow\}}(\Pi, \mathbf{A}')$, we have that $p \in S$ for some $S \subseteq atom(\Pi)$ such that $sup_{\mathbf{A}'}(\Pi, S, S) = \emptyset$. By Lemma A.11, we conclude that $sup_{\mathbf{A}}(\Pi, S, S) = \emptyset$. Thus, it holds that $\boldsymbol{T}p \notin \mathbf{A}$, which yields $\boldsymbol{F}p \in \mathbf{A}$ because $\mathbf{A}$ is total.

($U\!\downarrow$) If $\ell \in D_{\{U\downarrow\}}(\Pi, \mathbf{A}')$, we have that $\ell \in \{\boldsymbol{t}\beta\} \cup min_{\mathbf{A}'}(\alpha, S) \cup max_{\mathbf{A}'}(\beta, S)$ for some $S \subseteq atom(\Pi)$ such that $(\mathbf{A}')^{\boldsymbol{T}} \cap S \neq \emptyset$ and $sup_{\mathbf{A}'}(\Pi, S, S) = \{\alpha \leftarrow \beta\}$. Since $\mathbf{A}^{\boldsymbol{T}} \cap S \neq \emptyset$, it holds that $sup_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$. However, given that min and max are well-behaved for α and β, respectively, we also have that $(\alpha \leftarrow \beta) \notin sup_{\mathbf{A}' \cup \{\overline{\ell}\}}(\Pi, S, S)$. By Lemma A.11, we conclude that $sup_{\mathbf{A} \cup \{\overline{\ell}\}}(\Pi, S, S) \subseteq sup_{\mathbf{A}' \cup \{\overline{\ell}\}}(\Pi, S, S) \subseteq sup_{\mathbf{A}'}(\Pi, S, S) \setminus \{\alpha \leftarrow \beta\} = \emptyset$. That is, $sup_{\mathbf{A} \cup \{\overline{\ell}\}}(\Pi, S, S) = \emptyset \neq sup_{\mathbf{A}}(\Pi, S, S)$, which yields $\overline{\ell} \notin \mathbf{A}$. Finally, since $\mathbf{A}$ is total, $\overline{\ell} \notin \mathbf{A}$ implies $\ell \in \mathbf{A}$.

We have thus shown that, in every branch $(\Pi, \mathbf{A}')$ such that $\mathbf{A}' \subseteq \mathbf{A}$, any entry deducible by $I\!\uparrow$, $I\!\downarrow$, $N\!\uparrow$, $N\!\downarrow$, $U\!\uparrow$, or $U\!\downarrow$ belongs to $\mathbf{A}$, so that $D_{\{(a)\text{–}(f)\}}(\Pi, \mathbf{A}') \subseteq \mathbf{A}$. □

Finally, the next two lemmas show that, for a total assignment $\mathbf{A}$ such that the truth values of variables $v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi)$ match the valuation of $\tau[v]$ wrt $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, the language-specific tableau rules in Figure 5, 7, and 8, respectively, preserve the entries of $\mathbf{A}$ when applied wrt any assignment contained in $\mathbf{A}$.

LEMMA A.13. *Let Π be a disjunctive program, $X \subseteq atom(\Pi)$, and*

$$\begin{aligned}\mathbf{A} \;=\; & \{\boldsymbol{T}v \mid v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi), X \models \tau[v]\} \\ \cup\; & \{\boldsymbol{F}v \mid v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi), X \not\models \tau[v]\}.\end{aligned}$$

Then, for every $v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi)$, we have that

(1) $\boldsymbol{t}v \in \mathbf{A}$ *iff* $X \models \tau[v]$*;*

(2) $\boldsymbol{t}not\ v \in \mathbf{A}$ *iff* $X \models \tau[not\ v]$*;*

(*3*) $\boldsymbol{f}v \in \mathbf{A}$ *iff* $X \not\models \tau[v]$*;*

(*4*) $\boldsymbol{f}\mathit{not}\ v \in \mathbf{A}$ *iff* $X \not\models \tau[\mathit{not}\ v]$.

PROOF. By the definition of $\mathbf{A}$, for every $v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi)$:

(1) $\boldsymbol{t}v \in \mathbf{A}$ iff $\boldsymbol{T}v \in \mathbf{A}$ iff $X \models \tau[v]$;

(2) $\boldsymbol{t}\mathit{not}\ v \in \mathbf{A}$ iff $\boldsymbol{F}v \in \mathbf{A}$ iff $X \not\models \tau[v]$ iff $X \models \neg\tau[v]$ iff $X \models \tau[\mathit{not}\ v]$;

(3) $\boldsymbol{f}v \in \mathbf{A}$ iff $\boldsymbol{F}v \in \mathbf{A}$ iff $X \not\models \tau[v]$;

(4) $\boldsymbol{f}\mathit{not}\ v \in \mathbf{A}$ iff $\boldsymbol{T}v \in \mathbf{A}$ iff $X \models \tau[v]$ iff $X \not\models \neg\tau[v]$ iff $X \not\models \tau[\mathit{not}\ v]$.

We have thus shown that all items of the statement hold. □

LEMMA A.14. *Let* Π *be a disjunctive program,* $X \subseteq \mathit{atom}(\Pi)$*, and*

$$\begin{aligned}\mathbf{A} \;=\; & \{\boldsymbol{T}v \mid v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi), X \models \tau[v]\} \\ \cup\; & \{\boldsymbol{F}v \mid v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi), X \not\models \tau[v]\}.\end{aligned}$$

Then, for every $\mathbf{A}' \subseteq \mathbf{A}$*, we have that* $D_{\{(h)–(v)\}}(\Pi, \mathbf{A}') \subseteq \mathbf{A}$.

PROOF. By Lemma A.13, for every literal $l = v$ or $l = \mathit{not}\ v$, where $v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi)$, we have that $\boldsymbol{t}l \in \mathbf{A}$ iff $X \models \tau[l]$, and that $\boldsymbol{f}l \in \mathbf{A}$ iff $X \not\models \tau[l]$. Hence, we can treat such conditions as synonyms in the following consideration of some $\mathbf{A}' \subseteq \mathbf{A}$ and the tableau rules *(h)–(v)*:

($TC\!\uparrow$) If $\{l_1, \dots, l_n\} \in \mathit{conj}(\Pi)$ such that $\{\boldsymbol{t}l_1, \dots, \boldsymbol{t}l_n\} \subseteq \mathbf{A}'$, we have that $X \models \tau[l_1], \dots, X \models \tau[l_n]$. That is, $X \models (\tau[l_1] \wedge \cdots \wedge \tau[l_n])$, so that $\boldsymbol{T}\{l_1, \dots, l_n\} \in \mathbf{A}$.

($TC\!\downarrow$) If $\{l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\} \in \mathit{conj}(\Pi)$ such that $\{\boldsymbol{F}\{l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\}, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_{i-1}, \boldsymbol{t}l_{i+1}, \dots, \boldsymbol{t}l_n\} \subseteq \mathbf{A}'$, we have that $X \models \tau[l_1], \dots, X \models \tau[l_{i-1}], X \models \tau[l_{i+1}], \dots, X \models \tau[l_n]$ but $X \not\models (\tau[l_1] \wedge \cdots \wedge \tau[l_{i-1}] \wedge \tau[l_i] \wedge \tau[l_{i+1}] \wedge \cdots \wedge \tau[l_n])$. That is, $X \not\models \tau[l_i]$, so that $\boldsymbol{f}l_i \in \mathbf{A}$.

($FC\!\uparrow$) If $\{l_1, \dots, l_i, \dots, l_n\} \in \mathit{conj}(\Pi)$ such that $\boldsymbol{f}l_i \in \mathbf{A}'$, we have that $X \not\models \tau[l_i]$. That is, $X \not\models (\tau[l_1] \wedge \cdots \wedge \tau[l_i] \wedge \cdots \wedge \tau[l_n])$, so that $\boldsymbol{F}\{l_1, \dots, l_i, \dots, l_n\} \in \mathbf{A}$.

($FC\!\downarrow$) If $\{l_1, \dots, l_n\} \in \mathit{conj}(\Pi)$ such that $\boldsymbol{T}\{l_1, \dots, l_n\} \in \mathbf{A}'$, we have that $X \models (\tau[l_1] \wedge \cdots \wedge \tau[l_n])$. That is, $X \models \tau[l_1], \dots, X \models \tau[l_n]$, so that $\{\boldsymbol{t}l_1, \dots, \boldsymbol{t}l_n\} \subseteq \mathbf{A}$.

($TLU\!\uparrow$) If $j\{l_1, \dots, l_j, \dots, l_{k+1}, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{t}l_1, \dots, \boldsymbol{t}l_j, \boldsymbol{f}l_{k+1}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}'$, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $|L| < j$, we have that $\{l_1, \dots, l_j\} \not\subseteq L$, that is, $X \models (\bigvee_{l \in \{l_1, \dots, l_n\} \setminus L} \tau[l])$. Furthermore, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $k < |L|$, we have that $L \cap \{l_{k+1}, \dots, l_n\} \neq \emptyset$, that is, $X \not\models (\bigwedge_{l \in L} \tau[l])$. We obtain that

$$X \;\models\; \textstyle\bigwedge_{L \subseteq \{l_1, \dots, l_n\}, |L| < j \text{ or } k < |L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1, \dots, l_n\} \setminus L} \tau[l])\big),$$

so that $\boldsymbol{T}j\{l_1, \dots, l_j, \dots, l_{k+1}, \dots, l_n\}k \in \mathbf{A}$.

($TLU\!\downarrow$) If $j\{l_1, \dots, l_{j-1}, l_j, \dots, l_k, l_{k+1}, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{F}j\{l_1, \dots, l_{j-1}, l_j, \dots, l_k, l_{k+1}, \dots, l_n\}k, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_{j-1}, \boldsymbol{f}l_{k+1}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}'$, we have that

$$X \;\not\models\; \textstyle\bigwedge_{L \subseteq \{l_1, \dots, l_n\}, |L| < j \text{ or } k < |L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1, \dots, l_n\} \setminus L} \tau[l])\big).$$

However, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $k < |L|$, we have that $L \cap \{l_{k+1}, \dots, l_n\} \neq \emptyset$, that is, $X \not\models (\bigwedge_{l \in L} \tau[l])$. Furthermore, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $|L| < j$ and

$L \neq \{l_1, \dots, l_{j-1}\}$, we have that $\{l_1, \dots, l_{j-1}\} \not\subseteq L$, that is, $X \models (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])$. We obtain that

$$X \models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, (|L|<j \text{ and } L \neq \{l_1,\dots,l_{j-1}\}) \text{ or } k<|L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big).$$

That is, $X \not\models \big((\tau[l_1] \wedge \dots \wedge \tau[l_{j-1}]) \rightarrow (\tau[l_j] \vee \dots \vee \tau[l_k] \vee \tau[l_{k+1}] \vee \dots \vee \tau[l_n])\big)$, so that $X \not\models \tau[l_j], \dots, X \not\models \tau[l_k]$ and $\{\boldsymbol{f}l_j, \dots, \boldsymbol{f}l_k\} \subseteq \mathbf{A}$.

($T{\scriptstyle L}U\!\downarrow$) If $j\{l_1, \dots, l_j, l_{j+1}, \dots, l_{k+1}, l_{k+2}, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{F}j\{l_1, \dots, l_j, l_{j+1}, \dots, l_{k+1}, l_{k+2}, \dots, l_n\}k, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_j, \boldsymbol{f}l_{k+2}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}'$, we have that

$$X \not\models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, |L|<j \text{ or } k<|L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big).$$

However, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $|L| < j$, we have that $\{l_1, \dots, l_j\} \not\subseteq L$, that is, $X \models (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])$. Furthermore, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $k < |L|$ and $L \neq \{l_1, \dots, l_{k+1}\}$, we have that $L \cap \{l_{k+2}, \dots, l_n\} \neq \emptyset$, that is, $X \not\models (\bigwedge_{l \in L} \tau[l])$. We obtain that

$$X \models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, |L|<j \text{ or } (k<|L| \text{ and } L \neq \{l_1,\dots,l_{k+1}\})} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big).$$

That is, $X \not\models \big((\tau[l_1] \wedge \dots \wedge \tau[l_j] \wedge \tau[l_{j+1}] \wedge \dots \wedge \tau[l_{k+1}]) \rightarrow (\tau[l_{k+2}] \vee \dots \vee \tau[l_n])\big)$, so that $X \models \tau[l_{j+1}], \dots, X \models \tau[l_{k+1}]$ and $\{\boldsymbol{t}l_{j+1}, \dots, \boldsymbol{t}l_{k+1}\} \subseteq \mathbf{A}$.

($FL\!\uparrow$) If $j\{l_1, \dots, l_j, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{f}l_j, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}'$, for $L' = \{l \in \{l_1, \dots, l_n\} \mid X \models \tau[l]\}$, we have that $L' \subseteq \{l_1, \dots, l_{j-1}\}$ and $|L'| < j$, while $X \not\models \tau[l]$ for all $l \in \{l_1, \dots, l_n\} \setminus L'$. Hence, $X \not\models \big((\bigwedge_{l \in L'} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L'} \tau[l])\big)$ and

$$X \not\models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, |L|<j \text{ or } k<|L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big),$$

so that $\boldsymbol{F}j\{l_1, \dots, l_j, \dots, l_n\}k \in \mathbf{A}$.

($FL\!\downarrow$) If $j\{l_1, \dots, l_j, l_{j+1}, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{T}j\{l_1, \dots, l_j, l_{j+1}, \dots, l_n\}k, \boldsymbol{f}l_{j+1}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}'$, we have that

$$X \models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, |L|<j \text{ or } k<|L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big).$$

In particular, for every $1 \leq i \leq j$ and $L_i = \{l \in \{l_1, \dots, l_n\} \setminus \{l_i\} \mid X \models \tau[l]\}$, we have that $L_i \subseteq \{l_1, \dots, l_j\} \setminus \{l_i\}$ and $|L_i| < j$, while $X \not\models \tau[l]$ for every $l \in \{l_1, \dots, l_n\} \setminus (L_i \cup \{l_i\})$. Hence, $X \models \big((\bigwedge_{l \in L_i} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L_i} \tau[l])\big)$ but $X \not\models \big((\bigwedge_{l \in L_i} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus (L_i \cup \{l_i\})} \tau[l])\big)$. That is, $X \models \tau[l_i]$ for every $1 \leq i \leq j$, so that $\{\boldsymbol{t}l_1, \dots, \boldsymbol{t}l_j\} \subseteq \mathbf{A}$.

($FU\!\uparrow$) If $j\{l_1, \dots, l_{k+1}, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{t}l_1, \dots, \boldsymbol{t}l_{k+1}\} \subseteq \mathbf{A}'$, for $L' = \{l \in \{l_1, \dots, l_n\} \mid X \models \tau[l]\}$, we have that $\{l_1, \dots, l_{k+1}\} \subseteq L'$ and $k < |L'|$, while $X \not\models \tau[l]$ for all $l \in \{l_1, \dots, l_n\} \setminus L'$. Hence, $X \not\models \big((\bigwedge_{l \in L'} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L'} \tau[l])\big)$ and

$$X \not\models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, |L|<j \text{ or } k<|L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big),$$

so that $\boldsymbol{F}j\{l_1, \dots, l_{k+1}, \dots, l_n\}k \in \mathbf{A}$.

($FU\!\downarrow$) If $j\{l_1, \dots, l_k, l_{k+1}, \dots, l_n\}k \in \mathit{card}(\Pi)$ such that $\{\boldsymbol{T}j\{l_1, \dots, l_k, l_{k+1}, \dots, l_n\}k, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_k\} \subseteq \mathbf{A}'$, we have that

$$X \models \bigwedge_{L \subseteq \{l_1,\dots,l_n\}, |L|<j \text{ or } k<|L|} \big((\bigwedge_{l \in L} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\dots,l_n\} \setminus L} \tau[l])\big).$$

In particular, for every $k < i \leq n$ and $L_i = \{l \in \{l_1, \ldots, l_n\} \mid X \models \tau[l]\} \cup \{l_i\}$, we have that $\{l_1, \ldots, l_k\} \cup \{l_i\} \subseteq L_i$ and $k < |L_i|$, while $X \not\models \tau[l]$ for every $l \in \{l_1, \ldots, l_n\} \setminus L_i$. Hence, $X \models \big((\bigwedge_{l \in L_i} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\ldots,l_n\} \setminus L_i} \tau[l])\big)$ but $X \not\models \big((\bigwedge_{l \in L_i \setminus \{l_i\}} \tau[l]) \rightarrow (\bigvee_{l \in \{l_1,\ldots,l_n\} \setminus L_i} \tau[l])\big)$. That is, $X \not\models \tau[l_i]$ for every $k < i \leq n$, so that $\{\boldsymbol{f} l_{k+1}, \ldots, \boldsymbol{f} l_n\} \subseteq \mathbf{A}$.

($TD\!\uparrow$) If $\{l_1; \ldots; l_i; \ldots; l_n\} \in \mathit{disj}(\Pi)$ such that $\boldsymbol{t} l_i \in \mathbf{A}'$, we have that $X \models \tau[l_i]$. That is, $X \models (\tau[l_1] \vee \cdots \vee \tau[l_i] \vee \cdots \vee \tau[l_n])$, so that $\boldsymbol{T}\{l_1; \ldots; l_i; \ldots; l_n\} \in \mathbf{A}$.

($TD\!\downarrow$) If $\{l_1; \ldots; l_n\} \in \mathit{disj}(\Pi)$ such that $\boldsymbol{F}\{l_1; \ldots; l_n\} \in \mathbf{A}'$, we have that $X \not\models (\tau[l_1] \vee \cdots \vee \tau[l_n])$. That is, $X \not\models \tau[l_1], \ldots, X \not\models \tau[l_n]$, so that $\{\boldsymbol{f} l_1, \ldots, \boldsymbol{f} l_n\} \subseteq \mathbf{A}$.

($FD\!\uparrow$) If $\{l_1; \ldots; l_n\} \in \mathit{disj}(\Pi)$ such that $\{\boldsymbol{f} l_1, \ldots, \boldsymbol{f} l_n\} \subseteq \mathbf{A}'$, we have that $X \not\models \tau[l_1], \ldots, X \not\models \tau[l_n]$. That is, $X \not\models (\tau[l_1] \vee \cdots \vee \tau[l_n])$, so that $\boldsymbol{F}\{l_1; \ldots; l_n\} \in \mathbf{A}$.

($FD\!\downarrow$) If $\{l_1; \ldots; l_{i-1}; l_i; l_{i+1}; \ldots; l_n\} \in \mathit{disj}(\Pi)$ such that $\{\boldsymbol{T}\{l_1; \ldots; l_{i-1}; l_i; l_{i+1}; \ldots; l_n\}, \boldsymbol{f} l_1, \ldots, \boldsymbol{f} l_{i-1}, \boldsymbol{f} l_{i+1}, \ldots, \boldsymbol{f} l_n\} \subseteq \mathbf{A}'$, we have that $X \not\models \tau[l_1], \ldots, X \not\models \tau[l_{i-1}], X \not\models \tau[l_{i+1}], \ldots, X \not\models \tau[l_n]$ but $X \models (\tau[l_1] \vee \cdots \vee \tau[l_{i-1}] \vee \tau[l_i] \vee \tau[l_{i+1}] \vee \cdots \vee \tau[l_n])$. That is, $X \models \tau[l_i]$, so that $\boldsymbol{t} l_i \in \mathbf{A}$.

We have thus shown that, in every branch $(\Pi, \mathbf{A}')$ such that $\mathbf{A}' \subseteq \mathbf{A}$, any entry deducible by some of the tableau rules *(h)–(v)* belongs to $\mathbf{A}$, so that $D_{\{(h)\text{–}(v)\}}(\Pi, \mathbf{A}') \subseteq \mathbf{A}$. □

A.2.3 *Proofs of Soundness and Completeness.* The following theorem characterizes the answer sets of a disjunctive program in terms of total assignments $\mathbf{A}$ such that the generic tableau rules in Figure 4 do not yield a contradiction and the entries in $\mathbf{A}$ match the valuations of propositional formulas associated with their variables.

THEOREM A.15. *Let Π be a disjunctive program and $X \subseteq \mathit{atom}(\Pi)$. Then, we have that X is an answer set of Π iff*

$$\begin{aligned}\mathbf{A} \;=\; & \{\boldsymbol{T}v \mid v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi), X \models \tau[v]\} \\ \cup\; & \{\boldsymbol{F}v \mid v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi), X \not\models \tau[v]\}\end{aligned}$$

is such that $\boldsymbol{t}\beta \notin \mathbf{A}$ or $\boldsymbol{f}\alpha \notin \mathbf{A}$ for every $(\alpha \leftarrow \beta) \in \Pi$ and $\mathit{sup}_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$ or $\mathbf{A}^{\boldsymbol{T}} \cap S = \emptyset$ for every $S \subseteq \mathit{atom}(\Pi)$.

PROOF. By Lemma A.13, for every literal $l = v$ or $l = \mathit{not}\ v$, where $v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \cup \mathit{disj}(\Pi)$, we have that $\boldsymbol{t} l \in \mathbf{A}$ iff $X \models \tau[l]$, and that $\boldsymbol{f} l \in \mathbf{A}$ iff $X \not\models \tau[l]$. Hence, we can treat such conditions as synonyms in the following consideration of the implications of the statement.

($\Rightarrow$) Assume that X is an answer set of Π. Then, for every $(\alpha \leftarrow \beta) \in \Pi$, we have that $X \models \big(\tau[\beta] \rightarrow \tau[\alpha]\big)$ if $\alpha \notin \mathit{card}(\Pi)$, and that $X \models \big(\tau[\beta] \rightarrow \big(\tau[\alpha] \wedge \bigwedge_{p \in \mathit{atom}(\alpha)}(p \vee \neg p)\big)\big)$ if $\alpha \in \mathit{card}(\Pi)$. This implies that $X \not\models \tau[\beta]$ or $X \models \tau[\alpha]$, from which we conclude that $\boldsymbol{t}\beta \notin \mathbf{A}$ or $\boldsymbol{f}\alpha \notin \mathbf{A}$. Furthermore, for any $S \subseteq \mathit{atom}(\Pi)$ such that $\mathbf{A}^{\boldsymbol{T}} \cap S = X \cap S \neq \emptyset$, we have that $Y = X \setminus S \subset X$ is not a model of $(\tau[\Pi])^X$. That is, $Y \not\models \phi^X$ for some $\phi \in \tau[\Pi]$, where $\phi = \big(\tau[\beta] \rightarrow \tau[\alpha]\big)$ if $\alpha \notin \mathit{card}(\Pi)$ or $\phi = \big(\tau[\beta] \rightarrow \big(\tau[\alpha] \wedge \bigwedge_{p \in \mathit{atom}(\alpha)}(p \vee \neg p)\big)\big)$ if $\alpha \in \mathit{card}(\Pi)$ for some $(\alpha \leftarrow \beta) \in \Pi$. In view of $X \models \phi$ but $Y \not\models \phi^X$, we conclude that $\phi^X \neq \bot$, $Y \models (\tau[\beta])^X$, $X \models \tau[\beta]$, and $X \models \tau[\alpha]$. Furthermore, from $X \models \tau[\beta]$, we immediately obtain $\boldsymbol{f}\beta \notin \mathbf{A}$.

Given $Y \models (\tau[\beta])^X$, we first show that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta, S)$ holds. The following cases are possible:

(1) $\beta = \mathit{not}\ v$ for some $v \in \mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi)$, so that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta, S)$ holds.

(2) $\beta \in Y = X \setminus S$, so that $\beta \in \mathit{atom}(\Pi) \setminus S$ and $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta, S)$ hold.

(3) $\beta = j\{l_1,\dots,l_n\}k \in \mathit{card}(\Pi)$ and

$$(\tau[\beta])^X \;=\; \Big(\bigwedge\nolimits_{L\subseteq\{l_1,\dots,l_n\},|L|<j \text{ or } k<|L|}\big((\bigwedge\nolimits_{l\in L}\tau[l]) \rightarrow (\bigvee\nolimits_{l\in\{l_1,\dots,l_n\}\setminus L}\tau[l])\big)\Big)^X.$$

Since $Y \models (\tau[\beta])^X$, for any $L \subseteq \{l_1,\dots,l_n\}$ such that $|L| < j$, we have that $\{l \in L \mid \boldsymbol{f}l \in \mathbf{A}\} \neq \emptyset$, $L \cap S \neq \emptyset$, or $\{l \in \{l_1,\dots,l_n\} \setminus L \mid \boldsymbol{f}l \notin \mathbf{A}\} \not\subseteq S$. However, regarding $L' = \{l \in \{l_1,\dots,l_n\} \setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\}$, it holds that $\{l \in L' \mid \boldsymbol{f}l \in \mathbf{A}\} = \emptyset$, $L' \cap S = \emptyset$, and $\{l \in \{l_1,\dots,l_n\} \setminus L' \mid \boldsymbol{f}l \notin \mathbf{A}\} \subseteq S$. It follows that $|L'| \geq j$, so that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta,S)$ holds.

(4) $\beta = \{l_1,\dots,l_n\} \in \mathit{conj}(\Pi)$ and $(\tau[\beta])^X = (\bigwedge_{l\in\{l_1,\dots,l_n\}}\tau[l])^X = \bigwedge_{l\in\{l_1,\dots,l_n\}}(\tau[l])^X$. Since $Y \models (\tau[\beta])^X$, we conclude that $Y \models (\tau[l])^X$ for every $l \in \{l_1,\dots,l_n\}$. Given this, one of the first three cases applies to each $l \in \{l_1,\dots,l_n\}$, from which we conclude that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(l,S)$ holds, so that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta,S)$ holds as well.

We have thus shown that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta,S)$ holds.

We now turn to proving that $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$ holds. For this, note that, if $\alpha \notin \mathit{card}(\Pi)$, $X \models \tau[\alpha]$ but $Y \not\models (\tau[\alpha])^X$ yield $\alpha \in \mathit{atom}(\Pi) \cup \mathit{disj}(\Pi)$. Hence, the following cases are possible:

(1) $\alpha \in S$, so that $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$ holds.

(2) $\alpha = \{l_1;\dots;l_n\} \in \mathit{disj}(\Pi)$ and $\emptyset \neq \{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\} \subseteq S$. That is, $\{l_1,\dots,l_n\} \cap S \neq \emptyset$ and $\{l \in \{l_1,\dots,l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\} = \emptyset$, so that $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$ holds.

(3) $\alpha = j\{l_1,\dots,l_n\}k \in \mathit{card}(\Pi)$ and $(\{l_1,\dots,l_n\} \cap X) \cap S \neq \emptyset$ because $X \models \tau[\alpha]$ but $Y \not\models \big(\tau[\alpha] \wedge \bigwedge_{p\in \mathit{atom}(\alpha)}(p \vee \neg p)\big)^X$.[3] Furthermore, $X \models \tau[\alpha]$ implies $|\{l \in \{l_1,\dots,l_n\} \mid \boldsymbol{t}l \in \mathbf{A}\}| \leq k$. Along with $(\{l_1,\dots,l_n\} \cap X) \cap S \neq \emptyset$, that is, $\{l \in \{l_1,\dots,l_n\} \cap S \mid \boldsymbol{t}l \in \mathbf{A}\} \neq \emptyset$, we conclude that $|\{l \in \{l_1,\dots,l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\}| < k$, so that $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$ holds.

We have thus shown that $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$ holds. Along with the previous observations that $\boldsymbol{f}\beta \notin \mathbf{A}$ and that $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta,S)$ holds, we conclude that $(\alpha \leftarrow \beta) \in \mathit{sup}_{\mathbf{A}}(\Pi,S,S)$, so that $\mathit{sup}_{\mathbf{A}}(\Pi,S,S) \neq \emptyset$. Since the choice of $S \subseteq \mathit{atom}(\Pi)$ such that $\mathbf{A}^{\boldsymbol{T}} \cap S \neq \emptyset$ was arbitrary, this establishes that $\mathit{sup}_{\mathbf{A}}(\Pi,S,S) \neq \emptyset$ or $\mathbf{A}^{\boldsymbol{T}} \cap S = \emptyset$ for every $S \subseteq \mathit{atom}(\Pi)$.

($\Leftarrow$) Assume that X is not an answer set of Π. Then, there is either some $(\alpha \leftarrow \beta) \in \Pi$ such that $X \models \tau[\beta]$ and $X \not\models \tau[\alpha]$ or some $Y \subset X$ such that $Y \models (\tau[\Pi])^X$. In the former case, we have that $\boldsymbol{t}\beta \in \mathbf{A}$ and $\boldsymbol{f}\alpha \in \mathbf{A}$ for some $(\alpha \leftarrow \beta) \in \Pi$. In the latter case, let $S = X \setminus Y$. Then, it holds that $\emptyset \neq \mathbf{A}^{\boldsymbol{T}} \cap S = S$. For the sake of contradiction, assume that $\mathit{sup}_{\mathbf{A}}(\Pi,S,S) \neq \emptyset$, that is, $(\alpha \leftarrow \beta) \in \Pi$ such that $\boldsymbol{f}\beta \notin \mathbf{A}$, $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$, and $\overrightarrow{\mathit{sup}}_{\mathbf{A}}(\beta,S)$ hold.

In view of $\overleftarrow{\mathit{sup}}_{\mathbf{A}}(\alpha,S)$, the following cases are possible:

(1) $\alpha \in S$, $(\tau[\alpha])^X = \alpha$, and so

$$Y \;\not\models\; (\tau[\alpha])^X.$$

(2) $\alpha = \{l_1;\dots;l_n\} \in \mathit{disj}(\Pi)$, $\{l \in \{l_1,\dots,l_n\} \setminus S \mid \boldsymbol{t}l \in \mathbf{A}\} = \emptyset$, $(\tau[\alpha])^X \equiv \bigvee_{l\in\{l_1,\dots,l_n\}\cap S}\tau[l] = \bigvee_{p\in\{l_1,\dots,l_n\}\cap S}p$, and so

$$Y \;\not\models\; (\tau[\alpha])^X.$$

(3) $\alpha = j\{l_1,\dots,l_n\}k \in \mathit{card}(\Pi)$, $\{l_1,\dots,l_n\} \cap S = \mathit{atom}(\alpha) \cap S \neq \emptyset$, and so

$$Y \;\not\models\; \big(\tau[\alpha] \wedge \bigwedge\nolimits_{p\in \mathit{atom}(\alpha)}(p \vee \neg p)\big)^X.$$

[3] Note that all atoms occurring in $\big(\tau[\alpha] \wedge \bigwedge_{p\in \mathit{atom}(\alpha)}(p \vee \neg p)\big)^X$ belong to $\{l_1,\dots,l_n\} \cap X$.

We have thus shown that $Y \not\models (\tau[\alpha])^X$ if $\alpha \notin card(\Pi)$, and that $Y \not\models \big(\tau[\alpha] \wedge \bigwedge_{p\in atom(\alpha)}(p \vee \neg p)\big)^X$ if $\alpha \in card(\Pi)$.

We now turn to β, for which $\boldsymbol{f}\beta \notin \mathbf{A}$ implies $\boldsymbol{t}\beta \in \mathbf{A}$, that is, $X \models \tau[\beta]$. Furthermore, we have that $\overrightarrow{sup}_{\mathbf{A}}(\beta, S)$ holds, and the following cases are possible:

(1) $\beta = not\ v$ for some $v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi)$, $(\tau[\beta])^X = \neg\bot$, and so
$$Y \models (\tau[\beta])^X.$$

(2) $\beta \in atom(\Pi) \setminus S$, $(\tau[\beta])^X = \beta \in Y$, and so
$$Y \models (\tau[\beta])^X.$$

(3) $\beta = j\{l_1, \dots, l_n\}k \in card(\Pi)$ and
$$(\tau[\beta])^X = \Big(\bigwedge_{L\subseteq\{l_1,\dots,l_n\},|L|<j \text{ or } k<|L|}\big((\bigwedge_{l\in L}\tau[l]) \rightarrow (\bigvee_{l\in\{l_1,\dots,l_n\}\setminus L}\tau[l])\big)\Big)^X.$$
For any $L \subseteq \{l_1, \dots, l_n\}$ such that $k < |L|$, $X \models \tau[\beta]$ implies $(\bigwedge_{l\in L}\tau[l])^X = \bot$, so that $Y \not\models (\bigwedge_{l\in L}\tau[l])^X$. Furthermore, since $\overrightarrow{sup}_{\mathbf{A}}(\beta, S)$ holds, we have that $|\{l \in \{l_1, \dots, l_n\}\setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\}| \geq j$. Hence, for any $L \subseteq \{l_1, \dots, l_n\}$ such that $|L| < j$, it holds that $\{l \in \{l_1, \dots, l_n\}\setminus S \mid \boldsymbol{f}l \notin \mathbf{A}\} = \{l \in \{l_1, \dots, l_n\}\setminus S \mid \boldsymbol{t}l \in \mathbf{A}\} \not\subseteq L$ and $\{l \in \{l_1, \dots, l_n\}\setminus L \mid \boldsymbol{t}l \in \mathbf{A}\} \not\subseteq S$, so that $Y \models (\bigvee_{l\in\{l_1,\dots,l_n\}\setminus L}\tau[l])^X$. Combining the cases for $|L| < j$ and $k < |L|$ yields that
$$Y \models (\tau[\beta])^X.$$

(4) $\beta = \{l_1, \dots, l_n\} \in conj(\Pi)$ and $(\tau[\beta])^X = (\bigwedge_{l\in\{l_1,\dots,l_n\}}\tau[l])^X = \bigwedge_{l\in\{l_1,\dots,l_n\}}(\tau[l])^X$. For every $l \in \{l_1, \dots, l_n\}$, $X \models \tau[\beta]$ and $\overrightarrow{sup}_{\mathbf{A}}(\beta, S)$ imply $X \models \tau[l]$ and $\overrightarrow{sup}_{\mathbf{A}}(l, S)$. Given this, one of the first three cases applies to each $l \in \{l_1, \dots, l_n\}$, from which we conclude that $Y \models (\tau[l])^X$, and so
$$Y \models (\tau[\beta])^X.$$

We have thus shown that $Y \models (\tau[\beta])^X$. Along with $Y \not\models (\tau[\alpha])^X$ if $\alpha \notin card(\Pi)$ and $Y \not\models \big(\tau[\alpha] \wedge \bigwedge_{p\in atom(\alpha)}(p \vee \neg p)\big)^X$ if $\alpha \in card(\Pi)$, we further conclude that $Y \not\models \big(\tau[\beta] \rightarrow \tau[\alpha]\big)^X$ if $\alpha \notin card(\Pi)$ and $Y \not\models \big(\tau[\beta] \rightarrow \big(\tau[\alpha] \wedge \bigwedge_{p\in atom(\alpha)}(p \vee \neg p)\big)\big)^X$ if $\alpha \in card(\Pi)$. That is, $Y \not\models (\tau[\Pi])^X$, which is a contradiction to our initial assumption. This shows that $sup_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$ cannot be the case, so that $sup_{\mathbf{A}}(\Pi, S, S) = \emptyset$ must hold. In addition, $\emptyset \neq \mathbf{A}^{\boldsymbol{T}} \cap S = S$ holds by the choice of $S = X \setminus Y$. □

We are now ready to show Theorem 5.1, 5.2, 5.5, and 5.6, stating the soundness and completeness of tableau calculi for unary, conjunctive, cardinality, and disjunctive programs, respectively. Since disjunctive programs include unary, conjunctive, and cardinality programs, it is sufficient to prove Theorem 5.6.

THEOREM 5.6. *Let Π be a disjunctive program.*
Then, we have that the following holds for the tableau calculus consisting of the tableau rules (a)–(v):

(*1*) *Every incomplete tableau for Π and $\emptyset$ can be extended to a complete tableau for Π and $\emptyset$.*

(*2*) *Program Π has an answer set X iff every complete tableau for Π and $\emptyset$ has a unique non-contradictory branch $(\Pi, \mathbf{A})$ such that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) = X$.*

(*3*) *Program* Π *has no answer set iff every complete tableau for* Π *and* $\emptyset$ *is a refutation.*

PROOF. We separately consider the items of the statement:

(1) By applying *Cut*$[atom(\Pi)\cup conj(\Pi)\cup card(\Pi)\cup disj(\Pi)]$, an incomplete branch in a tableau for Π and $\emptyset$ can be extended to a subtableau such that, for every branch $(\Pi, \mathbf{A})$ in it, we have that $atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi) \subseteq \mathbf{A}^{\boldsymbol{T}} \cup \mathbf{A}^{\boldsymbol{F}}$. Furthermore, if $(\Pi, \mathbf{A})$ is not complete, then $D_{\{(a)–(f),(h)–(v)\}}(\Pi, \mathbf{A}) \not\subseteq \mathbf{A}$, so that the application of some of the tableau rules *(a)–(f)* in Figure 4 or *(h)–(v)* in Figure 5, 7, and 8 yields a contradictory and thus complete branch.

(2) By Theorem A.15, for every $X \subseteq atom(\Pi)$, we have that X is an answer set of Π iff the total assignment

$$\begin{aligned}\mathbf{A} \;=\; & \{\boldsymbol{T}v \mid v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi), X \models \tau[v]\}\\ \cup\; & \{\boldsymbol{F}v \mid v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi), X \not\models \tau[v]\}\end{aligned}$$

is such that $\boldsymbol{t}\beta \notin \mathbf{A}$ or $\boldsymbol{f}\alpha \notin \mathbf{A}$ for every $(\alpha \leftarrow \beta) \in \Pi$ and $sup_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$ or $\mathbf{A}^{\boldsymbol{T}} \cap S = \emptyset$ for every $S \subseteq atom(\Pi)$. Given this, we separately show the implications of the second item.
($\Rightarrow$) Assume that X is an answer set of Π. Then, Lemma A.10, A.12, and A.14 establish that $D_{\{(a)–(f),(h)–(v)\}}(\Pi, \mathbf{A}') \subseteq \mathbf{A}$ for every $\mathbf{A}' \subseteq \mathbf{A}$. Furthermore, for any application of *Cut*$[atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi)]$ on a branch $(\Pi, \mathbf{A}')$ such that $\mathbf{A}' \subseteq \mathbf{A}$, we have that the assignment in exactly one of the resulting branches is contained in $\mathbf{A}$. Along with $\emptyset \subseteq \mathbf{A}$, it follows that every complete tableau for Π and $\emptyset$ has a non-contradictory branch $(\Pi, \mathbf{A})$ such that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) = X$. By Lemma A.6, A.7, and A.8, we also have that $(\Pi, \mathbf{A})$ is the unique non-contradictory complete branch such that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) = X$.
($\Leftarrow$) Assume that $(\Pi, \mathbf{A})$ is a non-contradictory complete branch. Then, for every $v \in atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \cup disj(\Pi)$, Lemma A.6, A.7, and A.8 establish that $\boldsymbol{T}v \in \mathbf{A}$ iff $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) \models \tau[v]$. Furthermore, Lemma A.3 and A.4 show that $\boldsymbol{t}\beta \notin \mathbf{A}$ or $\boldsymbol{f}\alpha \notin \mathbf{A}$ for every $(\alpha \leftarrow \beta) \in \Pi$ and that $sup_{\mathbf{A}}(\Pi, S, S) \neq \emptyset$ or $\mathbf{A}^{\boldsymbol{T}} \cap S = \emptyset$ for every $S \subseteq atom(\Pi)$. By Theorem A.15, we conclude that $X = \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ is an answer set of Π.

(3) From the second item, if Π has an answer set, then every complete tableau for Π and $\emptyset$ has a non-contradictory branch; by the first item, there is some complete tableau for Π and $\emptyset$, so that some complete tableau for Π and $\emptyset$ is not a refutation. Conversely, if some complete tableau for Π and $\emptyset$ is not a refutation, it has a non-contradictory branch $(\Pi, \mathbf{A})$, and $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ is an answer set of Π, as shown in the proof of the second item.

We have thus shown that all items of the statement hold. □

A.2.4 *Proofs of Correspondences on Normal Programs.* We now show the correspondences stated in Proposition 5.3 and 5.4 between the basic tableau rules in Figure 1 and the (generic) tableau rules in Figure 4 and 5, respectively, on the common class of normal programs.

PROPOSITION 5.3. *Let* Π *be a normal program,* $\mathbf{A}$ *an assignment, and* F, G *any pair of a basic tableau rule* F *and a generic tableau rule* G *belonging to the same line in Table I.*
Then, we have that

(*1*) $D_{\{F\}}(\Pi, \mathbf{A}) = D_{\{G\}}(\Pi, \mathbf{A})$ *if* $F \notin \{BTA, WFJ[2^{atom(\Pi)}]\}$*;*

(*2*) $D_{\{BTA\}}(\Pi, \mathbf{A}) \supseteq D_{\{N\downarrow\}}(\Pi, \mathbf{A})$ *and, if* $D_{\{BTA\}}(\Pi, \mathbf{A}) \neq D_{\{N\downarrow\}}(\Pi, \mathbf{A})$*, then* $\mathbf{A} \cup D_{\{N\uparrow\}}(\Pi, \mathbf{A})$ *is contradictory;*

(*3*) $D_{\{\mathit{WFJ}[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \supseteq D_{\{U\downarrow\}}(\Pi,\mathbf{A})$ *and, if* $\boldsymbol{T}B \in D_{\{\mathit{WFJ}[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \setminus D_{\{U\downarrow\}}(\Pi,\mathbf{A})$*, then* $\mathbf{A} \cup D_{\{U\uparrow\}}(\Pi,\mathbf{A}\cup\{\boldsymbol{F}B\})$ *is contradictory.*

PROOF. The correspondences are obvious for the pairs *(c)*, *(a)*, *(d)*, *(b)*, *(a)*, *(h)*, *(b)*, *(i)*, *(e)*, *(j)*, and *(f)*, *(k)*. It remains to show the statement for the pairs *FFA*, $N\uparrow$, *BTA*, $N\downarrow$, *WFN*$[2^{atom(\Pi)}]$, $U\uparrow$, and *WFJ*$[2^{atom(\Pi)}]$, $U\downarrow$:

(*FFA*, $N\uparrow$) We have that $\boldsymbol{F}p \in D_{\{\mathit{FFA}\}}(\Pi,\mathbf{A})$ iff $p \in atom(\Pi)$ such that $body(p) \subseteq \mathbf{A}^{\boldsymbol{F}}$ iff $p \in atom(\Pi)$ such that $sup_{\mathbf{A}}(\Pi,\{p\},\emptyset) = \emptyset$ iff $\boldsymbol{F}p \in D_{\{N\uparrow\}}(\Pi,\mathbf{A})$.

(*BTA*, $N\downarrow$) If $\boldsymbol{T}B \in D_{\{N\downarrow\}}(\Pi,\mathbf{A})$, then $sup_{\mathbf{A}}(\Pi,\{p\},\emptyset) = \{p \leftarrow B\}$ for some $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, so that $\alpha \neq p$ or $\boldsymbol{F}\beta \in \mathbf{A}$ for every $(\alpha \leftarrow \beta) \in \Pi \setminus \{p \leftarrow B\}$. From this, we conclude that $body(p) \setminus \mathbf{A}^{\boldsymbol{F}} = \{B\}$, so that $\boldsymbol{T}B \in D_{\{\mathit{BTA}\}}(\Pi,\mathbf{A})$. Furthermore, if $\boldsymbol{T}B' \in D_{\{\mathit{BTA}\}}(\Pi,\mathbf{A}) \setminus D_{\{N\downarrow\}}(\Pi,\mathbf{A})$, then $body(p') \setminus \mathbf{A}^{\boldsymbol{F}} \subseteq \{B'\}$ for some $B' \in body(\Pi)$ and $p' \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, which implies that $sup_{\mathbf{A}}(\Pi,\{p'\},\emptyset) \subseteq \{p' \leftarrow B'\}$. However, $\boldsymbol{T}B' \notin D_{\{N\downarrow\}}(\Pi,\mathbf{A})$ yields that $(p' \leftarrow B') \notin sup_{\mathbf{A}}(\Pi,\{p'\},\emptyset)$. Hence, we have that $sup_{\mathbf{A}}(\Pi,\{p'\},\emptyset) = \emptyset$, and $\mathbf{A} \cup D_{\{N\uparrow\}}(\Pi,\mathbf{A})$ is contradictory because $p' \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$.

(*WFN*$[2^{atom(\Pi)}]$, $U\uparrow$) We have that $\boldsymbol{F}p \in D_{\{\mathit{WFN}[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A})$ iff $p \in S$ for some $S \subseteq atom(\Pi)$ such that $EB_{\Pi}(S) \subseteq \mathbf{A}^{\boldsymbol{F}}$ iff $p \in S$ for some $S \subseteq atom(\Pi)$ such that $sup_{\mathbf{A}}(\Pi,S,S) = \emptyset$ iff $\boldsymbol{F}p \in D_{\{U\uparrow\}}(\Pi,\mathbf{A})$.

(*WFJ*$[2^{atom(\Pi)}]$, $U\downarrow$) If $\boldsymbol{T}B \in D_{\{U\downarrow\}}(\Pi,\mathbf{A})$, then $sup_{\mathbf{A}}(\Pi,S,S) = \{p \leftarrow B\}$, where $p \in S$ for some $S \subseteq atom(\Pi)$ such that $\mathbf{A}^{\boldsymbol{T}} \cap S \neq \emptyset$. From this, we conclude that $EB_{\Pi}(S) \setminus \mathbf{A}^{\boldsymbol{F}} = \{B\}$, so that $\boldsymbol{T}B \in D_{\{\mathit{WFJ}[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A})$. Furthermore, if $\boldsymbol{T}B' \in D_{\{\mathit{WFJ}[2^{atom(\Pi)}]\}}(\Pi,\mathbf{A}) \setminus D_{\{U\downarrow\}}(\Pi,\mathbf{A})$, then $EB_{\Pi}(S') \setminus \mathbf{A}^{\boldsymbol{F}} \subseteq \{B'\}$ for some $B' \in body(\Pi)$ and $S' \subseteq atom(\Pi)$ such that $\mathbf{A}^{\boldsymbol{T}} \cap S' \neq \emptyset$, which implies that $sup_{\mathbf{A}}(\Pi,S',S') \subseteq \{p' \leftarrow B' \mid p' \in S'\}$. In view of Lemma A.10 and A.11, we have that $sup_{\mathbf{A}\cup\{\boldsymbol{F}B'\}}(\Pi,S',S') = \emptyset$, and $\mathbf{A} \cup D_{\{U\uparrow\}}(\Pi,\mathbf{A} \cup \{\boldsymbol{F}B'\})$ is contradictory because $\mathbf{A}^{\boldsymbol{T}} \cap S' \neq \emptyset$.

We have thus shown that the stated correspondences according to Table I hold. □

PROPOSITION 5.4. *Let* Π *be a normal program,* $\mathbf{A}$ *an assignment,* $\mathcal{T}$ *a tableau calculus containing any subset of the tableau rules in Figure 1 for* $\Omega = 2^{atom(\Pi)}$*, and* $\mathcal{T}'$ *the generic image of* $\mathcal{T}$*.*

If FFA $\in \mathcal{T}$ *or BTA* $\notin \mathcal{T}$ *and if WFJ*$[\Omega] \in \mathcal{T}$ *implies that* $\{\mathit{FTB}, \mathit{FFB}, \mathit{WFN}[\Omega], \mathit{Cut}[\Gamma]\} \subseteq \mathcal{T}$ *for* $\Gamma \subseteq atom(\Pi) \cup body(\Pi)$ *such that* $atom(\Pi) \subseteq \Gamma$ *or* $body(\Pi) \subseteq \Gamma$*, then we have that the following holds:*

(*1*) *For every complete tableau of* $\mathcal{T}$ *for* Π *and* $\mathbf{A}$ *with* n *branches, there is a complete tableau of* $\mathcal{T}'$ *for* Π *and* $\mathbf{A}$ *with the same non-contradictory branches and at most* $(\max\{|atom(\Pi)|, |body(\Pi)|\} + 1) * n$ *branches overall.*

(*2*) *Every (complete) tableau of* $\mathcal{T}'$ *for* Π *and* $\mathbf{A}$ *is a (complete) tableau of* $\mathcal{T}$ *for* Π *and* $\mathbf{A}$.

PROOF. Assume that *FFA* $\in \mathcal{T}$ or *BTA* $\notin \mathcal{T}$ and that *WFJ*$[\Omega] \in \mathcal{T}$ implies that $\{\mathit{FTB}, \mathit{FFB}, \mathit{WFN}[\Omega], \mathit{Cut}[\Gamma]\} \subseteq \mathcal{T}$ for $\Gamma \subseteq atom(\Pi) \cup body(\Pi)$ such that $atom(\Pi) \subseteq \Gamma$ or $body(\Pi) \subseteq \Gamma$. By Proposition 5.3, we immediately conclude that every (complete) tableau of $\mathcal{T}'$ for Π and $\mathbf{A}$ is a (complete) tableau of $\mathcal{T}$ for Π and $\mathbf{A}$ as well. Furthermore, in view of the first two items in the statement of Proposition 5.3, we have that any application of a tableau rule in $\mathcal{T}$ other than *WFJ*$[\Omega]$ on a branch $(\Pi, \mathbf{A}')$ extending $(\Pi, \mathbf{A})$ leads to the same result, in terms of deduced entries or a contradiction, respectively, by applying a corresponding tableau rule in $\mathcal{T}'$. Hence, it is sufficient to show that, if there is some $\boldsymbol{T}B \in D_{\{\mathit{WFJ}[\Omega]\}}(\Pi, \mathbf{A}') \setminus (\mathbf{A}' \cup D_{\{TC\uparrow, U\downarrow\}}(\Pi, \mathbf{A}'))$, there is a corresponding subtableau of $\mathcal{T}'$ that introduces at most $|(B^+ \cup B^-) \setminus ((\mathbf{A}')^{\boldsymbol{T}} \cup (\mathbf{A}')^{\boldsymbol{F}})|$ contradictory branches, while a single remaining branch includes $\boldsymbol{T}B$ (and possibly further entries belonging to any non-contradictory branch extending $(\Pi, \mathbf{A}' \cup \{\boldsymbol{T}B\})$ in a complete tableau of $\mathcal{T}$ for Π and $\mathbf{A}$). To this end, assume that $\boldsymbol{T}B \in D_{\{\mathit{WFJ}[\Omega]\}}(\Pi, \mathbf{A}') \setminus (\mathbf{A}' \cup D_{\{TC\uparrow, U\downarrow\}}(\Pi, \mathbf{A}'))$. Then, $EB_{\Pi}(S) \setminus (\mathbf{A}')^{\boldsymbol{F}} \subseteq \{B\}$ for some $S \subseteq \mathit{atom}(\Pi)$ such that $(\mathbf{A}')^{\boldsymbol{T}} \cap S \neq \emptyset$, $\mathit{sup}_{\mathbf{A}'}(\Pi, S, S) \subseteq \{p \leftarrow B \mid p \in S\}$, and $|\mathit{sup}_{\mathbf{A}'}(\Pi, S, S)| \neq 1$. Furthermore, one of the following cases applies:

(1) If $\mathit{sup}_{\mathbf{A}'}(\Pi, S, S) = \emptyset$, we have that $\boldsymbol{F}p \in D_{\{U\uparrow\}}(\Pi, \mathbf{A}')$ for every $p \in S$. Given that $(\mathbf{A}')^{\boldsymbol{T}} \cap S \neq \emptyset$, we conclude that $(\Pi, \mathbf{A}')$ can be extended to a contradictory branch by an application of $U\uparrow$.

(2) If $\mathit{sup}_{\mathbf{A}'}(\Pi, S, S) \neq \emptyset$, in view of Lemma A.10 and A.11, we have that $\mathit{sup}_{\mathbf{A}' \cup \{\boldsymbol{F}B\}}(\Pi, S, S) = \emptyset$, so that an application of $U\uparrow$ is sufficient to contradict any extension of $(\Pi, \mathbf{A}')$ including $\boldsymbol{F}B$. In particular, if $\boldsymbol{F}B \in D_{\{FC\uparrow\}}(\Pi, \mathbf{A}')$, we can extend $(\Pi, \mathbf{A}')$ to a contradictory branch without cutting. Otherwise, if $\mathit{Cut}[\Gamma] \in \mathcal{T}$ such that $\mathit{body}(\Pi) \subseteq \Gamma$, we can cut on B, contradict the branch for $\boldsymbol{F}B$ by applying $U\uparrow$, and proceed with the branch $(\Pi, \mathbf{A}' \cup \{\boldsymbol{T}B\})$, also obtained by applying $\mathit{WFJ}[\Omega]$. Alternatively, if $\mathit{Cut}[\Gamma] \in \mathcal{T}$ such that $\mathit{atom}(\Pi) \subseteq \Gamma$, we can successively cut on atoms in $(B^+ \cup B^-) \setminus ((\mathbf{A}')^{\boldsymbol{T}} \cup (\mathbf{A}')^{\boldsymbol{F}})$ and contradict a branch for $\boldsymbol{f}l$, where $l \in B$, by applying $FC\uparrow$ and $U\uparrow$. Provided that $B^+ \cap B^- = \emptyset$,[4] this strategy yields a single branch $(\Pi, \mathbf{A}' \cup \{\boldsymbol{t}l \mid l \in B\})$, which can be further extended to $(\Pi, \mathbf{A}' \cup \{\boldsymbol{t}l \mid l \in B\} \cup \{\boldsymbol{T}B\})$ by an application of $TC\uparrow$. Given that $\mathit{FFB} \in \mathcal{T}$, we also have that any non-contradictory branch extending $(\Pi, \mathbf{A}' \cup \{\boldsymbol{T}B\})$ in a complete tableau of $\mathcal{T}$ for Π and $\mathbf{A}$ contains $\boldsymbol{t}l$ for all $l \in B$.

We have thus shown that an entry $\boldsymbol{T}B \in D_{\{\mathit{WFJ}[\Omega]\}}(\Pi, \mathbf{A}') \setminus (\mathbf{A}' \cup D_{\{TC\uparrow, U\downarrow\}}(\Pi, \mathbf{A}'))$ can also be generated in the single (if any) non-contradictory branch in a subtableau of $\mathcal{T}'$ extending $(\Pi, \mathbf{A}')$ and admitting the same non-contradictory extensions as $(\Pi, \mathbf{A}' \cup \{\boldsymbol{T}B\})$ in a complete tableau of $\mathcal{T}$ for Π and $\mathbf{A}$, while introducing at most $\max\{|\mathit{atom}(\Pi)|, |\mathit{body}(\Pi)|\}$ contradictory branches overall along each branch in a complete tableau of $\mathcal{T}$ for Π and $\mathbf{A}$. $\square$

The previous results allow us to derive Theorem 3.1 as a consequence of Theorem 5.2 (i.e., Theorem 5.6 restricted to the class of conjunctive programs).

THEOREM 3.1. *Let Π be a normal program.*
Then, we have that the following holds for tableau calculi $\mathcal{T}_{\mathit{smodels}}$, $\mathcal{T}_{\mathit{nomore}}$, and $\mathcal{T}_{\mathit{nomore}^{++}}$:

(*1*) *Every incomplete tableau for Π and $\emptyset$ can be extended to a complete tableau for Π and $\emptyset$.*

(*2*) *Program Π has an answer set X iff every complete tableau for Π and $\emptyset$ has a unique non-contradictory branch $(\Pi, \mathbf{A})$ such that $\mathbf{A}^{\boldsymbol{T}} \cap \mathit{atom}(\Pi) = X$.*

(*3*) *Program Π has no answer set iff every complete tableau for Π and $\emptyset$ is a refutation.*

PROOF. By Proposition 5.3, $\mathcal{T}_{\mathit{smodels}}$, $\mathcal{T}_{\mathit{nomore}}$, and $\mathcal{T}_{\mathit{nomore}^{++}}$ admit the same non-contradictory complete branches as the tableau calculus consisting of the tableau rules *(a)–(k)* in Figure 4 and 5; in particular, if $\boldsymbol{T}B \in D_{\{U\downarrow\}}(\Pi, \mathbf{A})$ for a branch $(\Pi, \mathbf{A})$, we have that $\boldsymbol{T}B \in$

[4] If $B^+ \cap B^- \neq \emptyset$, all branches in a subtableau of $\mathcal{T}'$ obtained by successively cutting on atoms in $(B^+ \cup B^-) \setminus ((\mathbf{A}')^{\boldsymbol{T}} \cup (\mathbf{A}')^{\boldsymbol{F}})$ and contradicting branches for $\boldsymbol{f}l$, where $l \in B$, are contradictory. Given that $\mathit{FFB} \in \mathcal{T}$, any branch extending $(\Pi, \mathbf{A}' \cup \{\boldsymbol{T}B\})$ in a complete tableau of $\mathcal{T}$ for Π and $\mathbf{A}$ is contradictory too.

$D_{\{WFJ[2^{atom(\Pi)}]\}}(\Pi, \mathbf{A})$, so that $\mathbf{A} \cup D_{\{WFN[2^{atom(\Pi)}]\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{F}B\})$ is contradictory (cf. Figure 1).[5] Hence, from Theorem 5.2 and the fact that answer sets of $\tau[\Pi]$ match answer sets (as introduced in Section 2) of Π (cf. [Lifschitz 2008]), the result follows immediately for $\mathcal{T}_{nomore{+}{+}}$. Moreover, for $\mathcal{T}_{smodels}$ and $\mathcal{T}_{nomore}$, using $Cut[atom(\Pi)]$ and $Cut[body(\Pi)]$, respectively, in place of $Cut[atom(\Pi) \cup body(\Pi)]$, it is sufficient to show that the first item of the statement holds. Regarding $\mathcal{T}_{smodels}$, note that, for every $B \in body(\Pi)$, either $\boldsymbol{T}B \in D_{\{FTB\}}(\Pi, \mathbf{A})$ or $\boldsymbol{F}B \in D_{\{FFB\}}(\Pi, \mathbf{A})$ for any non-contradictory assignment $\mathbf{A}$ such that $atom(\Pi) \subseteq \mathbf{A}^{\boldsymbol{T}} \cup \mathbf{A}^{\boldsymbol{F}}$, so that the first item of the statement holds for $\mathcal{T}_{smodels}$. Regarding $\mathcal{T}_{nomore}$, for every $p \in atom(\Pi)$, either $\boldsymbol{T}p \in D_{\{FTA\}}(\Pi, \mathbf{A})$ or $\boldsymbol{F}p \in D_{\{FFA\}}(\Pi, \mathbf{A})$ for any non-contradictory assignment $\mathbf{A}$ such that $body(\Pi) \subseteq \mathbf{A}^{\boldsymbol{T}} \cup \mathbf{A}^{\boldsymbol{F}}$, so that the first item of the statement holds for $\mathcal{T}_{nomore}$ as well. □

Along with Lemma A.2 on different variants of tableau rule *WFN*, Theorem 3.1 yields Theorem 4.6.

THEOREM 4.6. *Let Π be a normal program.*
Then, we have that the following holds for tableau calculus $\mathcal{T}_{comp} \cup \{WFN[loop(\Pi)]\}$:

(1) *Every incomplete tableau for Π and $\emptyset$ can be extended to a complete tableau for Π and $\emptyset$.*
(2) *Program Π has an answer set X iff every complete tableau for Π and $\emptyset$ has a unique non-contradictory branch $(\Pi, \mathbf{A})$ such that $\mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi) = X$.*
(3) *Program Π has no answer set iff every complete tableau for Π and $\emptyset$ is a refutation.*

PROOF. By Lemma A.2, $\mathcal{T}_{nomore{+}{+}}$ and $\mathcal{T}_{comp} \cup \{WFN[loop(\Pi)]\}$ admit the same non-contradictory complete branches. Hence, the result follows immediately from Theorem 3.1. □

We have thus proven the formal results presented in Section 5, and also demonstrated Theorem 3.1 and 4.6.

A.3 Proofs of Results from Section 6

We below consider minimal refutations of tableau calculi $\mathcal{T}_{nomore}$, $\mathcal{T}_{smodels}$, $\mathcal{T}_{card}$, and $\mathcal{T}_{conj}$ for particular families of logic programs, thus showing exponential separations between $\mathcal{T}_{nomore}$ and $\mathcal{T}_{smodels}$ as well as between $\mathcal{T}_{card}$ and $\mathcal{T}_{conj}$.

PROPOSITION 6.1. *There is an infinite family $\{\Pi^n\}$ of normal programs such that*

(1) *the size of minimal refutations of $\mathcal{T}_{nomore}$ for Π^n is asymptotically linear in n;*
(2) *the size of minimal refutations of $\mathcal{T}_{smodels}$ for Π^n is asymptotically exponential in n.*

PROOF. Consider the following family $\{\Pi^n_a \cup \Pi^n_c\}$ of normal programs for $n \geq 1$:

$$\Pi^n_a \cup \Pi^n_c \;=\; \{x \leftarrow not\ x\} \cup \textstyle\bigcup_{1 \leq i \leq n} \{x \leftarrow a_i, b_i;\ a_i \leftarrow not\ b_i;\ b_i \leftarrow not\ a_i\}$$

The domain of assignments $\mathbf{A}$ is $dom(\mathbf{A}) = \{x, \{not\ x\}\} \cup \bigcup_{1 \leq i \leq n}\{a_i, b_i, \{not\ a_i\}, \{not\ b_i\}, \{a_i, b_i\}\}$, and we investigate minimal refutations of $\mathcal{T}_{nomore}$ and $\mathcal{T}_{smodels}$ for members of $\{\Pi^n_a \cup \Pi^n_c\}$.

An optimal strategy to construct a refutation of $\mathcal{T}_{nomore}$ for $\Pi^n_a \cup \Pi^n_c$ (cf. Figure 11) is as follows:

[5] Every non-contradictory complete branch has exactly one occurrence in any complete tableau of the tableau calculus containing *(a)–(k)*, $\mathcal{T}_{smodels}$, $\mathcal{T}_{nomore}$, or $\mathcal{T}_{nomore{+}{+}}$ for Π and $\emptyset$. For the former, this is established by Lemma A.3, A.4, A.7, A.10, A.12, and A.14 (along with the fact that *Cut* applications preserve non-contradictory complete branches). For $\mathcal{T}_{smodels}$, $\mathcal{T}_{nomore}$, and $\mathcal{T}_{nomore{+}{+}}$, it follows from the observation that $D_{\{(a)\text{–}(h), WFN[2^{atom(\Pi)}]\}}(\Pi, \mathbf{A}') \subseteq D_{\{(a)\text{–}(h), WFN[2^{atom(\Pi)}]\}}(\Pi, \mathbf{A})$ for every assignment $\mathbf{A}$ and every $\mathbf{A}' \subseteq \mathbf{A}$.

(1) Cut on $\{not\ x\}$, complete the branch for $\boldsymbol{T}\{not\ x\}$, using the deterministic tableau rules *BTB* and *FTA*, and deduce $\boldsymbol{T}x$ in the branch for $\boldsymbol{F}\{not\ x\}$, using the deterministic tableau rule *BFB*.

(2) Complete the branch containing $\boldsymbol{T}x$ (and $\boldsymbol{F}\{not\ x\}$), but none of $\boldsymbol{T}\{a_i, b_i\}$ for $1 \leq i \leq n$, if it contains $n-1$ entries of the form $\boldsymbol{F}\{a_i, b_i\}$, using the deterministic tableau rules *BTA* and *BTB*. Otherwise, if there are less than $n-1$ entries of the form $\boldsymbol{F}\{a_i, b_i\}$ in the branch, cut on some unassigned $\{a_i, b_i\}$ for $1 \leq i \leq n$ and complete the branch for $\boldsymbol{T}\{a_i, b_i\}$, using the deterministic tableau rules *BTA* and *BTB*.

In a nutshell, a refutation constructed in this way makes use of immediate contradictions obtained when assigning any of the bodies $\{a_i, b_i\}$ to true, so that each application of *Cut*$[body(\Pi_a^n \cup \Pi_c^n)]$ yields one branch that is completed without cutting any further. Hence, such a refutation of $\mathcal{T}_{nomore}$ for $\Pi_a^n \cup \Pi_c^n$ is of size linear in n.

An optimal strategy to construct a refutation of $\mathcal{T}_{smodels}$ for $\Pi_a^n \cup \Pi_c^n$ (cf. Figure 10) is as follows:

(1) Cut on x, complete the branch for $\boldsymbol{F}x$, using the deterministic tableau rules *FTB* and *BFA*, and deduce $\boldsymbol{F}\{not\ x\}$ in the branch for $\boldsymbol{T}x$, using the deterministic tableau rule *FFB*.

(2) Complete any of the branches containing $\boldsymbol{T}x$ (and $\boldsymbol{F}\{not\ x\}$) if the branch contains $n-1$ entries of the form $\boldsymbol{F}\{a_i, b_i\}$ for $1 \leq i \leq n$, using the deterministic tableau rules *BTA* and *BTB*. Otherwise, if there are less than $n-1$ entries of the form $\boldsymbol{F}\{a_i, b_i\}$ in a branch, cut on some unassigned a_i for $1 \leq i \leq n$ and deduce $\boldsymbol{F}\{a_i, b_i\}$ in the branch for $\boldsymbol{T}a_i$ as well as in the branch for $\boldsymbol{F}a_i$, using the deterministic tableau rules *BTA*, *BTB*, and *FFB*.

As the second step shows, cuts on atoms a_i (or b_i) for $1 \leq i \leq n$ yield symmetric alternatives, since $\boldsymbol{F}\{a_i, b_i\}$ is deduced in each of the resulting branches. That is, except for the initial cut on x, applications of *Cut*$[atom(\Pi_a^n \cup \Pi_c^n)]$ do not admit immediate contradictions and must thus be cascaded to form a perfect binary tree. Hence, a minimal refutation of $\mathcal{T}_{smodels}$ for $\Pi_a^n \cup \Pi_c^n$ is of size exponential in n.

We have thus shown that the asymptotic sizes of minimal refutations of $\mathcal{T}_{nomore}$ and $\mathcal{T}_{smodels}$ for $\Pi_a^n \cup \Pi_c^n$ are $O(n)$ and $O(2^n)$, respectively. Hence, $\mathcal{T}_{nomore}$ is not polynomially simulated by $\mathcal{T}_{smodels}$. $\square$

PROPOSITION 6.2. *There is an infinite family* $\{\Pi^n\}$ *of normal programs such that*

(*1*) *the size of minimal refutations of* $\mathcal{T}_{smodels}$ *for* Π^n *is asymptotically linear in n;*

(*2*) *the size of minimal refutations of* $\mathcal{T}_{nomore}$ *for* Π^n *is asymptotically exponential in* n.

PROOF. Consider the following family $\{\Pi_b^n \cup \Pi_c^n\}$ of normal programs for $n \geq 1$:

$$\begin{aligned}
\Pi_b^n \cup \Pi_c^n \;=\;& \{y \leftarrow c_1, \dots, c_n, not\ y\} \\
\cup\;& \textstyle\bigcup_{1 \leq i \leq n} \{c_i \leftarrow not\ a_i;\ c_i \leftarrow not\ b_i;\ a_i \leftarrow not\ b_i;\ b_i \leftarrow not\ a_i\}
\end{aligned}$$

The domain of assignments $\mathbf{A}$ is $dom(\mathbf{A}) = \{y, \{c_1, \dots, c_n, not\ y\}\} \cup \bigcup_{1 \leq i \leq n} \{a_i, b_i, c_i, \{not\ a_i\}, \{not\ b_i\}\}$, and we investigate minimal refutations of $\mathcal{T}_{smodels}$ and $\mathcal{T}_{nomore}$ for members of $\{\Pi_b^n \cup \Pi_c^n\}$.

An optimal strategy to construct a refutation of $\mathcal{T}_{smodels}$ for $\Pi_b^n \cup \Pi_c^n$ (cf. Figure 13) is as follows:

(1) Cut on y, complete the branch for $\boldsymbol{T}y$, using the deterministic tableau rules *BTA* and *FFB*, and deduce $\boldsymbol{F}\{c_1, \dots, c_n, not\ y\}$ in the branch for $\boldsymbol{F}y$, using the deterministic tableau rule *BFA*.

(2) Complete the branch containing $\boldsymbol{F}\{c_1, \ldots, c_n, not\ y\}$ (and $\boldsymbol{F}y$), but none of $\boldsymbol{F}c_i$ for $1 \leq i \leq n$, if it contains $n-1$ entries of the form $\boldsymbol{T}c_i$, using the deterministic tableau rules *BFB*, *BFA*, and *FFA*. Otherwise, if there are less than $n-1$ entries of the form $\boldsymbol{T}c_i$ in the branch, cut on some unassigned c_i for $1 \leq i \leq n$ and complete the branch for $\boldsymbol{F}c_i$, using the deterministic tableau rules *BFB*, *BFA*, and *FFA*.

In a nutshell, a refutation constructed in this way makes use of immediate contradictions obtained when assigning any of the atoms c_i to false, so that each application of *Cut*$[atom(\Pi_b^n \cup \Pi_c^n)]$ yields one branch that is completed without cutting any further. Hence, such a refutation of $\mathcal{T}_{smodels}$ for $\Pi_b^n \cup \Pi_c^n$ is of size linear in n.

An optimal strategy to construct a refutation of $\mathcal{T}_{nomore}$ for $\Pi_b^n \cup \Pi_c^n$ (cf. Figure 12) is as follows:

(1) Cut on $\{c_1, \ldots, c_n, not\ y\}$, complete the branch for $\boldsymbol{T}\{c_1, \ldots, c_n, not\ y\}$, using the deterministic tableau rules *FTA* and *BTB*, and deduce $\boldsymbol{F}y$ in the branch for $\boldsymbol{F}\{c_1, \ldots, c_n, not\ y\}$, using the deterministic tableau rule *FFA*.

(2) Complete any of the branches containing $\boldsymbol{F}\{c_1, \ldots, c_n, not\ y\}$ (and $\boldsymbol{F}y$) if the branch contains $n-1$ entries of the form $\boldsymbol{T}c_i$ for $1 \leq i \leq n$, using the deterministic tableau rules *BFB*, *BFA*, and *FFA*. Otherwise, if there are less than $n-1$ entries of the form $\boldsymbol{T}c_i$ in a branch, cut on some unassigned $\{not\ a_i\}$ for $1 \leq i \leq n$ and deduce $\boldsymbol{T}c_i$ in the branch for $\boldsymbol{T}\{not\ a_i\}$ as well as in the branch for $\boldsymbol{F}\{not\ a_i\}$, using the deterministic tableau rules *FTA*, *BFB*, and *BTA*.

As the second step shows, cuts on bodies $\{not\ a_i\}$ (or $\{not\ b_i\}$) for $1 \leq i \leq n$ yield symmetric alternatives, since $\boldsymbol{T}c_i$ is deduced in each of the resulting branches. That is, except for the initial cut on $\{c_1, \ldots, c_n, not\ y\}$, applications of *Cut*$[body(\Pi_b^n \cup \Pi_c^n)]$ do not admit immediate contradictions and must thus be cascaded to form a perfect binary tree. Hence, a minimal refutation of $\mathcal{T}_{nomore}$ for $\Pi_b^n \cup \Pi_c^n$ is of size exponential in n.

We have thus shown that the asymptotic sizes of minimal refutations of $\mathcal{T}_{smodels}$ and $\mathcal{T}_{nomore}$ for $\Pi_b^n \cup \Pi_c^n$ are $O(n)$ and $O(2^n)$, respectively. Hence, $\mathcal{T}_{smodels}$ is not polynomially simulated by $\mathcal{T}_{nomore}$. □

COROLLARY 6.3. *Tableau calculi $\mathcal{T}_{smodels}$ and $\mathcal{T}_{nomore}$ are efficiency-incomparable.*

PROOF. This result follows immediately from Proposition 6.1 and 6.2, since they show that neither $\mathcal{T}_{nomore}$ is polynomially simulated by $\mathcal{T}_{smodels}$, nor vice versa. □

COROLLARY 6.4. *Tableau calculus $\mathcal{T}_{nomore^{++}}$ is exponentially stronger than both $\mathcal{T}_{smodels}$ and $\mathcal{T}_{nomore}$.*

PROOF. This result follows immediately from Corollary 6.3, since $\mathcal{T}_{nomore}$ and $\mathcal{T}_{smodels}$ are both polynomially simulated by $\mathcal{T}_{nomore^{++}}$ (any tableau of $\mathcal{T}_{nomore}$ or $\mathcal{T}_{smodels}$ is a tableau of $\mathcal{T}_{nomore^{++}}$ as well), while $\mathcal{T}_{nomore}$ and $\mathcal{T}_{smodels}$ are not polynomially simulated by one another. □

PROPOSITION 6.5. *Tableau calculus $\mathcal{T}_{card}$ is exponentially stronger than $\mathcal{T}_{conj}$.*

PROOF. Consider the following family $\{\Pi_c^n \cup \Pi_d^n\}$ of cardinality programs for $n \geq 1$:

$$\Pi_c^n \cup \Pi_d^n \;=\; \{z \leftarrow 1\{a_1, b_1\}2, \ldots, 1\{a_n, b_n\}2, not\ z\} \cup \textstyle\bigcup_{1 \leq i \leq n} \{a_i \leftarrow not\ b_i;\ b_i \leftarrow not\ a_i\}$$

The domain of assignments $\mathbf{A}$ is $dom(\mathbf{A}) \;=\; \{z, \{1\{a_1, b_1\}2, \ldots, 1\{a_n, b_n\}2, not\ \ z\}\} \cup$

$\bigcup_{1\leq i\leq n}\{a_i, b_i, 1\{a_i, b_i\}2\}$,[6] and we investigate minimal refutations of $\mathcal{T}_{card}$ and $\mathcal{T}_{conj}$ for members of $\{\Pi^n_c \cup \Pi^n_d\}$.

An optimal strategy to construct a refutation of $\mathcal{T}_{card}$ for $\Pi^n_c \cup \Pi^n_d$ is as follows:

(1) Cut on z, complete the branch for $\boldsymbol{T}z$, using the deterministic tableau rules $N\!\downarrow$ and $FC\!\uparrow$, and deduce $\boldsymbol{F}\{1\{a_1, b_1\}2, \dots, 1\{a_n, b_n\}2, \mathit{not}\ z\}$ in the branch for $\boldsymbol{F}z$, using the deterministic tableau rule $I\!\downarrow$.

(2) Complete the branch containing $\boldsymbol{F}\{1\{a_1, b_1\}2, \dots, 1\{a_n, b_n\}2, \mathit{not}\ z\}$ (and $\boldsymbol{F}z$), but none of $\boldsymbol{F}1\{a_i, b_i\}2$ for $1 \leq i \leq n$, if it contains $n-1$ entries of the form $\boldsymbol{T}1\{a_i, b_i\}2$, using the deterministic tableau rules $TC\!\downarrow$, $TL\textsc{u}\!\downarrow$, and $I\!\downarrow$. Otherwise, if there are less than $n-1$ entries of the form $\boldsymbol{T}1\{a_i, b_i\}2$ in the branch, cut on some unassigned $1\{a_i, b_i\}2$ for $1 \leq i \leq n$ and complete the branch for $\boldsymbol{F}1\{a_i, b_i\}2$, using the deterministic tableau rules $TL\textsc{u}\!\downarrow$ and $I\!\downarrow$.

In a nutshell, a refutation constructed in this way makes use of immediate contradictions obtained when assigning any of the cardinality constraints $1\{a_i, b_i\}2$ to false, so that each application of *Cut*$[\mathit{atom}(\Pi^n_c \cup \Pi^n_d) \cup \mathit{conj}(\Pi^n_c \cup \Pi^n_d) \cup \mathit{card}(\Pi^n_c \cup \Pi^n_d)]$ yields one branch that is completed without cutting any further. Hence, such a refutation of $\mathcal{T}_{card}$ for $\Pi^n_c \cup \Pi^n_d$ is of size linear in n.

An optimal strategy to construct a refutation of $\mathcal{T}_{conj}$ for $\Pi^n_c \cup \Pi^n_d$ is as follows:

(1) Cut on z, complete the branch for $\boldsymbol{T}z$, using the deterministic tableau rules $N\!\downarrow$ and $FC\!\uparrow$, and deduce $\boldsymbol{F}\{1\{a_1, b_1\}2, \dots, 1\{a_n, b_n\}2, \mathit{not}\ z\}$ in the branch for $\boldsymbol{F}z$, using the deterministic tableau rule $I\!\downarrow$.

(2) Complete any of the branches containing $\boldsymbol{F}\{1\{a_1, b_1\}2, \dots, 1\{a_n, b_n\}2, \mathit{not}\ z\}$ (and $\boldsymbol{F}z$) if the branch contains $n-1$ entries of the form $\boldsymbol{T}1\{a_i, b_i\}2$, using the deterministic tableau rules $TC\!\downarrow$, $TL\textsc{u}\!\downarrow$, and $I\!\downarrow$. Otherwise, if there are less than $n-1$ entries of the form $\boldsymbol{T}1\{a_i, b_i\}2$ in a branch, cut on some unassigned a_i for $1 \leq i \leq n$ and deduce $\boldsymbol{T}1\{a_i, b_i\}2$ in the branch for $\boldsymbol{T}a_i$ as well as in the branch for $\boldsymbol{F}a_i$, using the deterministic tableau rules $TLU\!\uparrow$ and $I\!\downarrow$.

As the second step shows, cuts on atoms a_i (or b_i) for $1 \leq i \leq n$ yield symmetric alternatives, since $\boldsymbol{T}1\{a_i, b_i\}2$ is deduced in each of the resulting branches. That is, except for the initial cut on z, applications of *Cut*$[\mathit{atom}(\Pi^n_c \cup \Pi^n_d) \cup \mathit{conj}(\Pi^n_c \cup \Pi^n_d)]$ do not admit immediate contradictions and must thus be cascaded to form a perfect binary tree. Hence, a minimal refutation of $\mathcal{T}_{conj}$ for $\Pi^n_c \cup \Pi^n_d$ is of size exponential in n.

We have thus shown that the asymptotic sizes of minimal refutations of $\mathcal{T}_{card}$ and $\mathcal{T}_{conj}$ for $\Pi^n_c \cup \Pi^n_d$ are $O(n)$ and $O(2^n)$, respectively. Since $\mathcal{T}_{conj}$ is polynomially simulated by $\mathcal{T}_{card}$, this yields that $\mathcal{T}_{card}$ is exponentially stronger than $\mathcal{T}_{conj}$. □

Finally, we case by case show that the application of a tableau rule $R\!\downarrow$ can be simulated by means of *Cut* and $R\!\uparrow$, so that the inclusion or exclusion of $R\!\downarrow$ cannot (alone) be responsible for an exponential separation between tableau calculi.

PROPOSITION 6.6. *Let Π be a disjunctive program, $\mathcal{T}$ a tableau calculus containing any subset of the tableau rules (a)–(v), and $\mathcal{T}'$ an approximation of $\mathcal{T}$.*

If Cut$[\Gamma] \in \mathcal{T}'$ such that $\mathit{atom}(\Pi) \cup \mathit{conj}(\Pi) \cup \mathit{card}(\Pi) \subseteq \Gamma$, then we have that $\mathcal{T}$ is polynomially simulated by $\mathcal{T}'$.

[6]For convenience, we take $\mathit{not}\ a_i$ and $\mathit{not}\ b_i$ to be atomic literals, rather than elements of a (singleton) conjunction. The latter would also be possible and, in view of the deterministic tableau rules in Figure 5, not affect proof complexity.

PROOF. Assume that $\mathit{Cut}[\Gamma] \in \mathcal{T}'$ such that $atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \subseteq \Gamma$. Then, we show that deducing an entry ℓ by a tableau rule $R{\downarrow}$ can be simulated by cutting on the variable of ℓ and completing the branch for $\overline{\ell}$ by an application of $R{\uparrow}$. To demonstrate this, we consider all tableau rules $R{\downarrow}$ and show that $\mathbf{A} \cup D_{\{R\uparrow\}}(\Pi, \mathbf{A} \cup \{\overline{\ell}\})$ is contradictory if $\ell \in D_{\{R\downarrow\}}(\Pi, \mathbf{A})$:

($I{\downarrow}$) If $\boldsymbol{f}\beta \in D_{\{I\downarrow\}}(\Pi, \mathbf{A})$, we have that $\boldsymbol{f}\alpha \in \mathbf{A}$. Since $\boldsymbol{t}\alpha \in D_{\{I\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}\beta\})$, it holds that $\mathbf{A} \cup D_{\{I\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}\beta\})$ is contradictory.

($N{\downarrow}$) If $\ell \in D_{\{N\downarrow\}}(\Pi, \mathbf{A})$, we have that $\ell \in \{\boldsymbol{t}\beta\} \cup min_{\mathbf{A}}(\alpha, \{p\}) \cup max_{\mathbf{A}}(\beta, \emptyset)$ for some $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$ such that $sup_{\mathbf{A}}(\Pi, \{p\}, \emptyset) = \{\alpha \leftarrow \beta\}$. For $\ell = \boldsymbol{t}\beta$, we get that $(\alpha \leftarrow \beta) \notin sup_{\mathbf{A}\cup\{\boldsymbol{f}\beta\}}(\Pi, \{p\}, \emptyset) = \{(\alpha' \leftarrow \beta') \in \Pi \mid \boldsymbol{f}\beta' \notin \mathbf{A} \cup \{\boldsymbol{f}\beta\}, \overleftarrow{sup}_{\mathbf{A}\cup\{\boldsymbol{f}\beta\}}(\alpha', \{p\}), \overrightarrow{sup}_{\mathbf{A}\cup\{\boldsymbol{f}\beta\}}(\beta', \emptyset)\}$. For $\ell \in min_{\mathbf{A}}(\alpha, \{p\})$ or $\ell \in max_{\mathbf{A}}(\beta, \emptyset)$, Lemma A.10 yields that $\overleftarrow{sup}_{\mathbf{A}\cup\{\overline{\ell}\}}(\alpha, \{p\})$ or $\overrightarrow{sup}_{\mathbf{A}\cup\{\overline{\ell}\}}(\beta, \emptyset)$, respectively, does not hold, which as with $\ell = \boldsymbol{t}\beta$ implies that $(\alpha \leftarrow \beta) \notin sup_{\mathbf{A}\cup\{\overline{\ell}\}}(\Pi, \{p\}, \emptyset)$. By Lemma A.11, we further conclude that $sup_{\mathbf{A}\cup\{\overline{\ell}\}}(\Pi, \{p\}, \emptyset) \subseteq sup_{\mathbf{A}}(\Pi, \{p\}, \emptyset) \setminus \{\alpha \leftarrow \beta\} = \emptyset$. That is, $\boldsymbol{F}p \in D_{\{N\uparrow\}}(\Pi, \mathbf{A} \cup \{\overline{\ell}\})$ for some $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, so that $\mathbf{A} \cup D_{\{N\uparrow\}}(\Pi, \mathbf{A} \cup \{\overline{\ell}\})$ is contradictory.

($U{\downarrow}$) If $\ell \in D_{\{U\downarrow\}}(\Pi, \mathbf{A})$, we have that $\ell \in \{\boldsymbol{t}\beta\} \cup min_{\mathbf{A}}(\alpha, S) \cup max_{\mathbf{A}}(\beta, S)$ for some $S \subseteq atom(\Pi)$ such that $\mathbf{A}^{\boldsymbol{T}} \cap S \neq \emptyset$ and $sup_{\mathbf{A}}(\Pi, S, S) = \{\alpha \leftarrow \beta\}$. For $\ell = \boldsymbol{t}\beta$, we get that $(\alpha \leftarrow \beta) \notin sup_{\mathbf{A}\cup\{\boldsymbol{f}\beta\}}(\Pi, S, S) = \{(\alpha' \leftarrow \beta') \in \Pi \mid \boldsymbol{f}\beta' \notin \mathbf{A} \cup \{\boldsymbol{f}\beta\}, \overleftarrow{sup}_{\mathbf{A}\cup\{\boldsymbol{f}\beta\}}(\alpha', S), \overrightarrow{sup}_{\mathbf{A}\cup\{\boldsymbol{f}\beta\}}(\beta', S)\}$. For $\ell \in min_{\mathbf{A}}(\alpha, S)$ or $\ell \in max_{\mathbf{A}}(\beta, S)$, Lemma A.10 yields that $\overleftarrow{sup}_{\mathbf{A}\cup\{\overline{\ell}\}}(\alpha, S)$ or $\overrightarrow{sup}_{\mathbf{A}\cup\{\overline{\ell}\}}(\beta, S)$, respectively, does not hold, which as with $\ell = \boldsymbol{t}\beta$ implies that $(\alpha \leftarrow \beta) \notin sup_{\mathbf{A}\cup\{\overline{\ell}\}}(\Pi, S, S)$. By Lemma A.11, we further conclude that $sup_{\mathbf{A}\cup\{\overline{\ell}\}}(\Pi, S, S) \subseteq sup_{\mathbf{A}}(\Pi, S, S) \setminus \{\alpha \leftarrow \beta\} = \emptyset$. That is, $\boldsymbol{F}p \in D_{\{U\uparrow\}}(\Pi, \mathbf{A} \cup \{\overline{\ell}\})$ for some $p \in \mathbf{A}^{\boldsymbol{T}} \cap atom(\Pi)$, so that $\mathbf{A} \cup D_{\{U\uparrow\}}(\Pi, \mathbf{A} \cup \{\overline{\ell}\})$ is contradictory.

($TC{\downarrow}$) If $\boldsymbol{f}l_i \in D_{\{TC\downarrow\}}(\Pi, \mathbf{A})$, we have that $\{\boldsymbol{F}C, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_{i-1}, \boldsymbol{t}l_{i+1}, \dots, \boldsymbol{t}l_n\} \subseteq \mathbf{A}$ for $C = \{l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\} \in conj(\Pi)$. Since $\boldsymbol{T}C \in D_{\{TC\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_i\})$, it holds that $\mathbf{A} \cup D_{\{TC\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_i\})$ is contradictory.

($FC{\downarrow}$) If $\boldsymbol{t}l_i \in D_{\{FC\downarrow\}}(\Pi, \mathbf{A})$, we have that $\boldsymbol{T}C \in \mathbf{A}$ for $C = \{l_1, \dots, l_i, \dots, l_n\} \in conj(\Pi)$. Since $\boldsymbol{F}C \in D_{\{FC\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_i\})$, it holds that $\mathbf{A} \cup D_{\{FC\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_i\})$ is contradictory.

($TLU{\downarrow}$) If $\boldsymbol{f}l_j \in D_{\{TLU\downarrow\}}(\Pi, \mathbf{A})$, we have that $\{\boldsymbol{F}B, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_{j-1}, \boldsymbol{f}l_{k+1}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}$ for $B = j\{l_1, \dots, l_j, \dots, l_{k+1}, \dots, l_n\}k \in card(\Pi)$. Since $\boldsymbol{T}B \in D_{\{TLU\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_j\})$, it holds that $\mathbf{A} \cup D_{\{TLU\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_j\})$ is contradictory.

($TLU{\downarrow}$) If $\boldsymbol{t}l_{k+1} \in D_{\{TLU\downarrow\}}(\Pi, \mathbf{A})$, we have that $\{\boldsymbol{F}B, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_j, \boldsymbol{f}l_{k+2}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}$ for $B = j\{l_1, \dots, l_j, \dots, l_{k+1}, \dots, l_n\}k \in card(\Pi)$. Since $\boldsymbol{T}B \in D_{\{TLU\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_{k+1}\})$, it holds that $\mathbf{A} \cup D_{\{TLU\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_{k+1}\})$ is contradictory.

($FL{\downarrow}$) If $\boldsymbol{t}l_j \in D_{\{FL\downarrow\}}(\Pi, \mathbf{A})$, we have that $\{\boldsymbol{T}B, \boldsymbol{f}l_{j+1}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}$ for $B = j\{l_1, \dots, l_j, \dots, l_n\}k \in card(\Pi)$. Since $\boldsymbol{F}B \in D_{\{FL\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_j\})$, it holds that $\mathbf{A} \cup D_{\{FL\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_j\})$ is contradictory.

($FU{\downarrow}$) If $\boldsymbol{f}l_{k+1} \in D_{\{FU\downarrow\}}(\Pi, \mathbf{A})$, we have that $\{\boldsymbol{T}B, \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_k\} \subseteq \mathbf{A}$ for $B = j\{l_1, \dots, l_{k+1}, \dots, l_n\}k \in card(\Pi)$. Since $\boldsymbol{F}B \in D_{\{FU\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_{k+1}\})$, it holds that $\mathbf{A} \cup D_{\{FU\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_{k+1}\})$ is contradictory.

($TD{\downarrow}$) If $\boldsymbol{f}l_i \in D_{\{TD\downarrow\}}(\Pi, \mathbf{A})$, we have that $\boldsymbol{F}D \in \mathbf{A}$ for $D = \{l_1; \dots; l_i; \dots; l_n\} \in disj(\Pi)$. Since $\boldsymbol{T}D \in D_{\{TD\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_i\})$, it holds that $\mathbf{A} \cup D_{\{TD\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{t}l_i\})$ is contradictory.

($FD{\downarrow}$) If $\boldsymbol{t}l_i \in D_{\{FD\downarrow\}}(\Pi, \mathbf{A})$, we have that $\{\boldsymbol{T}D, \boldsymbol{f}l_1, \dots, \boldsymbol{f}l_{i-1}, \boldsymbol{f}l_{i+1}, \dots, \boldsymbol{f}l_n\} \subseteq \mathbf{A}$ for $D = \{l_1; \dots; l_{i-1}; l_i; l_{i+1}; \dots; l_n\} \in disj(\Pi)$. Since $\boldsymbol{F}D \in D_{\{FD\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_i\})$, it holds that $\mathbf{A} \cup D_{\{FD\uparrow\}}(\Pi, \mathbf{A} \cup \{\boldsymbol{f}l_i\})$ is contradictory.

We have thus shown that deducing ℓ by a tableau rule $R{\downarrow}$ can be simulated by means of applying *Cut* and $R{\uparrow}$. As each such simulation introduces only two additional entries, $\overline{\ell}$ and the complement of some entry belonging to the branch at hand, every tableau of $\mathcal{T}$ can be transformed into a tableau of $\mathcal{T}'$ having approximately similar size, provided that the *Cut* applications needed for simulations are admitted by $\mathcal{T}'$. In fact, the variable of an entry deducible by a tableau rule $R{\downarrow}$ cannot be a disjunction, so that all simulations are possible if $\mathit{Cut}[\Gamma] \in \mathcal{T}'$ such that $atom(\Pi) \cup conj(\Pi) \cup card(\Pi) \subseteq \Gamma$. $\square$

We have thus proven the formal results presented in Section 6.