Maps in Multiple Belief Change

PAVLOS PEPPAS*
University of Patras
and
COSTAS D. KOUTRAS†
University of Peloponnese
and
MARY-ANNE WILLIAMS‡
The University of Technology, Sydney

Multiple Belief Change extends the classical AGM framework for Belief Revision introduced by Alchourron, Gardenfors, and Makinson in the early '80s. The extended framework includes epistemic input represented as a (possibly infinite) *set of sentences*, as opposed to a *single sentence* assumed in the original framework. The transition from single to multiple epistemic input worked out well for the operation of belief revision. The AGM postulates and the system-of-spheres model were adequately generalized and so was the representation result connecting the two. In the case of belief contraction however, the transition was not as smooth. The generalized postulates for contraction, which were shown to correspond precisely to the generalized *partial meet model*, failed to match up to the generalized *epistemic entrenchment model*. The mismatch was fixed with the addition of an extra postulate, called the *limit postulate*, that relates contraction by multiple epistemic input to a series of contractions by single epistemic input. The new postulate however creates problems on other fronts. Firstly, the limit postulate needs to be mapped into appropriate constraints in the partial meet model. Secondly, via the Levi and Harper Identities, the new postulate translates into an extra postulate for multiple revision, which in turn needs to be characterized in terms of systems of spheres. Both these open problems are addressed in this paper. In addition, the limit postulate is compared with a similar condition in the literature, called (K*F), and is shown to be strictly weaker than it. An interesting aspect of our results is that they reveal a profound connection between rationality in multiple belief change and the notion of an *elementary set* of possible worlds (closely related to the notion of an elementary class of models from classical logic).

Categories and Subject Descriptors: I.2.3 [**Artificial Intelligence**]: Deduction and Theorem Proving—*Nonmonotonic Reasoning and* **Belief Revision**

* Dept of Business Administration, University of Patras, Patras 265 00, Greece, pavlos@upatras.gr
† Dept of Computer Science and Technology, University of Peloponnese, 221 00 Tripolis, Greece, ckoutras@uop.gr
‡ Faculty of Information Technology, The University of Technology, Sydney, NSW 2007, Australia, Mary-Anne@it.uts.edu.au
Permission to make digital/hard copy of all or part of this material without fee for personal or classroom use provided that the copies are not made or distributed for profit or commercial advantage, the ACM copyright/server notice, the title of the publication, and its date appear, and notice is given that copying is by permission of the ACM, Inc. To copy otherwise, to republish, to post on servers, or to redistribute to lists requires prior specific permission and/or a fee.
© 2011 ACM 1529-3785/2011/0700-0001 $5.00

1. INTRODUCTION

Since the publication of the celebrated AGM article in 1985, [Alchourron et al. 1985],[1] *Belief Revision* has grown to be a central area of research in *Knowledge Representation*. Belief Revision studies the process by which a rational agent changes her initial belief set in the light of new information.

In the original formal framework developed in [Alchourron et al. 1985], the new information is modeled as a *single logical sentence* φ and two types of belief change are studied; belief *revision*, after which the whole area took its name,[2] and belief *contraction*. More recent work generalized this framework to include epistemic input encoded as a (possibly infinite) set of logical sentences Γ, thus introducing the processes of *multiple revision* and *multiple contraction* (see [Fuhrmann and Hansson 1994], [Peppas 1996], [Zhang and Foo 2001], [Peppas 2004]).

The transition to multiple epistemic input was not without complications and some important questions remain unanswered. In this article we address three of the most challenging open problems in multiple belief change. All three problems relate to well known conditions that reduce multiple revision and contraction to a series of sentence-revisions and sentence-contractions respectively.[3]

To describe the three problems in more detail, some background is necessary. The process of belief revision is defined as the type of rational belief change for which the epistemic input needs to be incorporated into the agent's initial belief set K, possibly at the expense of some of the original beliefs in K. In belief contraction on the other hand, the epistemic input represents information that needs to be removed from the initial belief set K. Once again, beliefs different from the epistemic input may also be affected in the process.

Belief revision and contraction have been described both axiomatically and constructively, in what is now known as the *AGM paradigm*. In terms of axiomatic models, the so-called *AGM postulates* for revision and contraction are widely acknowledged to have captured much of what is the essence of these two types of belief change. Among constructive models, the three most popular ones are the *system of spheres model*, the *partial meet model*, and the *epistemic entrenchments* model. The system of spheres model is used to construct revision functions, while the other two are used in contraction. Representation results have been established that prove the equivalence between the AGM postulates and the corresponding constructions (see [Peppas 2008] for a recent survey on Belief Revision).

As noted above, the AGM paradigm has recently been extended to include multiple epistemic input. The AGM postulates for revision have been modified accordingly, [Lindstrom 1991], and so was the corresponding system of spheres model, [Peppas 2004]. In the case of belief contraction, things were more complicated as there are at least three different ways of interpreting the contraction of a belief set K by a set of sentences Γ, known

[1]Named so after the initials of its authors, *Carlos Alchourron*, *Peter Gardenfors*, and *David Makinson*.
[2]To distinguish the research area from the process, we shall use the capitalized term *Belief Revision* for the former and the same term in lower case letter (i.e. *belief revision*) for the latter.
[3]We shall often refer to the original AGM revision and contraction operators as sentence-revision and sentence-contraction, to distinguish them from their multiple counterparts.

as *package contraction*, *choice contraction* and *set contraction*, [Fuhrmann and Hansson 1994], [Zhang and Foo 2001]. In this article we consider only the third and most recent generalization of contraction.

Set contraction is defined as the process of rationally contracting K to make it *consistent* with the epistemic input Γ (notice the difference in aim with sentence contraction). Zhang and Foo, [Zhang and Foo 2001], proposed a generalization of the AGM postulates for set contraction and considered corresponding generalizations of the constructive models for contraction. In the case of the partial meet model things worked out well. Generalizing the epistemic entrenchment model however proved more challenging. A new structure called a *nicely ordered partition* was introduced for this purpose which however didn't quite match up to the postulates: the functions constructed from nicely ordered partition are a *proper* subset of those satisfying the postulates for set contraction. The mismatch was fixed with the introduction of an extra postulate, called the *limit postulate* for set contraction, which associates contraction by a set of sentences Γ with contractions by *finite* subsets of Γ (the symbol $\subseteq_f$ below stands for *finite subset*; i.e. $C \subseteq_f D$ means that C is a finite subset of D):

$$(\dot{-}\text{LP}) \qquad K \dot{-} \Gamma = \bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} K \dot{-} (A \cup B)$$

With the addition of ($\dot{-}$LP), Zhang and Foo obtained an exact match between the postulates for set contraction and the functions induced from nicely ordered partitions, and provided arguments in support of the intuitive appeal of the new postulate.

However the introduction of ($\dot{-}$LP) generates gaps on other fronts. Firstly, condition ($\dot{-}$LP) needs to be mapped into appropriate constraints in the partial meet model. This is an important open problem already identified in [Zhang and Foo 2001].

Secondly, because of the close relation between contraction and revision via the Levi and Harper Identities (see section 8 below), the limit postulate for set contraction gives rise to a corresponding extra postulate for multiple revision:

$$(\text{*LP}) \qquad K * \Gamma = \bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} K * (A \cup B)$$

Like ($\dot{-}$LP), the new postulate (*LP) creates the need for extra constraints on systems of spheres. Formulating these conditions and proving their equivalence to (*LP) is still an open problem.

A third open problem in multiple belief change is the relationship between (*LP) and another very similar condition also reducing multiple revision to sentence revision proposed and studied in [Peppas 1996], [Peppas 2004]:[4]

$$(\text{K*F}) \qquad K * \Gamma = \bigcap_{A \subseteq_f \Gamma} ((K * A) + \Gamma)$$

Our aim in this paper is to address all three open problems mentioned above. *We formulate constraints for the partial meet and system of spheres models which characterize precisely*

[4]In condition (K*F), + denotes the operation of *expansion*, i.e. union followed by logical closure – see section 2 for details)

*the postulates ($\dot{-}$LP) and (*LP) respectively. Moreover, we prove that (K*F) is strictly stronger than (*LP) by explicitly constructing a multiple revision function that violates the former but satisfies the latter*. These results, together with the ones in [Zhang and Foo 2001] and [Peppas 2011], complete the picture of the effects that the limit postulate(s) have on the multiple belief change landscape.

The article is structured as follows. In the next section we introduce the necessary notation and terminology. In section 3 we review the postulates for multiple revision and the associated system of spheres model. In section 4 we compare conditions (*LP) and (K*F). In section 5 we present our system-of-spheres characterization of (*LP) and we prove that (*LP) is strictly weaker than (K*F). Next we turn to set contraction. We introduce the relevant background in section 6, and in section 7 we present our partial-meet-characterization of ($\dot{-}$LP). Section 8 contains some concluding remarks.

2. PRELIMINARIES

Throughout this paper we shall be working with a formal language L governed by a logic which is identified by its consequence relation $\vdash$. Very little is assumed about L and $\vdash$. In particular, L is taken to be closed under all Boolean connectives, and $\vdash$ has to satisfy the following properties:

(i) $\vdash \varphi$ for all truth-functional tautologies A (supraclassicality).

(ii) If $\vdash (\varphi \rightarrow y)$ and $\vdash \varphi$, then $\vdash y$ (modus ponens).

(iii) $\vdash$ is consistent, i.e. $\nvdash \varphi \wedge \neg\varphi$.

(iv) $\vdash$ satisfies the deduction theorem, that is, $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \vdash y$ iff $\vdash \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \rightarrow y$.

(v) $\vdash$ is compact.

For a *finite* set of sentences $A = \{\varphi_1, \ldots, \varphi_n\}$, of L we shall use $\wedge A$ to denote the conjunction of all elements of A, i.e. the sentence $\varphi_1 \wedge \cdots \wedge \varphi_n$. For a set of sentences Γ of L, $Cn(\Gamma)$ denotes the set of all logical consequences of Γ, i.e. $Cn(\Gamma) = \{\varphi \in L\colon \Gamma \vdash \varphi\}$. Whenever A is a *finite* subset of Γ, we write $A \subseteq_f \Gamma$.

A theory K of L is any set of sentences of L closed under $\vdash$, i.e. $K = Cn(K)$. We shall denote the set of all theories of L by $\mathcal{K}_L$. A theory K of L is complete iff for all sentences $\varphi \in L$, $\varphi \in K$ or $\neg\varphi \in K$. We shall denote the set of all consistent complete theories of L by $\mathcal{M}_L$. In the context of Belief Revision, consistent complete theories play the role of *possible worlds* and therefore we shall use the two terms interchangeably. For a set of sentences Γ of L, $[\Gamma]$ denotes the set of all consistent complete theories of L that contain Γ. Often we shall use the notation $[\varphi]$ for a sentence $\varphi \in L$, as an abbreviation of $[\{\varphi\}]$. For a theory K and a set of sentences Γ, we shall denote by $K+\Gamma$ the closure under $\vdash$ of $K \cup \Gamma$, i.e. $K + \Gamma = Cn(K \cup \Gamma)$. For a sentence $\varphi \in L$ we shall often write $K + \varphi$ as an abbreviation of $K + \{\varphi\}$. For two sets of sentences Γ, Δ, we define $\Gamma \vdash \Delta$ iff $\Gamma \vdash \delta$ for all $\delta \in \Delta$. Finally, the symbols $\top$ and $\bot$ will be used to denote an arbitrary (but fixed) tautology and contradiction of L respectively.

3. MULTIPLE BELIEF REVISION REVIEW

Multiple belief revision was defined in [Lindstrom 1991], as a function $* : \mathcal{K}_L \times 2^L \mapsto \mathcal{K}_L$, mapping $\langle K, \Gamma \rangle$ to $K * \Gamma$, that satisfies the following postulates:

($K * 1$) $K * \Gamma$ is a theory of L.
($K * 2$) $\Gamma \subseteq K * \Gamma$.
($K * 3$) $K * \Gamma \subseteq K + \Gamma$.
($K * 4$) If $K + \Gamma \neq L$ then $K + \Gamma \subseteq K * \Gamma$.
($K * 5$) $K * \Gamma \vdash \perp$ iff $\Gamma \vdash \perp$.
($K * 6$) If $Cn(\Gamma) = Cn(\Delta)$ then $K * \Gamma = K * \Delta$.
($K * 7$) $K * (\Gamma \cup \Delta) \subseteq (K * \Gamma) + \Delta$.
($K * 8$) If $(K * \Gamma) + \Delta \neq L$ then $(K * \Gamma) + \Delta \subseteq K * (\Gamma \cup \Delta)$.

The above postulates are a straightforward generalization of the AGM postulates for sentence revision (i.e. revision by a single sentence φ rather than a *set* of sentence Γ). The reader is referred to [Gardenfors 1988] and [Peppas 2008] for a detailed discussion on the motivation of these postulates.

To improve readability, in this article we shall ignore the limiting cases of revising by an empty or an inconsistent set, and we assume that the epistemic input Γ is always a *non-empty* and *consistent* set of sentences.

It turns out that ($K * 1$) - ($K * 8$) are satisfied not by one, but by a whole family of revision functions. This family can be constructed with the aid of a structure called a *system of spheres* introduced in [Grove 1988] originally for sentence revision, but later generalized in [Peppas 2004] for multiple revision.

Given a theory K, Grove defines a system of spheres S centered on $[K]$, to be a collection of subsets of $\mathcal{M}_L$, the elements of which are called *spheres*, that satisfies the following conditions:

(S1) S is totally ordered with respect to set inclusion.
(S2) The smallest sphere in S is $[K]$; that is $[K] \in S$, and if $U \in S$ then $[K] \subseteq U$.
(S3) $\mathcal{M}_L \in S$ (and therefore $\mathcal{M}_L$ is the largest sphere in S).
(S4) For every consistent $\varphi \in L$, there is a smallest sphere in S intersecting $[\varphi]$.

A system of spheres S is essentially a preorder on possible worlds (alias, consistent complete theories) representing comparative plausibility: the closer a world is to the center of the system $[K]$ the more plausible it is. With this reading in mind, Grove proposed an intuitive construction of sentence revision functions based on systems of spheres (see condition (S*) below) and proves that his method is sound and complete with respect to the AGM postulates for sentence revision. Peppas, [Peppas 2004], later generalized this result for multiple revision. The generalization however required two further constraints on systems of spheres. The first constraint is a straightforward generalization of (S4):

(SM) For every nonempty consistent $\Gamma \subseteq L$, there exists a smallest sphere in S, denoted $c(\Gamma)$, intersecting $[\Gamma]$.

The second constraint relates the notion of an *elementary* set of possible worlds, inspired by the notion of an elementary class of models from classical logic (see [Chang and Keisler 1991]), and adequately adjusted in the present context. In particular, we shall say that a set V of consistent complete theories is *elementary* iff $V = [\bigcap V]$.[5] In other words, V is elementary if no world outside V is compatible with the theory $\bigcap V$. As shown in [Peppas et al. 1995], not all sets of consistent complete theories are elementary; in fact for every language L with infinitely many logical equivalent classes, there are infinitely many non-elementary subsets of $\mathcal{M}_L$. Condition (SD) below requires that in a system of spheres S, the spheres are so arranged that the set $c(\Gamma) \cap [\Gamma]$ is always elementary:

(SD) For every nonempty consistent $\Gamma \subseteq L$, $c(\Gamma) \cap [\Gamma]$ is elementary.

Although initially it strikes us as a technical condition, (SD) can also be understood intuitively once we consider the special role of the set $c(\Gamma) \cap [\Gamma]$ in belief revision. Notice that under the intended reading of S, the set $c(\Gamma) \cap [\Gamma]$ contains the most plausible Γ-worlds, which in turn are the very worlds used in the system-of-spheres construction of revision functions:

(S*) $K * \Gamma = \bigcap(c(\Gamma) \cap [\Gamma])$

Condition (S*) defines the result of revision as the theory corresponding to the most plausible worlds compatible with the epistemic input. This is precisely the method proposed by Grove in [Grove 1988] and later generalized by Peppas in [Peppas 2004]. In view of (S*) let us now re-examine (SD). Consider a nonempty consistent set of sentences Γ and suppose that, contrary to (SD), there is a world z compatible with $\bigcap(c(\Gamma) \cap [\Gamma])$, that is not in $c(\Gamma) \cap [\Gamma]$. This entails that $z \notin c(\Gamma)$. Hence z is strictly less plausible than all worlds in $c(\Gamma) \cap [\Gamma]$. So when revising by Γ we end up with a belief set compatible with a "sub-optimal" (i.e. not maximally plausible) Γ-world; putting it differently, the epistemic loss induced from the revision by Γ is over and above what is necessitated by Γ itself. This is clearly in violation with one of the defining features of rationality in belief revision, known as the *principle of minimal change*, which loosely speaking, dictates that epistemic loss should be minimized during this process. Hence the need for (SD).

We shall say that a system of spheres S is *well ranked* iff in addition to (S1) - (S4), it satisfies the conditions (SM) and (SD). In [Peppas 2004] it was shown that the functions produced by well ranked systems of spheres are precisely those satisfying the postulates $(K * 1)$ - $(K * 8)$ for multiple revision mentioned above:

THEOREM 1. *[Peppas 2004]. Let K be a theory and S a well ranked system of spheres centered on $[K]$. The function $*$ defined from S via (S*) satisfies $(K * 1)$ - $(K * 8)$.*

[5] If $V = \emptyset$, we define $\bigcap V = L$, from which is follows that the empty set is elementary.

THEOREM 2. *[Peppas 2004]. Let K be a theory and $*$ a multiple revision function satisfying $(K*1)$ - $(K*8)$. There exists a well ranked system of spheres S centered on $[K]$ such that (S*) is satisfied.*

4. COMPARING (*LP) WITH (K*F)

Having reviewed the necessary background on multiple revision, let us now turn to the open problems mentioned in the introduction, starting with the relationship between the conditions (*LP) and (K*F), repeated below:

$$\text{(*LP)} \qquad K * \Gamma = \bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} K * (A \cup B)$$

$$\text{(K*F)} \qquad K * \Gamma = \bigcap_{A \subseteq_f \Gamma} ((K * A) + \Gamma)$$

Both these conditions can be viewed as methods of reducing multiple revision to sentence revision (since revision by a finite set A is equivalent to revision by the sentence $\wedge A$). Moreover in [Zhang and Foo 2001] it was shown that in the presence of (K*1) - (K*8), (*LP) is *equivalent* to the condition (rLP) below, which at first sight looks very similar to (K*F):

$$\text{(rLP)} \qquad K * \Gamma = (\bigcap_{A \subseteq_f Cn(\Gamma)} (K * A)) + \Gamma$$

The two conditions look very similar indeed. In fact there are only two differences between (K*F) and (rLP). Firstly, in (rLP) the initial belief set K is revised by all finite subsets of the *closure* of Γ, whereas in (K*F) only the finite subsets of Γ itself as used. Secondly, in (rLP) this series of sentence revisions is first intersected and then expanded by Γ, whereas in (K*F) it's the other way around (the result of each sentence revision is first expanded by Γ and then all expanded theories are intersected). These two differences, however small as they may appear, suffice to make (K*F) strictly stronger than (rLP), and therefore strictly stronger than (*LP).

Our first result shows that (K*F) entails (*LP). We break down the proof into a lemma and a corollary since the lemma will be used independently later in the article.[6]

LEMMA 1. *Let K be a theory, $\Gamma \subseteq L$ a nonempty consistent set of sentences, and $*$ a multiple revision function satisfying (K*1) - (K*8). If there is a $C \subseteq_f \Gamma$ such that $K * C$ is consistent with Γ, then $K * \Gamma = \bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} K * (A \cup B)$.*

Proof. Assume that there is a $C \subseteq_f \Gamma$ such that $K * C$ is consistent with Γ. Then from $(K * 7)$, $(K * 8)$ we derive that $K * (C \cup \Gamma) = (K * C) + \Gamma$, which again by $(K * 6)$ entails that $K * \Gamma = (K * C) + \Gamma$.

Consider now any $\varphi \in K * \Gamma$. Then $\varphi \in (K * C) + \Gamma$ and consequently, by compactness, there is a $D \subseteq_f \Gamma$ such that $(\wedge D \rightarrow \varphi) \in K * C$. Moreover, since Γ is consistent with $(K * C)$, D

[6]Alternatively, we can use (rLP) as an intermediate to establish the same result.

is also consistent with $(K * C)$. Hence by $(K * 7)$ - $(K * 8)$, $K * (C \cup D) = (K * C) + D$ and consequently $\varphi \in K * (C \cup D)$. Now call A the set $C \cup D$. It is not hard to verify that for any $B \subseteq_f Cn(\Gamma)$, $K * (A \cup B) = (K * A) + B$.[7] Consequently, since $\varphi \in K * A$, it follows that $\varphi \in K * (A \cup B)$. Since B was chosen arbitrarily, this entails that $\varphi \in \bigcap_{B \subseteq_f Cn(\Gamma)} (K * (A \cup B))$, which in turn implies that $K * \Gamma \subseteq \bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} (K * (A \cup B))$.

For the converse, let φ be any sentence in $\bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} (K * (A \cup B))$. Then there is a $A \subseteq_f \Gamma$ such that $\varphi \in K * (A \cup B)$ for all $B \subseteq_f Cn(\Gamma)$. Consequently, $\varphi \in K * (A \cup C)$. Next notice that, since $K * C$ is consistent with Γ, it is also consistent with A, and therefore by $(K * 7)$ - $(K * 8)$, $K * (A \cup C) = (K * C) + A$, which again entails $\varphi \in (K * C) + \Gamma$. Hence, from $K * \Gamma = (K * C) + \Gamma$ we derive that $\varphi \in K * \Gamma$ as desired. ■

COROLLARY 1. *Let K be a theory of L, and $*$ a multiple revision function satisfying (K*1) - (K*8). If $*$ satisfies (K*F) then it also satisfies (*LP).*

Proof. Assume that $*$ satisfies (K*F), and let Γ be any nonempty consistent set of sentences. By $(K * 5)$, $K * \Gamma$ is consistent, and therefore since $K * \Gamma = \bigcap_{A \subseteq_f \Gamma} ((K * A) + \Gamma)$, there is at least one $A \subseteq_f \Gamma$, such that $K * A$ is consistent with Γ. Then by Lemma 1, $*$ satisfies (*LP). ■

Our next result shows that the converse is not true; i.e. (K*F) is *strictly stronger* than (*LP). Theorem 3 below proves this through the construction of a well ranked system of spheres whose induced multiple revision function is shown to satisfy (*LP) but violate (K*F).

THEOREM 3. *There exists a consistent theory K and a multiple revision function $*$ satisfying (K*1) - (K*8), such that $*$ satisfies (*LP) but violates (K*F) at K.*

Proof. For the purpose of this proof (and only for this proof), we shall fix the details of the language L as follows: we take L to be a propositional language with *infinitely many* propositional variables denoted $p_0, p_1, p_2, \ldots$. Define T_0, T_1, T_2, ..., to be the following theories:

$$
\begin{aligned}
T_0 &= Cn(\{p_0, p_1, \ldots\}) \\
T_1 &= Cn(\{p_1, p_2, \ldots\}) \\
T_2 &= Cn(\{p_2, p_3, \ldots\}) \\
&\vdots \\
T_j &= Cn(\{p_j, p_{j+1}, \ldots\}) \\
&\vdots
\end{aligned}
$$

We set $K = T_0$ and define S to be the following system of spheres centered at $[K]$ (see Figure 1):

$$S = \{[T_i] : i \in \mathbb{N}_0\} \cup \{\mathcal{M}_L\}^8$$

[7] Notice that $K * A = (K * C) + D$, and since $K * C$ is consistent with Γ, it follows that $K * A$ is consistent with B, which by $(K * 7)$ - $(K * 8)$ entails that $K * (A \cup B) = (K * A) + B$.
[8] By $\mathbb{N}_0$ we denote the set of all non-negative integers; i.e. $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

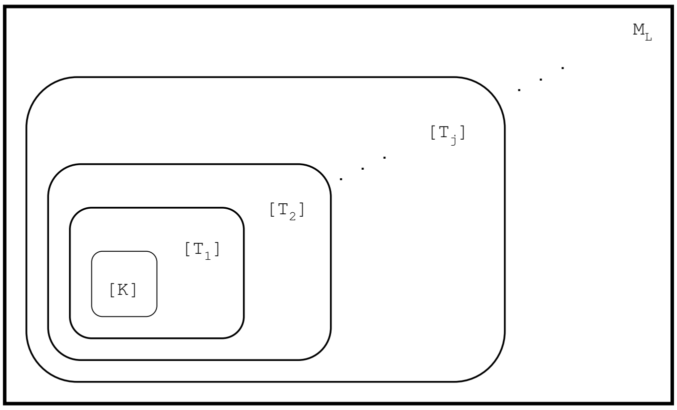

Fig. 1. (*LP) $\not\Rightarrow$ (K*F)

We will show that S is a well ranked system of spheres, and moreover that the multiple revision function $*$ induced by S satisfies (*LP) but violates (K*F).

Starting with the well-rankness of S, notice that from its construction, it is straightforward to verify that conditions (S1) - (S3) are true. Next for (SM), let Γ be an arbitrary nonempty consistent set of sentences. If none of $[T_i]$ (with $i \in \mathbb{N}_0$) intersects $[\Gamma]$ then (SM) is trivially true. Assume therefore that for some $j \in \mathbb{N}_0$, $[T_j] \cap [\Gamma] \neq \emptyset$. Since there are only finitely many spheres in S smaller than $[T_j]$ (namely, $[T_0], [T_1], \ldots [T_{j-1}]$), it follows that there is a smallest sphere in S intersecting $[\Gamma]$; thus (SM) is true, and therefore (S4) is also true (notice that (SM) entails (S4)). For (SD), notice that *all* spheres in S are (by construction) elementary. Hence for all nonempty, consistent sets of sentences Γ, $c(\Gamma)$ is elementary, and therefore $c(\Gamma) \cap [\Gamma]$ is also elementary as desired.

For (*LP), let Γ be a nonempty consistent set of sentences. We distinguish between two cases: $c(\Gamma) \neq \mathcal{M}_L$ and $c(\Gamma) = \mathcal{M}_L$. Starting with the first case, assume that $c(\Gamma) \neq \mathcal{M}_L$. Then there is a $j \in \mathbb{N}_0$ such that $c(\Gamma) = [T_j]$. Consequently, there are only *finitely many* spheres smaller than $c(\Gamma)$ in S. Moreover notice that for all $A \subseteq_f Cn(\Gamma)$, $c(A) \subseteq c(\Gamma)$ and therefore, there is a $Z \subseteq_f Cn(\Gamma)$ such that $c(A) \subseteq c(Z)$, for all $A \subseteq_f Cn(\Gamma)$. Next we show that $c(Z) = c(\Gamma)$. Assume towards contradiction that $c(Z) \subset c(\Gamma)$. Then $c(Z) \cap [\Gamma] = \emptyset$, and therefore by compactness and since $c(Z)$ is elementary, it follows that there is a $B \subseteq_f Cn(\Gamma)$ such that $\bigcap c(Z) \vdash \neg(\wedge B)$. This again entails that $c(Z) \subset c(B \cup Z)$, contradicting our initial assumption about Z. Hence $c(Z) = c(\Gamma)$, and given that $[\Gamma] \subseteq [Z]$ it follows that $K * Z$ is consistent with Γ. Then, from Lemma 1 we derive (*LP).

Consider now the second case where $c(\Gamma) = \mathcal{M}_L$. If there is a $Z \subseteq_f Cn(\Gamma)$ such that $c(Z) = \mathcal{M}_L$, then clearly $K * Z$ is consistent with Γ and, like before, (*LP) follows from Lemma 1. Assume therefore that $c(A) \subset \mathcal{M}_L$ for all $A \subseteq_f Cn(\Gamma)$. Notice that since $c(\Gamma) = \mathcal{M}_L$, (S*) entails that $K * \Gamma = \Gamma$. We will prove (*LP) by proving the equivalent condition (rLP). This in turn is done by showing that, under the assumptions of the case, all sentences φ in $\bigcap_{A \subseteq_f Cn(\Gamma)} K * A$ are tautologies.

Assume towards contradiction that for some $\varphi \in \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$, there is a world $z \in \mathcal{M}_L$ that falsifies φ; i.e. $z \vdash \neg\varphi$. Let p_j be the propositional variable with the highest index in φ, and let u be the world that agrees with z on $p_0, p_1, \ldots, p_j$ and satisfies all remaining variables $p_{j+1}, p_{j+2}, \ldots$; i.e. for all $0 \leq i \leq j$, $u \vdash p_i$ iff $z \vdash p_i$, and $u \vdash p_i$ for all $i > j$. Clearly $u \vdash \neg\varphi$. Next observe that u falsifies at least one of $p_0, \ldots, p_j$. For assume that $u \vdash p_i$, for all $0 \leq i \leq j$. Then $u = T_0 = K$, and consequently $\varphi \notin K$, which again entails that $\varphi \notin \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$.[9] This of course contradicts our initial assumption about φ. Hence there is a propositional variable in $\{p_0, \ldots, p_j\}$, falsified by u. Let p_m be the maximun such variable; i.e. $u \vdash \neg p_m$ with $0 \leq m \leq j$, and $u \vdash p_i$ for all $i > m$. Notice that since $c(\Gamma) = \mathcal{M}_L$, Γ is inconsistent with T_m. Hence by compactness, there is an $l \in \mathbb{N}_0$ such that $\Gamma \vdash \neg(p_m \wedge \cdots \wedge p_{m+l})$. Call B the singleton $\{\neg p_m \vee \cdots \vee \neg p_{m+l}\}$. It is not hard to see that $c(B) = [T_{m+1}]$ and therefore $[K * B] = [T_{m+1}] \cap [\neg p_m \vee \cdots \vee \neg p_{m+l}]$. This again entails that $u \in [K * B]$, and therefore, since $\neg\varphi \in u$, $\varphi \notin K * B$. Consequently, $\varphi \notin \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$, contradicting our initial assumption about φ. Hence all sentences φ in $\bigcap_{A \subseteq_f Cn(\Gamma)} K * A$ are tautologies, and consequently $(\bigcap_{A \subseteq_f Cn(\Gamma)} K * A) + \Gamma = \Gamma$. Given that, under the assumptions of the case, $K * \Gamma = \Gamma$, we derive that (rLP) is satisfied, and consequently so is (*LP).

We conclude the proof by showing that $*$ violates (K*F) at K. Let Γ be the set $\Gamma = \{\neg p_0, \neg p_1, \ldots\}$. Clearly $c(\Gamma) = \mathcal{M}_L$. Let A be any finite subset of Γ, and let p_k be the propositional variable with the highest index in A. It is not hard to see that $c(A) = [T_{k+1}]$, and therefore $p_{k+1} \in K * A$. This again entails that Γ is inconsistent with $K * A$, and therefore $(K * A) + \Gamma = \perp$. Given that A was chosen arbitrarily, it follows that $\bigcap_{A \subseteq_f \Gamma}((K * A) + \Gamma) \vdash \perp$. On the other hand, since Γ is consistent, from $(K * 5)$ it follows that $K * \Gamma \nvdash \perp$. Hence $K * \Gamma \neq \bigcap_{A \subseteq_f \Gamma}((K * A) + \Gamma)$, violating (K*F). ■

We conclude this section with an alternative way to approach the relationship between (K*F) and (*LP).[10]

Zhang and Foo define $\bigcup_{A \subseteq_f \Gamma} \bigcap_{B \subseteq_f Cn(\Gamma)} K * (A \cup B)$ as the *lower limit* of the set family $\{K * A : A \subseteq_f \Gamma\}$. Hence (*LP) esentially equates multiple revision by Γ with the lower limit of the set family $\{K * A : A \subseteq_f \Gamma\}$. It is natural to consider the other side of the coin as well; i.e. the condition that equates multiple revision with the *upper limit* of the same set family (also defined in [Zhang and Foo 2001]):

(u*LP) $\quad K * \Gamma = \bigcap_{A \subseteq_f \Gamma} \bigcup_{B \subseteq_f Cn(\Gamma)} K * (A \cup B)$

It turns out that (K*F) entails (u*LP) as well:

Lemma 2. *Let K be a theory of L, and $*$ a multiple revision function satisfying (K*1) - (K*8). If $*$ satisfies (K*F) then it satisfies (u*LP).*

Proof. Assume that $*$ satisfies (K*F). Let Γ be nonempty consistent set of sentences. From (K*F) it follows that there is a $C \subseteq_f \Gamma$ such that Γ is consistent with $K * C$. Then by (K*6)

[9]To see this simply observe that $p_0 \vee \neg p_0$ (like any other tautology) belongs to $Cn(\Gamma)$, and moreover $K = K * \{p_0 \vee \neg p_0\}$. Therefore since $\varphi \notin K$, it follows that $\varphi \notin \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$.

[10]We are grateful to the reviewer for pointing out this alternative approach.

- (K*8) we derive that $(K * C) + \Gamma = K * \Gamma$. Moreover it is not hard to see that for any $B \subseteq_f Cn(\Gamma)$, B is consistent with $K*C$ and therefore $K*(C \cup B) = (K*C)+B \subseteq (K*C)+\Gamma = K*\Gamma$. Hence $\bigcup_{B \subseteq_f Cn(\Gamma)} K*(C \cup B) \subseteq K*\Gamma$ and consequently, $\bigcap_{A \subseteq_f \Gamma} \bigcup_{B \subseteq_f Cn(\Gamma)} K*(A \cup B) \subseteq K*\Gamma$.

For the converse, let φ be any sentence in $K * \Gamma$. Then, since $K * \Gamma = (K * C) + \Gamma$, by compactness we derive that there is a $\Delta \subseteq_f \Gamma$ such that $((\wedge\Delta) \rightarrow \varphi) \in K * C$. Consider now any finite subset A of Γ. Clearly, $A \cup \Delta$ is consistent with $K * C$, and therefore from (K*7) - (K*8) we derive that $K * (A \cup C \cup \Delta) = (K * C) + (A \cup \Delta)$, which in turn implies that $\varphi \in K * (A \cup C \cup \Delta)$. Hence, for any $A \subseteq_f \Gamma$, $\varphi \in \bigcup_{B \subseteq_f Cn(\Gamma)} K * (A \cup B)$. Therefore $\varphi \in \bigcap_{A \subseteq_f \Gamma} \bigcup_{B \subseteq_f Cn(\Gamma)} K*(A \cup B)$, and consequently, $K*\Gamma \subseteq \bigcap_{A \subseteq_f \Gamma} \bigcup_{B \subseteq_f Cn(\Gamma)} K*(A \cup B)$. ■

According to Corollary 1 and Lemma 2, whenever (K*F) is true, both (*LP) and (u*LP) are true. Therefore another way to prove Theorem 3 would be to show that there is a multiple revision function satisfying (*LP) but not (u*LP). To this end, one can utilize a counterexample from [Zhang and Foo 2001] showing that there is a set contraction function $\dot{-}$ satisfying ($\dot{-}$LP) for which the lower limit of the family $\{K \dot{-} A : A \subseteq_f \Gamma\}$ is different from the upper limit. To this result, one can apply the Levi identity (with some adjustments) to get the desired conclusion. The advantage of our proof for Theorem 3 over this alternative line of reasoning is that in our proof we provide an explicit system-of-spheres construction that differentiates (*LP) from (K*F), thus helping to get a better grasp of the essence of the two conditions.

5. THE LIMIT POSTULATE IN THE SYSTEM OF SPHERES MODEL

How would (*LP) look like in the realm of systems of spheres? In the previous section we showed that (*LP) is strictly weaker that (K*F), so perhaps the system-of-spheres characterization of (K*F) is a good place to start with.

Let K be a theory, S a system of spheres centered on $[K]$, and $*$ the revision function induced from S via (S*). In [Peppas 2004] it was shown that $*$ satisfies (K*F) iff S satisfies the following condition:

(SF) For all $Q \subseteq S$, $\bigcup Q$ is elementary.

How can (SF) be weakened to match (*LP)? An obvious way to weaken (SF) would be to require only individual spheres of S (and not arbitrary unions of them) to be elementary. This is essentially what condition (EL) below says, except that it restricts the elementarity request to *proper* spheres of S.

By a proper sphere we mean any sphere $V \in S$ that contains at least one world outside all spheres smaller than V. More precisely, for a sphere $V \in S$, we define the *core* of V, denoted V^c, to be set $V^c = \bigcup\{U \in S : U \subset V\}$. We shall say that a sphere $V \in S$ is *proper* iff $V \neq V^c$. Notice that a non-proper sphere U is a *degenerate* sphere as far as multiple revision is concerned, in the sense that there is no set of sentences Γ such that $U = c(\Gamma)$. It is easy to show that all such non-proper spheres can be removed from a system of spheres S without affecting the induced multiple revision function. Hence the restriction to proper spheres in condition (EL) below:

(EL) All proper spheres in S are elementary.

Condition (EL) is a natural constraint on a system of spheres S that can be justified along similar lines to (SD).[11] Intuitively a proper sphere V of S can be understood as a "fallback" position that the agent can retreat to if her initial belief set K is challenged (see for example [Rott 2004]). All possible worlds outside V are strictly less plausible than those in V. Suppose now that V is not elementary. Then the agent's fallback theory $K' = \bigcap V$ admits a world z that is strictly less plausible than what is necessitated by the her retreat to V (i.e. $z \in [K']$ and yet $z \notin V$); clearly an undesired side-effect.

Unfortunately, despite its intuitive appeal, there is a mismatch between (EL) and (*LP):

Lemma 3. *There exists a consistent theory K and a well ranked system of spheres S centered on $[K]$ such that S satisfies (EL) and yet the multiple revision function $*$ induced from S violates (*LP).*

Proof. Let L be the language produced from the boolean connectives over the propositional variables q, p_0, p_1, p_2, ... Define the theories $Y_0, Y_1, Y_2, \ldots$ as follows:[12]

$$
\begin{aligned}
Y_0 &= Cn(\{q, p_0, p_1, \ldots\})\\
Y_1 &= Cn(\{q, p_1, p_2, \ldots\})\\
&\vdots\\
Y_j &= Cn(\{q, p_j, p_{j+1}, \ldots\})\\
&\vdots
\end{aligned}
$$

Let $K = Y_0$ and define S to be the system of spheres $S = \{[Y_i] : i \in \mathbb{N}_0\} \cup \{\mathcal{M}_L\}$. It is not hard to verify that S is indeed a well ranked system of spheres centered at $[K]$ (the argument is exactly the same as the one in the proof of Theorem 3). Moreover by construction, all spheres in S are elementary and therefore (EL) is satisfied. Next we show that the revision function $*$ induced from S at K violates (*LP).

Let Γ be the set $\Gamma = \{\neg q \vee \neg p_0, \neg q \vee \neg p_1, \neg q \vee \neg p_2, \ldots\}$. It is not hard to verify that $[\Gamma] = [\neg q] \cup [\{\neg p_0, \neg p_1, \neg p_2, \ldots\}]$. Hence, the smallest sphere intersecting Γ is $\mathcal{M}_L$, which by construction, is also the smallest sphere intersecting $[\neg q]$, i.e. $c(\Gamma) = c(\neg q) = \mathcal{M}_L$. Consequently, $q \notin K * \Gamma$. To prove that $*$ violates (*LP) it suffices to show that $q \in \bigcup_{A \subseteq_f Cn(\Gamma)} K * A$ (this would violate (rLP) which as already stated, is equivalent to (*LP)).

Consider any finite subset A of $Cn(\Gamma)$. Then by compactness there exist finitely many variables $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$ with $i_1 < i_2 < \cdots < i_n$, such that $\{\neg q \vee \neg p_{i_1}, \neg q \vee \neg p_{i_2}, \ldots, \neg q \vee \neg p_{i_n}\} \vdash A$, or equivalently, $\neg q \vee (\neg p_{i_1} \wedge \neg p_{i_2} \wedge \ldots \wedge \neg p_{i_n}) \vdash A$. It is not hard to see that the theory Y_{i_n+1} is compatible with $\neg q \vee (\neg p_{i_1} \wedge \neg p_{i_2} \wedge \ldots \wedge \neg p_{i_n})$ and therefore it is also compatible with A. Consequently the smallest sphere intersecting $[A]$ is no larger

[11] Notice that (EL) entails (SD).

[12] $Y_0, Y_1, Y_2, \ldots$ are essentially extensions by the new variable q, of the theories $T_0, T_1, T_2, \ldots$ respectively in the proof of Theorem 3.

than $[Y_{i_n+1}]$; i.e. $c(A) \subseteq [Y_{i_n+1}]$. Given that, by construction, all worlds in $[Y_{i_n+1}]$ satisfy q we then derive that $q \in K * A$. Since A was chosen arbitrarily, it then follows that $q \in \bigcup_{A \subseteq_f Cn(\Gamma)} K * A$. Hence (rLP) is violated, and consequently so is (*LP). ■

The above result proves that "vanilla" (EL) is not sufficient to characterize (*LP) while on the other hand (SF) is too strong. The notion of an elementary set of worlds however remains central to the characterization. What is needed is something in between (EL) and (SF) (with a twist).

First one definition. Let K be a theory and S a system of spheres centered on $[K]$. We shall say that a sphere in V is *finitely reachable* in S iff there exists a consistent sentence $\varphi \in L$ such that $c(\varphi) = V$.

Consider now the following restrictions on a system of spheres S, where V is an arbitrary sphere in S:

(R1) If V is finitely reachable then V is elementary.

(R2) If V is finitely reachable then V^c is elementary.

(R3) If $V^c \neq V$ then $[\bigcap V^c] \subseteq V$.

Condition (R1) is a weaker version of (EL) since it requires only finitely reachable spheres to be elementary. Similarly, (R2) is a weaker version of (SF): only collections of spheres strictly smaller from a finitely reachable one are required to have an elementary union.

The last condition, (R3), is perhaps the most interesting of the three. Although it may not be apparent at first sight, (R3) is also a weaker version of (SF). Let V be a non-finitely-reachable sphere and consider the principle case where V is different from its core V^c. Moreover let z be any world in $[\bigcap V^c]$. Where can z be placed in the system of spheres S? Condition (SF) confines the location of z to V^c. Condition (R3) on the other hand is not as strict. It allows z to move away from V^c *as long as it doesn't go beyond* V; i.e. V is a boundary that no $\bigcap V^c$-world can pass. Hence although V^c may not be elementary as (SF) would require, it is, in a sense, *almost* elementary since any "runaway" world (i.e. any world in $[\bigcap V^c] - V^c$) doesn't go that far away after all; it stays within V.

Conditions (R1) - (R3) turn out to be the system-of-spheres counterpart of (*LP):

THEOREM 4. *Let K be a theory, S a well-ranked system of spheres centered at $[K]$, and $*$ the multiply revision function induced from S at K via (S*). Then $*$ satisfies (*LP) iff S satisfies (R1) - (R3).*

Proof.

$(\Rightarrow)$

Assume that (*LP) holds. We show that (R1) - (R3) are satisfied.

Starting with (R1), suppose towards contradiction that there is a sphere $V \in S$ that is finitely reachable but not elementary. Then there is an $y \in L$ such that $c(y) = V$ and there is a world $z \in [\bigcap V] - V$. Define Γ to be the set $\Gamma = \{x \vee y \in L : x \in z\}$. We derive a contradiction by showing that $\neg y \notin K * \Gamma$ and yet $\neg y \in \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$ (which of course violates (rLP) and therefore it violates (*LP)).

It is not hard to verify that $[\Gamma] = [y] \cup \{z\}$ and therefore, since $z \notin V$, $c(\Gamma) = V = c(y)$. This again entails that $\neg y \notin K * \Gamma$.

For the second part of the argument, consider an arbitrary finite subset A of $Cn(\Gamma)$. By compactness there exist $x_1, \cdots x_n \in z$ such that $\{(x_1 \vee y) \wedge \cdots \wedge (x_n \vee y)\} \vdash A$ and therefore $\{y \vee (x_1 \wedge \cdots \wedge x_n)\} \vdash A$. Define B to be the set $B = \{x_1, \cdots, x_n\}$. Clearly then, $B \subseteq_f z$ and $c(A) \subseteq c(B)$. Now observe that to prove $\neg y \in K * A$ it suffices to show that $c(B) \subset V$.[13] Clearly, since $B \subseteq_f z$ and $z \in [\bigcap V]$, there is a B-world in V and therefore $c(B) \subseteq V$. Hence from $z \notin V$ we derive that $z \notin c(B) \cap [B]$, and consequently from (SD) it follows that z contradicts $K * B$. Therefore there is a $\varphi \in z$ such that $\neg\varphi \in K * B$. Consider now the set $B \cup \{\varphi\}$. Clearly $B \cup \{\varphi\} \subseteq_f z$, and therefore, given that $z \in [\bigcap V]$, $c(B \cup \{\varphi\}) \subseteq V$. Moreover, since $\neg\varphi \in K * B$ it follows that $c(B) \subset c(B \cup \{\varphi\})$. Consequently, $c(B) \subset V$, as desired. This again, as mentioned above, proves that $\neg y \in K * A$.

We have thus shown that $\neg y \notin K * \Gamma$ and yet for all finite subsets A of $Cn(\Gamma)$, $\neg y \in K * A$. This clearly violates (rLP) and therefore it violates (*LP). Hence (R1) is satisfied.

For (R2), let V be any finitely reachable sphere in S. Then for some consistent $\varphi \in L$, $c(\varphi) = V$. Assume towards contradiction that the core of V is not elementary and consequently there is a $z \in [\bigcap V^c]$ such that $z \notin V^c$. Let Δ be the set $\Delta = \{\varphi \vee x : x \in z\}$. It is not hard to verify that $[\Delta] = [\varphi] \cup \{z\}$, and therefore $c(\Delta) = c(\varphi)$, which by the construction of Δ entails that $\neg\varphi \notin K * \Delta$. Then by (rLP) it follows that there is a finite subset A of $Cn(\Delta)$ such that $\neg\varphi \notin K * A$, which again entails that $c(\varphi) \subseteq c(A)$. Moreover, by the construction of Δ and compactness we derive that there is a $x \in z$, such that $\{\varphi \vee x\} \vdash A$. Therefore $c(\varphi) \subseteq c(A) \subseteq c(\{\varphi \vee x\})$. This again entails that $V \subseteq c(\{x\})$. Hence all worlds in the core of V are $\neg x$-worlds and therefore $\neg x \in \bigcap V^c$. This however contradicts our assumption that $z \in [\bigcap V^c]$. Hence (R2) holds.

Finally for (R3), assume on the contrary that there is a sphere $V \in S$ such that $V^c \subset V$ and yet for some $z \in [\bigcap V^c]$, $z \notin V$. Let w be any world in $V - V^c$ and let Γ be the set, $\Gamma = \{x \vee y : x \in w \text{ and } y \in z\}$. It is not hard to see that $[\Gamma] = \{w, z\}$ and therefore $c(\Gamma) = V$, which again entails $K * \Gamma = w$.

Next we show that $\bigcap_{A \subseteq_f Cn(\Gamma)} K * A \subseteq Cn(\Gamma)$. Assume on the contrary that there is a $\varphi \in \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$ such that $\varphi \notin Cn(\Gamma)$. By (rLP) and $K * \Gamma = w$ we derive that $\varphi \in w$. Therefore, since $\varphi \notin Cn(\Gamma)$, from the construction of Γ it follows that $\neg\varphi \in z$. Applying (rLP) at $K * z$ we get, $K * z = z = z + \bigcap_{B \subseteq_f z} K * B$. Therefore from $\neg\varphi \in z$ it follows that there is a $B \subseteq_f z$ such that $\varphi \notin K * B$. Notice that since $B \subseteq_f z$ and $z \in [\bigcap V^c]$, there is at least one B-world in V^c, and consequently $c(B) \subset V = c(\Gamma)$. This, combined with the fact that $c(B)$ is elementary – recall that we have already shown (R1) – gives us that Γ contradicts $\bigcap c(B)$. Hence by compactness there is a $y \in Cn(\Gamma)$ such that $\neg y \in \bigcap c(B)$.

[13] For in this case, $c(A) \subset V = c(\varphi)$, and therefore all worlds in $c(A)$ entail $\neg y$.

Consider now the sentence $(\wedge B) \vee y$. Clearly $(\wedge B) \vee y \in Cn(\Gamma)$. Moreover it is not hard to verify that, because $\neg y \in \bigcap c(B)$, $c(\{(\wedge B) \vee y\}) = c(B)$, and therefore $K * \{(\wedge B) \vee y\} = K * B$. Hence $\varphi \notin K * \{(\wedge B) \vee y\}$, which however contradicts $\varphi \in \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$. Thus $\bigcap_{A \subseteq_f Cn(\Gamma)} K * A \subseteq Cn(\Gamma)$ as desired.

From $\bigcap_{A \subseteq_f Cn(\Gamma)} K * A \subseteq Cn(\Gamma)$ we get that $\Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K * A = Cn(\Gamma)$, and given that $Cn(\Gamma) \subset w = K * \Gamma$, we derive $\Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K * A \subset K * \Gamma$, which of course contradicts (rLP). Thus (R3) is satisfied.

$(\Leftarrow)$

Assume that (R1) - (R3) are true and let Γ be an arbitrary nonempty consistent set of sentences. If $c(\Gamma)$ is finitely reachable, then by (R2), the core of $c(\Gamma)$, call it V, is elementary. Hence by compactness there is a $A \subseteq_f Cn(\Gamma)$ such that $\neg(\wedge A) \in \bigcap V$. This again entails that $c(A) = c(\Gamma)$, which again entails that $K * A$ is consistent with Γ, and consequently by Lemma 1, (*LP) holds.

Assume therefore that $c(\Gamma)$ is not finitely reachable. First we show that $K * \Gamma \subseteq \Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$, or equivalently that $[\Gamma] \cap [\bigcap_{A \subseteq_f Cn(\Gamma)} K * A] \subseteq [K * \Gamma]$. Given that $[K * \Gamma] = c(\Gamma) \cap [\Gamma]$, it suffices to show that $[\bigcap_{A \subseteq_f Cn(\Gamma)} K * A] \subseteq c(\Gamma)$. Notice that all Γ-worlds in $c(\Gamma)$ are outside its core; i.e. $c(\Gamma)^c \subset c(\Gamma)$. Moreover it is not hard to verify that $[\bigcap_{A \subseteq_f Cn(\Gamma)} K * A] = [\bigcap(\bigcup_{A \subseteq_f Cn(\Gamma)} [K * A])]$. Next observe that, since $c(\Gamma)$ is not finitely reachable in S, $c(A) \subset c(\Gamma)$ for all $A \subseteq_f Cn(\Gamma)$. Then given that, by (SD), $c(A) \cap [A]$ is elementary, $[K * A]$ is always a subset of the core of $c(\Gamma)$, and consequently, $\bigcup_{A \subseteq_f Cn(\Gamma)} [K * A] \subseteq c(\Gamma)^c$. This again entails that $[\bigcap(\bigcup_{A \subseteq_f Cn(\Gamma)} [K * A])] \subseteq [\bigcap c(\Gamma)^c]$ and therefore by (R3), $[\bigcap_{A \subseteq_f Cn(\Gamma)} K * A] \subseteq c(\Gamma)$ as desired. This proves that $K * \Gamma \subseteq \Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$.

To conclude the proof we also need to show the converse. Consider therefore any sentence y in $\Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$. By compactness there is a $x \in Cn(\Gamma)$ such that $(x \rightarrow y) \in \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$. We next show that all worlds in $c(\Gamma)$ satisfy $(x \rightarrow y)$. Assume towards contradiction that there is a $(x \wedge \neg y)$-world in $c(\Gamma)$. Then, since we have assumed that $c(\Gamma)$ is not finitely reachable in S, $c(\{x \wedge \neg y\}) \subset c(\Gamma)$, which again entails by (R1) that $\bigcap c(\{x \wedge \neg y\})$ contradicts Γ. Consequently, by compactness, there is a $\varphi \in Cn(\Gamma)$ such that $\neg\varphi \in \bigcap c(\{x \wedge \neg y\})$. Consider now the sentence $\varphi \vee (x \wedge \neg y)$. Clearly $\varphi \vee (x \wedge \neg y) \in Cn(\Gamma)$ and moreover $c(\{\varphi \vee (x \wedge \neg y)\}) = c(\{x \wedge \neg y\})$. From this it follows that $K * \{\varphi \vee (x \wedge \neg y)\} = K * \{x \wedge \neg y\}$, and therefore, $(x \wedge \neg y) \in K * \{\varphi \vee (x \wedge \neg y)\}$, which of course contradicts $(x \rightarrow y) \in \bigcap_{A \subseteq_f Cn(\Gamma)} K * A$. Hence $(x \rightarrow y) \in \bigcap c(\Gamma)$ and therefore $y \in K * \Gamma$ as desired. ■

The last result of this section relates to condition (EL). As already noted, neither (K*F) nor (*LP) corresponds precisely to (EL). Yet condition (EL) is an intuitive constraint on systems-of-spheres and therefore a question that naturally poses itself is to identify its multiple-revision counterpart.

Let $\Gamma, \Delta \subseteq L$ be any two nonempty and consistent sets of sentences. We define $\Gamma \vee \Delta$ to be the set $\Gamma \vee \Delta = \{x \vee y \in L\colon x \in \Gamma \text{ and } y \in \Delta\}$. Consider condition (*CM) below:

(*CM) If $(K * (\Gamma \vee \Delta)) + \Gamma \vdash \bot$ then there is a $A \subseteq_f \Gamma$ such that $(K * (A \vee \Delta)) + A \vdash \bot$.

Condition (*CM) can be thought of the counterpart of *compactness* in the context of multiple belief revision. Essentially (*CM) says that if Γ is less plausible than Δ (as indicated by the fact that Γ is inconsistent with $K * (\Gamma \vee \Delta)$), then there is a finite subset A of Γ that is less plausible than Δ (i.e. A is inconsistent with $K * (A \vee \Delta)$). Theorem 5 shows that (*CM) is an exact axiomatization of (EL).

THEOREM 5. *Let K be a theory, S a well-ranked system of spheres centered at $[K]$, and $*$ the multiply revision function induced from S at K via (S*). Then $*$ satisfies (*CM) iff S satisfies (EL).*

Proof.

$(\Rightarrow)$

Assume $*$ satisfies (*CM) and let V be any proper sphere in S. If V is empty or $V = \mathcal{M}_L$ then clearly V is elementary. Assume therefore that $\emptyset \neq V \subset \mathcal{M}_L$, and let z be any world outside V; i.e. $z \in \mathcal{M}_L - V$. To prove that V is elementary it suffices to show that $z \notin [\bigcap V]$. Since V is proper, there is a world $w \in V - V^c$ (recall that V^c denotes the core of V), or in other words, $c(w) = V$. Then $c(w) \subset c(z)$ and consequently, z is inconsistent with $K * (z \vee w)$. By (*CM) we then derive that there is an $A \subseteq_f z$ such that A is inconsistent with $K * (A \vee w)$, which again entails that $c(w) \subset c(A)$. Consequently, all worlds in V contain $\neg(\wedge A)$, or equivalently, $\neg(\wedge A) \in \bigcap V$. Hence, since $A \subseteq_f z$, we derive that $z \notin [\bigcap V]$ as desired.

$(\Leftarrow)$

Assume that S satisfies (EL) and let $\Gamma, \Delta \subseteq L$ be any two nonempty and consistent sets of sentences. Moreover assume that Γ is inconsistent with $K * (\Gamma \vee \Delta)$. Given that $[\Gamma \vee \Delta] = [\Gamma] \cup [\Delta]$, it is not hard to verify that $c(\Gamma \vee \Delta) = c(\Delta) \subset c(\Gamma)$. Moreover, notice that since $K * (\Gamma \vee \Delta)$ is consistent, $c(\Delta) - c(\Delta)^c$ is nonempty and therefore $c(\Delta)$ is proper. Hence by (EL), $c(\Delta)$ is elementary. Consequently, $[\bigcap c(\Delta)] \cap [\Gamma] = \emptyset$. By compactness we then derive that there is an $A \subseteq_f \Gamma$ such that $\neg(\wedge A) \in \bigcap c(\Delta)$. Consider now that the set $A \vee \Delta$. Clearly, $c(A \vee \Delta) = c(\Delta) \subset c(A)$. Therefore, A is inconsistent with $K * (A \vee \Delta)$ as desired.
■

6. SET CONTRACTION REVIEW

So far we have addressed two of the three open problems mentioned in the introduction. The rest of the paper is devoted to the third problem; i.e. to provide a characterization of ($\dot{-}$LP) in the partial meet model. In this section we review the necessary literature on set contraction and the partial meet model, and the next section contains our representation result for ($\dot{-}$LP).

Zhang and Foo, [Zhang and Foo 2001], define set contraction as a function $\dot{-} : \mathcal{K}_L \times 2^L \mapsto \mathcal{K}_L$, mapping $\langle K, \Gamma \rangle$ to $K \dot{-} \Gamma$, that satisfies the following postulates:

($K\dot{-}1$) $K \dot{-} \Gamma$ is a theory of L.

($K\dot{-}2$) $K \dot{-} \Gamma \subseteq K$.

($K\dot{-}3$) If Γ is consistent with K then $K\dot{-}\Gamma = K$.
($K\dot{-}4$) If Γ is consistent, then Γ is consistent with $K\dot{-}\Gamma$.
($K\dot{-}5$) If $\varphi \in K$ and $\Gamma \vdash \neg\varphi$ then $K \subseteq (K\dot{-}\Gamma) + \varphi$.
($K\dot{-}6$) If $Cn(\Gamma) = Cn(\Delta)$ then $K\dot{-}\Gamma = K\dot{-}\Delta$.
($K\dot{-}7$) If $\Gamma \subseteq \Delta$ then $K\dot{-}\Delta \subseteq (K\dot{-}\Gamma) + \Delta$.
($K\dot{-}8$) If $\Gamma \subseteq \Delta$ and Δ is consistent with $K\dot{-}\Gamma$, then $K\dot{-}\Gamma \subseteq K\dot{-}\Delta$.

We note the different aims of set contraction and AGM sentence contraction: in the first case the initial belief set K is contracted in order to *become consistent* with the epistemic input (encoded as a set of sentences Γ), while in the latter, K is contracted so that it *fails to entail* the epistemic input (represented as a single sentence φ).

Like with multiple revision, we shall focus only on set contraction by *nonempty and consistent* sets of sentences.

The constructive model for set contraction we shall consider herein is the *partial meet* model. This model is based on the notion of a *remainder* of a belief set. More precisely, let K be a theory and Γ a nonempty consistent set of sentences. A *remainder* of K with respect to Γ, also called a Γ-*remainder* for short, is any maximal subset of K that is consistent with Γ, [Zhang and Foo 2001]; the set of all Γ-remainders is denoted by $K \perp\!\!\!\perp \Gamma$. By $\mathcal{R}_K$ we shall denote the set of all remainders of K with respect to any nonempty consistent Γ; i.e. $\mathcal{R}_K = \bigcup\{K \perp\!\!\!\perp \Gamma : \emptyset \neq \Gamma \subseteq L \text{ and } \Gamma \nvdash \bot\}$.

Consider now a preorder $\preccurlyeq$ in $\mathcal{R}_K$. For any nonempty set of remainders $\Phi \subseteq \mathcal{R}_K$, by $max_{\preccurlyeq}(\Phi)$ we shall denote the maximal elements of Φ with respect to $\preccurlyeq$, i.e. $max_{\preccurlyeq}(\Phi) = \{H \in \Phi : \text{for all } D \in \Phi, D \preccurlyeq H\}$. When the underlying preorder $\preccurlyeq$ is understood from the context, we shall drop the index $\preccurlyeq$ from *max*.

A preorder $\preccurlyeq$ on $\mathcal{R}_K$ essentially encodes preference between remainders with the better remainders appearing higher in the preorder. Given this reading, the partial meet model defines the (set) contraction of K by Γ as the theory resulting from the intersection of the best Γ-remainders:

(SC) $K\dot{-}\Gamma = \bigcap max(K \perp\!\!\!\perp \Gamma)$

It turns out that the functions induced by (SC) are a superset of those satisfying the postulates for set contraction ($K\dot{-}1$) - ($K\dot{-}8$). To obtain an *exact* match between the two, two extra constraints are needed on $\preccurlyeq$. The first guarantees that the set $max(K \perp\!\!\!\perp \Gamma)$ is always well defined:

($\preccurlyeq$ 1) $K \perp\!\!\!\perp \Gamma$ has a maximal element.

For the second constraint we need an extra definition. We define the *closure* of a set of remainders $\Phi \subseteq \mathcal{R}_K$, denoted $[\![\Phi]\!]$, to be the set $[\![\Phi]\!] = \{H \in \mathcal{R}_K : \bigcap \Phi \subseteq H\}$. The second constraint on $\preccurlyeq$ requires that $max(K \perp\!\!\!\perp \Gamma)$ is always equal to its closure:

($\preccurlyeq$ 2) $max(K \perp\!\!\!\perp \Gamma) = [\![max(K \perp\!\!\!\perp \Gamma)]\!]$.

In [Zhang and Foo 2001], it was shown that contraction functions generated from total preorders $\preccurlyeq$ satisfying ($\preccurlyeq$ 1) - ($\preccurlyeq$ 2) via (SC), coincide precisely with the class of function satisfying the postulates ($K\dot{-}1$) - ($K\dot{-}8$).[14] In addition to these results, Zhang and Foo proved that the well known relationships between sentence revision and contraction, described by the Levi and Harper Identities (see [Gardenfors 1988]), also hold (with adequate modifications) between multiple revision and set contraction. In particular, consider the following conditions:

$$K * \Gamma = (K\dot{-}\Gamma) + \Gamma \qquad \text{(Generalized Levi Identity)}$$

$$K\dot{-}\Gamma = K \cap (K * \Gamma) \qquad \text{(Generalized Harper Identity)}$$

It was shown in [Zhang and Foo 2001] that every set contraction function satisfying ($K\dot{-}1$) - ($K\dot{-}8$), induces via the Generalized Levi Identity a multiple revision function satisfying ($K * 1$) - ($K * 8$); conversely, every multiple revision function satisfying ($K * 1$) - ($K * 8$), produces via the Generalized Harper Identity a set contraction function satisfying ($K\dot{-}1$) - ($K\dot{-}8$). In fact, the relationship between multiple revision and set contraction is even stronger: for any set contraction function $\dot{-}$, the successive application of the Generalized Levi and Harper Identities leads us back to $\dot{-}$ itself.[15] The same is true for multiple revision: starting with a multiple revision function $*$ one makes a full circle back to $*$ when successively applying the Generalized Harper and Levi Identities.

A final result reported in [Zhang and Foo 2001] on the relationship between multiple revision and set contraction is related to the limit postulate(s). Consider any set contraction function $\dot{-}$ and let $*$ be the multiple revision function produced from $\dot{-}$ via the Generalized Levi Identity. Zhang and Foo proved that $*$ satisfying ($*LP$) iff $\dot{-}$ satisfies ($\dot{-}LP$).

We conclude our review of set contraction with a final note on the relationship between set contractions and preorders on remainders. Observe that for each set contraction function $\dot{-}$ there is in principle more than one preorder on remainders $\preccurlyeq$ corresponding to $\dot{-}$ via (SC). To see this notice that the location of the initial belief set K in $\preccurlyeq$ is irrelevant as far as the induced set contraction function $\dot{-}$ is concerned: if $K \in K \perp\!\!\!\perp \Gamma$ then K is the *only* remainder in $K \perp\!\!\!\perp \Gamma$ and therefore $K\dot{-}\Gamma = K$ regardless of K's location in $\preccurlyeq$. Hence for a given preorder $\preccurlyeq$ in $\mathcal{R}_K$, there is a whole family of preorders $\preccurlyeq'$, that differ from $\preccurlyeq$ only in the relative location of K, all of which give rise to the same set contraction function $\dot{-}$. We shall call *canonical* the member of this family that places K at the very top. More precisely, we shall say that a preorder $\preccurlyeq$ in $\mathcal{R}_K$ is *canonical* iff it is total, it satisfies ($\preccurlyeq$1) - ($\preccurlyeq$2), and moreover, for all $K' \in \mathcal{R}_K$, $K' \preccurlyeq K$. Based on the preceding discussion it is not hard to verify that canonical preorders suffice to generate *all* set contraction functions $\dot{-}$ satisfying ($K\dot{-}1$) - ($K\dot{-}8$). Hence assuming canonicity for $\preccurlyeq$ comes at no cost to generality, and this is what we will do in the next section to simplify the presentation of our results.

[14] To be precise, the results in [Zhang and Foo 2001] were stated slightly differently. Most importantly, in the original version, condition ($\preccurlyeq$ 1), connectivity, reflexivity, and totality of $\preccurlyeq$ were all tacitly assumed; only ($\preccurlyeq$ 2) was stated explicitly. Nevertheless, the two versions are equivalent.

[15] The proof is not explicitly stated in [Zhang and Foo 2001] but it follows immediately from the proof of Proposition 4.14.

7. THE LIMIT POSTULATE IN THE PARTIAL MEET MODEL

In the partial meet model, remainders play a role similar to that of possible worlds in the system of spheres model. This view is further supported by the following result.

LEMMA 4. *Let K be a theory. For any remainder $H \in \mathcal{R}_K$ there is a possible world $z \in \mathcal{M}_L$ such that $H = K \cap z$. Conversely, for any $z \in \mathcal{M}_L$, $K \cap z \in \mathcal{R}_K$.*

Proof. Let H be any remainder in $\mathcal{R}_K$. Then there is a nonempty consistent set of sentences Γ such that $H \in K \perp\!\!\!\perp \Gamma$. If $H = K$ then for any $z \in [K]$, $H = K \cap z$. Assume therefore that $H \neq K$. By definition, H is consistent with Γ, and therefore there exists at least one world $z \in [H] \cap [\Gamma]$. Clearly, since $H \subseteq K$ and $z \in [H]$, $H \subseteq K \cap z$. Moreover notice that $K \cap z$ is a subset of K that is consistent with Γ. Since H is a *maximal* such subset of K, it follows that $K \cap z$ can not be larger than H. Thus $H = K \cap z$ as desired.

For the second part of the lemma, let $z \in \mathcal{M}_L$ be any possible world. Call H the theory $H = K \cap z$. We show that $H \in K \perp\!\!\!\perp z$. Clearly H is a subset of K that is consistent with z. Hence all we need to show is that H is a maximal such subset. This however follows easily from the construction of H: for any sentence $y \in (K - z)$, $\neg y \in z$, and therefore $H \cup \{y\}$ is inconsistent with z. ■

Given this strong connection between possible worlds and remainders, it should not be surprising that the partial-meet characterization of ($\dot{-}$LP) resembles that of (*LP) in the system of spheres model.

First some extra definitions and notations. Let K be a theory and $\preccurlyeq$ a preorder in $\mathcal{R}_K$. We shall say that a remainder $H \in \mathcal{R}_K$ is *finitely accessible* (with respect to $\preccurlyeq$) iff there is a consistent sentence $\varphi \in L$ such that H is *maximal* in $K \perp\!\!\!\perp \{\varphi\}$ with respect to $\preccurlyeq$. Moreover, for any remainder $H \in \mathcal{R}_K$, by $H^{\preccurlyeq}$ we denote the set of all remainders that are greater or equal to H (wrt $\preccurlyeq$); i.e. $H^{\preccurlyeq} = \{D \in \mathcal{R}_K : H \preccurlyeq D\}$. Similarly, for the strict part of $\preccurlyeq$[16] we define $H^{\prec}$ to be the set $H^{\prec} = \{D \in \mathcal{R}_K : H \prec D\}$. Consider now the following conditions:

(PM1) If H is finitely accessible then $[\![H^{\preccurlyeq}]\!] \subseteq H^{\preccurlyeq}$.

(PM2) If $H \neq K$ and H is finitely accessible then $[\![H^{\prec}]\!] \subseteq H^{\prec}$.

(PM3) If $H \neq K$ then $[\![H^{\prec}]\!] \subseteq H^{\preccurlyeq}$.

To understand the intuition behind (PM1) - (PM3) one needs to look at the analogy between a world z in a system of spheres S on one hand, and a remainder H in a preorder $\preccurlyeq$ on the other. Given the connection between worlds and remainders as expressed by Lemma 4, it is not hard to see that a connection emerges between the smallest sphere V containing z, and $H^{\preccurlyeq}$: each set contains the items (worlds and remainders respectively) that are more or equally plausible to z and H respectively. With this reading in mind, it is not hard to

[16]The strict part $\prec$ of a preorder $\preccurlyeq$ is defined as follows: $x \prec y$ iff $x \preccurlyeq y$ and $y \not\preccurlyeq x$.

see that conditions (PM1) - (PM3) are directly analogous to conditions (R1) - (R3) and therefore can be justified on the same grounds.

A final note before presenting our last representation result. Like its analogue for multiple revision, condition ($\dot{-}$LP) – repeated below for convenience – is equivalent to a simpler condition (cLP):

($\dot{-}$LP) $\quad K\dot{-}\Gamma = \bigcup_{A\subseteq_f \Gamma} \bigcap_{B\subseteq_f Cn(\Gamma)} K\dot{-}(A \cup B)$

(cLP) $\quad K\dot{-}\Gamma = (K \cap Cn(\Gamma)) + \bigcap_{A\subseteq_f Cn(\Gamma)} K\dot{-}A$

In [Zhang and Foo 2001] it was shown that, in the presence of postulates ($K\dot{-}$1) - ($K\dot{-}$8), for set contraction, conditions ($\dot{-}$LP) and (cLP) are equivalent; this equivalence, like Lemma 4, will be used extensively in the proof of our final result. Equally useful will be the following two corollaries that are immediate consequences of Lemma 4:

Corollary 2. *Let K be a theory, and H, H' two distinct remainder in $\mathcal{R}_K$. If $H' \neq K$, then $H \nsubseteq H'$.*

Corollary 3. *Let K be a theory, H a remainder in $\mathcal{R}_K$, and Γ a nonempty consistent set of sentences such that $K + \Gamma \vdash \bot$. If Γ is consistent with H, then $H \in K \perp\!\!\!\perp \Gamma$.*

Theorem 6 below addresses the last open problem mentioned in the introduction. It provides a characterization of ($\dot{-}$LP) in the partial meet model:

Theorem 6. *Let K be a theory and $\preccurlyeq$ a canonical preorder in $\mathcal{R}_K$. The set contraction function $\dot{-}$ defined from $\preccurlyeq$ by means of (SC) satisfies ($\dot{-}$LP) iff $\preccurlyeq$ satisfies (PM1) - (PM3).*

Proof.

($\Rightarrow$)

Assume that $\dot{-}$ satisfies ($\dot{-}$LP), and thus the equivalent condition (cLP). Since $\preccurlyeq$ is canonical, from [Zhang and Foo 2001] it follows that $\dot{-}$ also satisfies ($K\dot{-}$1) - ($K\dot{-}$8).

For (PM1), assume that H is finitely accessible and let φ be a consistent sentence in L such that $H \in K \perp\!\!\!\perp \varphi$. If $H = K$ then $H^{\preccurlyeq} = \{K\}$ and therefore (PM1) trivially holds. Assume therefore that $H \neq K$. Then $\neg\varphi \in K$. Moreover assume towards contradiction that there is a $D \in \mathcal{R}_K$ such that $\bigcap H^{\preccurlyeq} \subseteq D$ and $D \prec H$. By Lemma 4, there is a world z such that $[D] = [K] \cup \{z\}$ and $z \notin [K]$. Consider now the set $\Gamma = \{\varphi \vee y : y \in z\}$. We will derive the desired contradiction by showing that on the one hand $\neg\varphi \notin K\dot{-}\Gamma$ and yet on the other hand $\neg\varphi \in \bigcap_{A\subseteq_f Cn(\Gamma)} K\dot{-}A$ (thus contradicting (cLP)).

To show $\neg\varphi \notin K\dot{-}\Gamma$ it suffices to show that $H \in max(K \perp\!\!\!\perp \Gamma)$. Clearly since H is consistent with φ, it is also consistent with Γ, and therefore by Corollary 3, $H \in K \perp\!\!\!\perp \Gamma$. Consider now any $H' \in K \perp\!\!\!\perp \Gamma$. If $\neg\varphi \notin H'$ then, by Corollary 3, $H' \in K \perp\!\!\!\perp \varphi$ and therefore, since H is maximal in $K \perp\!\!\!\perp \varphi$, $H' \preccurlyeq H$. Suppose on the other hand that $\neg\varphi \in H'$. Given that H' is consistent with Γ, it is not hard to verify that in this case $H' + \Gamma = z$, and therefore from Lemma 4 it follows that $H' = D$. Consequently once again we derive that $H' \preccurlyeq H$. Hence no remainder in $K \perp\!\!\!\perp \Gamma$ is strictly greater than H which makes H maximal in $K \perp\!\!\!\perp \Gamma$. Since H does not contain $\neg\varphi$ it then follows that $\neg\varphi \notin K \dot{-} \Gamma$.

For the second part of the argument, consider any $A \subseteq_f Cn(\Gamma)$, and let E be any remainder in $max(K \perp\!\!\!\perp A)$. We show that $\neg\varphi \in E$. If $E = K$ this is clearly true. Assume therefore that $E \neq K$, which again entails that $\neg(\wedge A) \in K$. Since H is consistent with Γ it is also consistent with A, and therefore (by Corollary 3), $H \in K \perp\!\!\!\perp A$, which again entails $H \preccurlyeq E$. Recall that $D \prec H$ and therefore, $D \notin max(K \perp\!\!\!\perp A)$. Since $max(K \perp\!\!\!\perp A)$ is closed (because of the completeness of $\preccurlyeq$), this entails that there is a $y \in \bigcap max(K \perp\!\!\!\perp A)$ such that $y \notin D$. Hence, one can easily verify, $\neg y \in z$. Given that $z \models A$ it follows that $\neg((\wedge A) \wedge \neg y) \notin D$. From $\bigcap H^{\preccurlyeq} \subseteq D$ we then derive that for some $G \in H^{\preccurlyeq}$, $\neg((\wedge A) \wedge \neg y) \notin G$, and therefore $G \in K \perp\!\!\!\perp (A \cup \{\neg y\})$. Consider now any $G' \in max(K \perp\!\!\!\perp (A \cup \{\neg y\})$. By the construction of G it follows that $H \preccurlyeq G \preccurlyeq G'$. Moreover, since G' is consistent with A, by Corollary 3 it follows that $G' \in K \perp\!\!\!\perp A$. Yet, since $y \in \bigcap max(K \perp\!\!\!\perp A)$ and G' is compatible with $\neg y$, we derive that $G' \notin max(K \perp\!\!\!\perp A)$. Putting together the above it follows that $H \preccurlyeq G' \prec E$, and therefore $H \prec E$. Since H is maximal in $K \perp\!\!\!\perp \varphi$ this entails that E contains $\neg\varphi$. Since E was chosen arbitrarily, it follows that all remainders in $max(K \perp\!\!\!\perp A)$ contain $\neg\varphi$ and therefore $\neg\varphi \in K \dot{-} A$. Since A was chosen as an arbitrary finite subset of $Cn(\Gamma)$ it follows that $\neg\varphi \in \bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A$. Earlier however we have shown that $\neg\varphi \notin K \dot{-} \Gamma$. Combined with (cLP) we derive a contradiction.

For (PM2), let $H \in \mathcal{R}_K$ be a finitely accessible remainder different from K, and let $D \in \mathcal{R}_K$ be such that $D \preccurlyeq H$. We will show that $D \notin [\![H^{\prec}]\!]$.

If $D \prec H$ this follows trivially from (PM1) proved above (observe that $[\![H^{\preccurlyeq}]\!] \subseteq [\![H^{\prec}]\!]$). Assume therefore that $H \preccurlyeq D$. Next we show that there is a finite set of sentences A such that $D \in max(K \perp\!\!\!\perp A)$ (i.e. D is finitely accessible).

Since H is finitely accessible, there is a consistent $\varphi \in L$, such that $\neg\varphi \in K$ and $H \in max(K \perp\!\!\!\perp \varphi)$. If $\neg\varphi \notin D$ then from Corollary 3, and $H \preccurlyeq D$, it follows that $D \in max(K \perp\!\!\!\perp \varphi)$, and therefore D is also finitely accessible. Assume therefore $\neg\varphi \in D$. Since $D \in \mathcal{R}_K$, by Lemma 4, there is a $z \in \mathcal{M}_L - [K]$ such that $[D] = [K] \cup \{z\}$. It is not hard to verify that D is the *only* remainder in $K \perp\!\!\!\perp z$. Define Γ to be the set $\Gamma = \{\varphi \vee x : x \in z\}$. Clearly Γ is inconsistent with K and consistent with both H and D. Therefore from Corollary 3, $H, D \in K \perp\!\!\!\perp \Gamma$. We next show that D is maximal in $K \perp\!\!\!\perp \Gamma$. Consider any $H' \in K \perp\!\!\!\perp \Gamma$. If $\neg\varphi \notin H'$ then from Corollary 3, $H' \in K \perp\!\!\!\perp \varphi$ and therefore $H' \preccurlyeq H$. Hence from $H \preccurlyeq D$ and transitivity, $H' \preccurlyeq D$. On the other hand, if $\neg\varphi \in H'$, then the consistency of H' with Γ entails that H' is consistent with z, and therefore $H' \in K \perp\!\!\!\perp z$. Given that D is the only remainder in $K \perp\!\!\!\perp z$, it follows that $H' = D$ and reflexivity entails that, once again, $H' \preccurlyeq D$. Hence D is indeed maximal in $K \perp\!\!\!\perp \Gamma$, and therefore, from $D \preccurlyeq H$, so is H. Given that $\neg\varphi \notin H$, this entails that $\neg\varphi \notin K \dot{-} \Gamma$. From (cLP) it then clearly follows that $\neg\varphi \notin \bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A$. Hence there is a $A \subseteq_f Cn(\Gamma)$ such that $\neg\varphi \notin K \dot{-} A$. This set A is the one that makes D finitely accessible. Indeed, since $\neg\varphi \notin K \dot{-} A$, there is a $H' \in max(K \perp\!\!\!\perp A)$ such that $\neg\varphi \notin H'$. Moreover, since D is consistent with Γ, D is also consistent with A and therefore by Corollary 3, $D \in K \perp\!\!\!\perp A$. Finally, since H' is consistent with φ, Corollary 3 entails that $H' \in K \perp\!\!\!\perp \varphi$, and therefore $H' \preccurlyeq H \preccurlyeq D$. Hence, by transitivity, D is maximal in

$K \perp\!\!\!\perp A$ as desired.

Consider now any $E \in H^{\prec}$. From $D \preccurlyeq H$ we derive that $D \prec E$. Then, since $D \in max(K \perp\!\!\!\perp A)$, we derive that $\neg(\wedge A) \in E$. Since E was chosen arbitrarily, it follows that $\neg(\wedge A) \in \bigcap H^{\prec}$. Given that D is consistent with A we then derive that $D \notin [\![H^{\prec}]\!]$ as desired.

For (PM3), assume towards contradiction that there are $H, D \in \mathcal{R}_K$ such that $H \neq K$, $\bigcap H^{\prec} \subseteq D$, and $D \notin H^{\preccurlyeq}$. Clearly then $D \prec H \prec K$. Moreover, from Lemma 4 it follows that there exist $z, u \in \mathcal{M}_L - [K]$ such that $[H] = [K] \cup \{z\}$ and $[D] = [K] \cup \{u\}$. Define Γ to be the set $\Gamma = \{x \vee y : x \in z \text{ and } y \in u\}$. We will derive the desired contradiction by shown that there is a sentence $\varphi \in K \dot{-} \Gamma$ such that $\varphi \notin ((K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A)$ (thus contradicting (cLP)).

It is not hard to verify that $K \perp\!\!\!\perp \Gamma = \{D, H\}$ and therefore, since $D \prec H$, $K \dot{-} \Gamma = H$. Hence, since $H \nsubseteq D$ (by Corollary 2), there is a $\varphi \in K \dot{-} \Gamma$ such that $\varphi \notin D$. With the aid of Lemma 4 it is not hard to see that the only remainder in $\mathcal{R}_K$ compatible with u is D, i.e. $K \perp\!\!\!\perp u = \{D\}$, and therefore $K \dot{-} u = D$. Hence from $\varphi \notin D$ and (cLP) it follows that $\varphi \notin (K \cap u) + \bigcap_{B \subseteq_f u} K \dot{-} B$. Consequently by compactness, for any $A \subseteq_f K \cap u$, $(\wedge A \rightarrow \varphi) \notin \bigcap_{B \subseteq_f u} K \dot{-} B$. Consider an arbitrary such $A \subseteq_f K \cap u$. Then there is a $B \subseteq_f u$ such that $(\wedge A \rightarrow \varphi) \notin K \dot{-} B$. Since D is compatible with u, it is also compatible with B, and therefore $\neg(\wedge B) \notin D$. Hence, from $\bigcap H^{\prec} \subseteq D$ we derive that there is an $E \in H^{\prec}$ such that E is compatible with B. This again entails that all maximal elements of $K \perp\!\!\!\perp B$ are in $H^{\prec}$. Consider now any $E \in max(K \perp\!\!\!\perp B)$. Clearly $H \prec E$ and since E is finitely accessible, by (PM1) – which we have already shown to be true – we derive that $E^{\preccurlyeq} = [\![E^{\preccurlyeq}]\!]$ and therefore $\bigcap E^{\preccurlyeq} \nsubseteq H$. Hence there is a $y \in \bigcap E^{\preccurlyeq}$ such that $y \notin H$. Since all remainders in $E^{\preccurlyeq}$ contain y and E is a maximal B-remainder, it is not hard to see that $max(K \perp\!\!\!\perp \{(\wedge B) \vee \neg y\}) = max(K \perp\!\!\!\perp B)$. Therefore $K \dot{-} \{(\wedge B) \vee \neg y\} = K \dot{-} B$ and consequently, $(\wedge A \rightarrow \varphi) \notin K \dot{-} \{(\wedge B) \vee \neg y\}$. We are only one step away from contradiction. Notice that since $y \in \bigcap E^{\preccurlyeq}$, it follows $y \in K$ and therefore from $y \notin H$ we derive that $\neg y \in z$ and consequently $(\wedge B) \vee \neg y \in Cn(\Gamma)$. Hence $(\wedge A \rightarrow \varphi) \notin \bigcap_{C \subseteq_f Cn(\Gamma)} K \dot{-} C$. Since A was chosen as an arbitrary finite subset of $K \cap u$, we derive that $\varphi \notin ((K \cap u) + \bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A$. Finally, since $Cn(\Gamma) \subseteq u$ it clearly follows that $\varphi \notin ((K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A$. This, together with $\varphi \in K \dot{-} \Gamma$, contradicts (cLP).

$(\Leftarrow)$

Assume that $\preccurlyeq$ satisfies (PM1) - (PM3). If Γ is compatible with K then clearly $K \dot{-} \Gamma = K$ and for all $A \subseteq_f Cn(\Gamma)$, $K \dot{-} A = K$, from which (cLP) trivially follows. Assume therefore that Γ is inconsistent with K. We distinguish between two cases, depending on whether the elements of $max(K \perp\!\!\!\perp \Gamma)$ are finitely accessible.

Case-I:

Assume that there is an $H \in max(K \perp\!\!\!\perp \Gamma)$ such that H is finitely accessible. To prove ($\dot{-}$LP), we first need to show that there is a finite subset A of Γ such that:

$$H \in max(K \perp\!\!\!\perp A) \qquad (1)$$

$$K \dot{-} \Gamma = K \cap (\Gamma + K \dot{-} A) \qquad (2)$$

Starting with (1), consider any world $z \in [\bigcap H^{\prec}]$. Define D to be the theory $D = K \cap z$. It is not hard to verify that, by construction, $\bigcap H^{\prec} \subseteq D$. Therefore, by (PM2), $H \prec D$. Since $H \in max(K \perp\!\!\!\perp \Gamma)$ this entails that D is inconsistent with Γ and therefore, so is z. Since z was chosen arbitrarily, this shows that $\bigcap H^{\prec}$ is inconsistent with Γ. By compactness we then derive that there is an $A \subseteq_f \Gamma$ such that $\neg(\wedge A) \in \bigcap H^{\prec}$. Notice that since H is consistent with Γ it is also consistent with A and therefore $H \in K \perp\!\!\!\perp A$. Then from $\neg(\wedge A) \in \bigcap H^{\prec}$ it follows that $H \in max(K \perp\!\!\!\perp A)$ as desired.

Next we show that $K \dot{-} \Gamma = K \cap (\Gamma + K \dot{-} A)$. For $LHS \subseteq RHS$ consider any sentence φ in $K \dot{-} \Gamma$. Clearly $\varphi \in K$ and therefore it suffices to show that $\varphi \in \Gamma + K \dot{-} A$. Observe that since $H \in max(K \perp\!\!\!\perp A)$ and H is consistent with Γ, $[\Gamma + K \dot{-} A] \neq \emptyset$. Let z be any world in $[\Gamma + K \dot{-} A]$. We show that $z \vdash \varphi$. If $z \in [K]$ this is trivially true (since $\varphi \in K$). Assume therefore that $z \notin [K]$. This entails that $z \in [\Gamma]$. Define D to be the set $D = K \cap z$. It is not hard to see that D is a z-remainder and moreover, $K \dot{-} A \subseteq D$. Since $\preccurlyeq$ is complete, we then derive that $D \in max(K \perp\!\!\!\perp A)$ and consequently $H \preccurlyeq D$. Moreover, since $z \in [\Gamma]$, D is consistent with Γ and therefore, by Corollary 3, $D \in K \perp\!\!\!\perp \Gamma$. Hence, from $H \preccurlyeq D$, we derive that D is a maximal Γ-remainder, and therefore, since $\varphi \in K \dot{-} \Gamma$, it contains φ. Clearly then $\varphi \in z$ and since z was chosen arbitrarily, it follows that all worlds in $[\Gamma + K \dot{-} A]$ are φ-worlds and therefore $\varphi \in \Gamma + K \dot{-} A$. Hence $K \dot{-} \Gamma \subseteq K \cap (\Gamma + K \dot{-} A)$.

For the converse, it suffices to show that $[K \dot{-} \Gamma] \subseteq [K \cap (\Gamma + K \dot{-} A)]$. Consider any world $z \in [K \dot{-} \Gamma]$. Notice that $[K \cap (\Gamma + K \dot{-} A)] = [K] \cup [(\Gamma + K \dot{-} A)]$. Hence if $z \in [K]$ we are done. Assume therefore that $z \notin [K]$. Then there is a $\varphi \in K$ such that $\neg\varphi \in z$. Define D to be $D = K \cap z$. It is not hard to see that D is a z-remainder and moreover, $K \dot{-} \Gamma \subseteq D$. Then, by the completeness of $\preccurlyeq$, D is a maximal Γ-remainder. This entails, firstly, that $z \in [\Gamma]$, and secondly that $H \preccurlyeq D$. Then given that H is also a maximal A-remainder and moreover D is consistent with A, we derive (with the aid of Corollary 3) that D is a maximal A-remainder. This again entails that $z \in [K \dot{-} A]$. Hence $z \in [\Gamma] \cap [K \dot{-} A]$, and therefore $z \in [K] \cup [(\Gamma + K \dot{-} A)]$ as desired. We have thus shown that $K \dot{-} \Gamma = K \cap (\Gamma + K \dot{-} A)$.

Having proved (1) and (2) we can now proceed with the proof of ($\dot{-}$LP). We do so by proving that the left hand side of ($\dot{-}$LP) is a subset of the right hand side, and vice versa. Starting $LHS \subseteq RHS$, let φ be any sentence in $K \dot{-} \Gamma$. Then $\varphi \in K$ and, by (2), for some $B \subseteq_f \Gamma$, $((\wedge B) \to \varphi) \in K \dot{-} A$. Moreover, since, by (1), H is both a maximal A-remainder and Γ-remainder, it follows that $K \dot{-} A$ is consistent with Γ. Hence $K \dot{-} A$ is consistent with B. Then by ($K \dot{-}$8), $K \dot{-} A \subseteq K \dot{-} (A \cup B)$ and therefore $((\wedge B) \to \varphi) \in K \dot{-} (A \cup B)$. Now let D be any maximal $(A \cup B)$-remainder. By Lemma 4 there exists a $(A \cup B)$-world z such that $[D] = [K] \cup \{z\}$. Since $((\wedge B) \to \varphi) \in K \dot{-} (A \cup B)$, it follows that $((\wedge B) \to \varphi) \in z$. Moreover, by construction, $B \subseteq z$, and therefore $\varphi \in z$. Given that φ is also in K, from $[D] = [K] \cup \{z\}$ we derive that $\varphi \in D$, and since D was chosen arbitrarily, $\varphi \in K \dot{-} (A \cup B)$. Next observe that by construction D is consistent with A and therefore, by Corollary 3, it is a A-remainder. Hence, $D \preccurlyeq H$. Moreover it is not hard to verify that H is a $A \cup B$-remainder, and therefore from $D \preccurlyeq H$ it follows that H is a *maximal* $A \cup B$-remainder. This entails that $K\dot{-}(A \cup B) \subseteq H$ and since H is consistent with Γ, we derive that $K\dot{-}(A \cup B)$ is also consistent with Γ. Consequently, by $(K\dot{-}8)$, $K\dot{-}(A \cup B) \subseteq K\dot{-}(A \cup B \cup C)$, for any $C \subseteq_f \Gamma$. Then since $\varphi \in K\dot{-}(A \cup B)$ we derive that $\varphi \in \bigcap_{C \subseteq_f \Gamma} K\dot{-}(A \cup B \cup C)$. Given that $A \cup B \subseteq_f \Gamma$ we then derive that $LHS \subseteq RHS$ as desired.

Conversely, let φ be any sentence in $\bigcup_{B \subseteq_f \Gamma} \bigcap_{C \subseteq_f \Gamma} K\dot{-}(B \cup C)$. Then for some $B \subseteq_f \Gamma$, $\varphi \in \bigcap_{C \subseteq_f \Gamma} K\dot{-}(B \cup C)$. Hence $\varphi \in K\dot{-}(B \cup A)$. Therefore $\varphi \in K$ and moreover, by $(K\dot{-}7)$, $\varphi \in (K\dot{-}A) + (A \cup B)$. Consequently, $\varphi \in K \cap ((K\dot{-}A) + \Gamma)$, and therefore, by (2), $\varphi \in K\dot{-}\Gamma$ as desired.

<u>*Case-II:*</u>

Assume now that no member of $max(K \perp\!\!\!\perp \Gamma)$ is finitely accessible. Firstly observe that $(K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f \Gamma} K\dot{-}A \subseteq K$.[17] Next we show that $(K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f \Gamma} K\dot{-}A \subseteq K\dot{-}\Gamma$.

Assume towards contradiction that there is a $y \in ((K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f \Gamma} K\dot{-}A)$ such that $y \notin K\dot{-}\Gamma$. Clearly, $y \in K$. Moreover, since $y \in ((K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f \Gamma} K\dot{-}A)$, by compactness, there is a $x \in (K \cap Cn(\Gamma))$ such that $(x \to y) \in \bigcap_{A \subseteq_f \Gamma} K\dot{-}A$. Moreover, from $y \notin K\dot{-}\Gamma$ it follows that there is a $H \in max(K \perp\!\!\!\perp \Gamma)$, such that $y \notin H$. It is not hard to verify that, since $x \in (K \cap Cn(\Gamma))$ and H is a Γ-remainder, $x \in H$. Therefore H is consistent with $x \wedge \neg y$ and consequently by Corollary 3, $H \in K \perp\!\!\!\perp \{x \wedge \neg y\}$. Since we have assumed that H is not finitely accessible, there is a $D \in max(K \perp\!\!\!\perp \{x \wedge \neg y\})$ such that $H \prec D$. This makes D strictly greater than all Γ-remainders. Next we show that $\bigcap D^{\preccurlyeq}$ is inconsistent with Γ, or equivalently, that no Γ-world belongs to $[\bigcap D^{\preccurlyeq}]$. Let z be an arbitrary Γ-world. Define H' to be the theory $H' = K \cap z$. It is not hard to verify that H' is a Γ-remainder and therefore $H' \prec D$. Moreover, since D is finitely accessible, by (PM1) we derive that $D^{\preccurlyeq}$ is closed, which combined with $H' \prec D$ entails that $\bigcap D^{\preccurlyeq} \nsubseteq H'$. Consequently there is a $\psi \in \bigcap D^{\preccurlyeq}$ such that $\psi \notin H'$. Given the construction of H' it follows that $\neg\psi \in z$ and therefore $z \notin [\bigcap D^{\preccurlyeq}]$. Hence we have shown that Γ is inconsistent with $\bigcap D^{\preccurlyeq}$. Therefore there is a $\varphi \in Cn(\Gamma)$ such that $\neg\varphi \in \bigcap D^{\preccurlyeq}$, which again entails (because of the canonicity of $\preccurlyeq$) that $\neg\varphi \in K$. Consider now the sentence $\varphi \vee (x \wedge \neg y)$. Clearly, $\neg(\varphi \vee (x \wedge \neg y)) \in K$, and therefore, by Corollary 3, $D \in K \perp\!\!\!\perp \{\varphi \vee (x \wedge \neg y)\}$, and given that $\neg\varphi \in \bigcap D^{\preccurlyeq}$, it follows that $\neg\varphi \in \bigcap max(K \perp\!\!\!\perp \{\varphi \vee (x \wedge \neg y)\})$, and therefore $(x \to y) \notin K\dot{-}\{\varphi \vee (x \wedge \neg y)\}$. Moreover notice that $\{\varphi \vee (x \wedge \neg y)\} \subseteq_f Cn(\Gamma)$. Hence we derive that $(x \to y) \notin \bigcap_{A \subseteq_f \Gamma} K\dot{-}A$. This of course contradicts our initial assumption about $x \to y$. Thus $(K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f \Gamma} K\dot{-}A \subseteq K\dot{-}\Gamma$ as desired.

For the converse, observe that $(K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A = K \cap (\Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A)$, and therefore, given that $K\dot{-}\Gamma \subseteq K$, it suffices to show that $K\dot{-}\Gamma \subseteq \Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A$, which in turn is equivalent to $[\Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A] \subseteq [K\dot{-}\Gamma]$. Consider therefore an arbitrary world $z \in [\Gamma + \bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A]$. If $z \in [K]$ then clearly, $z \in [K\dot{-}\Gamma]$. Assume therefore that $z \notin [K]$. Define H to be the set $H = K \cap z$. It is not hard to see that H is a Γ-remainder and that $[H] = [K] \cup \{z\}$. Then since $[K] \subseteq [\bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A]$ (see footnote 17) and moreover $z \in [\bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A]$, we derive that $[K] \cup \{z\} \subseteq [\bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A]$ and therefore, $\bigcap_{A \subseteq_f Cn(\Gamma)} K\dot{-}A \subseteq H$. To conclude the proof it suffices to show that $H \in max(K \perp\!\!\!\perp \Gamma)$.

[17]By $(K\dot{-}2)$, $K\dot{-}A \subseteq K$ for all $A \subseteq_f \Gamma$, and therefore $\bigcap_{A \subseteq_f \Gamma} K\dot{-}A \subseteq K$. Hence $(K \cap Cn(\Gamma)) + \bigcap_{A \subseteq_f \Gamma} K\dot{-}A \subseteq K$.

Assume on the contrary that there is a $D \in max(K \perp\!\!\!\perp \Gamma)$ such that $H \prec D$. Consider now any $A \subseteq_f Cn(\Gamma)$. We show that $\bigcap D^{\prec} \subseteq K \dot{-} A$. If A is consistent with K, then this is clearly true (recall that Γ is inconsistent with K and therefore $K \in D^{\prec}$). Assume therefore that $\neg(\wedge A) \in K$. By Corollary 3, $D \in K \perp\!\!\!\perp A$. Moreover by the assumption of the case, D is not finitely accessible, and therefore we derive that while D is a A-remainder, is not a *maximal* A-remainder. Hence all maximal A-remainders are in $D^{\prec}$. This again entails that $\bigcap D^{\prec} \subseteq K \dot{-} A$. Since A was chosen arbitrarily, it follows that $\bigcap D^{\prec} \subseteq \bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A$. Notice however that from $H \prec D$ and (PM3) we derive that $\bigcap D^{\prec} \nsubseteq H$, and therefore $\bigcap_{A \subseteq_f Cn(\Gamma)} K \dot{-} A \nsubseteq H$. This of course contradicts our earlier conclusion about H. Hence we have shown that $H \in max(K \perp\!\!\!\perp \Gamma)$, which again entails that $K \dot{-} \Gamma \subseteq H$ and therefore from the construction of H we derive that $z \in [K \dot{-} \Gamma]$ as desired. ■

8. SUMMARY AND DISCUSSION

In this article we have addressed three open problems in the multiple belief change literature. Namely, we provided characterizations of the limit postulates (*LP) and ($\dot{-}$LP) in the system-of-spheres and the partial meet models respectively, and we proved that (*LP) is strictly weaker than (K*F).

These results, together with the ones in [Zhang and Foo 2001] and [Peppas 2011] that relate the limit postulate to (generalizations of) the epistemic entrenchment model, complete the picture of the effects that the limit postulate has on all three major constructive models in Belief Revision. The obtained characterizations reveal a deep connection between the limit postulate and the notion of an *elementary* set of possible worlds.

There is also another side to these results. They provide further insight on the soundness of the limit postulate. As already stated, the limit postulate was primarily introduced to close the gap between the postulates (K$\dot{-}$1) - (K$\dot{-}$8) for set contraction and the nicely ordered partition model. It is important however to assess the limit postulate independently of its service to a particular model. Can the limit postulate be considered a general feature of rationality in multiple belief change, alongside the AGM postulates? In our view, the results of this article lead us to a negative answer. One can think of natural situations where conditions (R1) - (R3) (or alternatively (PM1) - (PM3)) are violated. Consider for example the system of spheres S in Figure 1, which as we have shown induces a revision function $*$ satisfying (*LP). The two worlds $z = Cn(\{p_0, \neg p_1, p_2, \neg p_3, \ldots\})$ and $u = Cn(\{\neg p_0, \neg p_1, \neg p_2, \neg p_3, \ldots\})$ belong to no sphere in S other than $\mathcal{M}_L$. Suppose now that we decide that z is in fact more plausible than u and to that affect we introduce an extra sphere $U = (\bigcup_{i \in \mathbb{N}_0} [T_i]) \cup \{z\}$ to S. This simple addition breaks down the compliance with (*LP); the new system of spheres violates (R3) and consequently its induced revision function violates (*LP).

In our view the results of this article strongly indicate that the limit postulate should be treated as the identifying condition of an important special case of set contraction (and multiple revision) functions, rather than a general feature of rational belief change.

ACKNOWLEDGMENT

We are grateful to Dongmo Zhang and Norman Foo for many discussions on the relationship between (K*F) and (*LP). We are also grateful to the anonymous reviewers for their valuable comments.

REFERENCES

Alchourron, C., Gardenfors, P., and Makinson, D. 1985. On the logic of theory change: Partial meet functions for contraction and revision. *Journal of Symbolic Logic 50,* 510–530.

Chang, C. and Keisler, H. 1991. *Model Theory*. Elsevier Science Publications.

Fuhrmann, A. and Hansson, S. 1994. A survey on multiple contraction. *Journal of Logic, Language, and Information 3*, 39–76.

Gardenfors, P. 1988. *Knowledge in Flux*. MIT Press.

Grove, A. 1988. Two modellings for theory change. *Journal of Philosophical Logic 17*, 157–170.

Lindstrom, S. 1991. A semantic approach to nonmonotonic reasoning: Inference operations and choice. In *Technical report, Dept of Philosophy, Uppsala University*.

Peppas, P. 1996. Well behaved and multiple belief revision. In *Proceedings of the 12th European Conference on Artificial Intelligence (ECAI-96)*. 90–94.

Peppas, P. 2004. The limit assumption and multiple revision. *Journal of Logic and Computation 14,* 3, 355–371.

Peppas, P. 2008. Belief revision. In *Handbook of Knowledge Representation*, F. van Harmelen, V. Lifschitz, and B. Porter, Eds. Elsevier, 90–94.

Peppas, P. 2011. Comparative possibility in set contraction. *Journal of Philosophical Logic*.

Peppas, P., Foo, N., and Williams, M. 1995. On the expressibility of propositions. *Logique et Analyse 139-149,* 251–272.

Rott, H. 2004. Stability, strength, and sensitivity: Converting belief into knowledge. *Erkenntnis 61*, 469–493.

Zhang, D. and Foo, N. 2001. Infinitary belief revision. *Journal of Philosophical Logic 30,* 6, 525–570.