# Online Appendix: Fuzzy Equilibrium Logic: Declarative Problem Solving in Continuous Domains

STEVEN SCHOCKAERT
Cardiff University
JEROEN JANSSEN
Vrije Universiteit Brussel
DIRK VERMEIR
Vrije Universiteit Brussel

## APPENDIX: PROOFS

### Proof of Proposition 2

Let $\alpha_1$ be a formula from $N5$ logic, and let $\alpha_2$ be the formula that is obtained from $\alpha_1$ by replacing conjunction, disjunction and implication by respectively a t-norm $\otimes$, a t-conorm $\oplus$ and an implicator $\rightarrow$. Noting that $V_2^-(w, a) \in \{0, 1\}$ and $V_2^+(w, a) \in \{0, 1\}$ for all $a \in At$ and $w \in \{h, t\}$, it follows easily from the definitions (17)–(23) that $V_1(w, \alpha_1) = 1$ iff $V_2(w, \alpha_2) = [1, 1]$, $V_1(w, \alpha_1) = 1$ iff $V_2(w, \alpha_2) = [1, 1]$, $V_1(w, \alpha_1) = -1$ iff $V_2(w, \alpha_2) = [0, 0]$ and $V_1(w, \alpha_1) = 0$ iff $V_2(w, \alpha_2) = [0, 1]$ (using structural induction). From this, we immediately find that $V_1$ is an $N5$ model of $\Theta_1$ iff $V_2$ is a fuzzy $N5$ model of $\Theta_2$.

### Proof of Corollary 1

Let $V_2$ be a fuzzy equilibrium model of $\Theta_2$ and assume that $V_1$ is not an equilibrium model of $\Theta_1$. We know from Proposition 2 that $V_1$ is an $N5$ model of $\Theta_1$, which means that $V_1$ cannot be $h$-minimal. Hence, there exists an $N5$ model $W_1$ of $\Theta_1$ which is 'smaller' than $V_1$. However, by Proposition 2, $W_1$ corresponds to a fuzzy $N5$ model $W_2$ of $\Theta_2$, which is smaller than $V_2$, meaning that $V_2$ cannot be $h$-minimal, a contradiction.

Author's address: S. Schockaert, School of Computer Science & Informatics, Cardiff University, 5 The Parade, CF24 3AA Cardiff, UK, s.schockaert@cs.cardiff.ac.uk; J. Janssen and D. Vermeir, Department of Computer Science, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium, {jeroen.janssen,dvermeir}@vub.ac.be.

Proof of Proposition 3

We show the proof by contraposition. Let $V$ be a fuzzy $N5$ model of $\Theta$ such that at least for one $a \in At$, $V(h,a) \notin \{[0,0],[1,1],[0,1]\}$. We now define a fuzzy $N5$ valuation $W$ as follows:

$$W(h,a) = \begin{cases} V(h,a) & \text{if } V(h,a)=[0,0] \text{ or } V(h,a)=[1,1] \\ [0,1] & \text{otherwise} \end{cases}$$

and $W(t,a) = V(t,a)$ for all $a \in At$. We now show that for each formula $\alpha$ we have

$$V^-(h,\alpha) = 1 \Rightarrow W^-(h,\alpha) = 1 \tag{76}$$

$$V^+(h,\alpha) = 1 \Rightarrow W^+(h,\alpha) = 1 \tag{77}$$

$$V^-(h,\alpha) = 0 \Rightarrow W^-(h,\alpha) = 0 \tag{78}$$

$$V^+(h,\alpha) = 0 \Rightarrow W^+(h,\alpha) = 0 \tag{79}$$

To show (76)–(79), we proceed by structural induction. Clearly (76)–(79) hold for atoms and constants. Now assume that (76)–(79) holds for the formulas $\alpha$ and $\beta$, then it is easy to show that (76)–(79) also holds for $\alpha \otimes_m \beta$, $\alpha \oplus_m \beta$ and $\sim\alpha$. Furthermore, we find

$$\begin{aligned} & V^-(h, not\,\alpha) = 1 \\ & \quad \Leftrightarrow 1 - V^-(t,\alpha) = 1 \\ & \quad \Leftrightarrow 1 - W^-(t,\alpha) = 1 \\ & \quad \Leftrightarrow W^-(h, not\,\alpha) = 1 \end{aligned} \qquad \begin{aligned} & V^+(h, not\,\alpha) = 1 \\ & \quad \Leftrightarrow V^-(h,\alpha) = 0 \\ & \quad \Rightarrow W^-(h,\alpha) = 0 \\ & \quad \Leftrightarrow W^+(h, not\,\alpha) = 1 \end{aligned}$$

and analogously for (78)–(79). For a rule $\alpha \to \beta$, we obtain

$$\begin{aligned} & V^-(h, \alpha \to \beta) = 1 \\ & \quad \Leftrightarrow \min(V^-(h,\alpha) \to V^-(h,\beta), V^-(t,\alpha) \to V^-(t,\beta)) = 1 \\ & \quad \Leftrightarrow \min(V^-(h,\alpha) \to V^-(h,\beta), W^-(t,\alpha) \to W^-(t,\beta)) = 1 \\ & \quad \Leftrightarrow (V^-(h,\alpha) = 0 \vee V^-(h,\beta) = 1) \wedge (W^-(t,\alpha) \to W^-(t,\beta)) = 1) \\ & \quad \Rightarrow (W^-(h,\alpha) = 0 \vee W^-(h,\beta) = 1) \wedge (W^-(t,\alpha) \to W^-(t,\beta)) = 1) \\ & \quad \Leftrightarrow W^-(h, \alpha \to \beta) = 1 \end{aligned}$$

and

$$\begin{aligned} V^+(h, \alpha \to \beta) = 1 & \Leftrightarrow V^-(h,\alpha) = 0 \vee V^+(h,\beta) = 1 \\ & \Rightarrow W^-(h,\alpha) = 0 \vee W^+(h,\beta) = 1 \\ & \Leftrightarrow W^+(h, \alpha \to \beta) = 1 \end{aligned}$$

and analogously for (78)–(79).

From (76), together with the fact that $V$ is a fuzzy $N5$ model of $\Theta$, we find that $W$ is also a fuzzy $N5$ model of $\Theta$, which means that $V$ cannot be $h$-minimal.

Proof of Corollary 2

Let $V_1$ be an equilibrium model of $\Theta_1$, and assume that $V_2$ is not a fuzzy equilibrium model of $\Theta_2$. From Proposition 2 we know that $V_2$ is a model of $\Theta_2$, which means that it is not $h$-minimal. Let $W_2$ be an $h$-minimal model of $\Theta_2$ that coincides with $V_2$ in world $t$. From Proposition 3 we know that $W_2(h,a) \in \{[0,0],[1,1],[0,1]\}$ for every atom $a$. From Proposition 2, we know that the corresponding classical $N5$ valuation $W_1$ is a model of $\Theta_1$, which means that $V_1$ cannot be $h$-minimal, a contradiction.

Proof of Proposition 4

(1) First assume that $P$ is a definite program. Clearly, by definition of fuzzy equilibrium model, and using the property that for a residual implicator $\rightarrow$ it holds that $W(x \rightarrow y) = 1$ iff $W(x) \geq W(y)$, we immediately have that the interpretation $W$ is a model of $P$. Now assume that there exists another model $M$ such that $M(a) \leq W(a)$ for all atoms $a$ in $At$, and such that there is an $a_0$ in $At$ satisfying $M(a_0) < W(a_0)$. Now let the fuzzy $N5$ valuation $V'$ be defined for $a$ in $At$ by

$$V'(h,a) = [M(a), V^+(h,a)] \qquad\qquad V'(t,a) = V(t,a)$$

We show that $V'$ is also a fuzzy $N5$ model of $P$, and therefore that $V$ cannot be $h$-minimal. Indeed, because $P$ is positive, we easily find for each rule $a \leftarrow \alpha$ that

$$\begin{aligned} V'^-(h, a \leftarrow \alpha) &= \min(V'^-(h,a) \leftarrow V'^-(h,\alpha), V'^-(t,a) \leftarrow V'^-(t,\alpha)) \\ &= \min([a \leftarrow \alpha]_M, V^-(t,a) \leftarrow V^-(t,\alpha)) \\ &= 1 \end{aligned}$$

where the last step follows from the fact that $V$ is a fuzzy $N5$ model of $P$ and $M$ is a model of $P$.

(2) Now let $P$ be an arbitrary constraint-free program. It is sufficient to show that $W$ is an answer set of $P^W$. Assume that $At$ is given by $\{a_1, a_2, \ldots, a_n\}$. Without loss of generality, we may assume that each rule $r$ from $P$ is of the form $c \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)$ where $f_r$ is a monotonically increasing function in the first $n$ arguments, and monotonically decreasing in the last $n$ arguments, and $c \in At$. The reduct $r^W$ of a rule $r : a_r \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)$ is given by $a_r \leftarrow f_r(a_1, \ldots, a_n, W(a_1), \ldots, W(a_n))$ and for any fuzzy $N5$ valuation $V'$ satisfying $W(a) = V'^-(t,a)$, it holds that

$$\begin{aligned} & V'^-(h, f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)) \\ &\quad = V'^-(h, f_r(a_1, \ldots, a_n, V'^-(t,a_1), \ldots, V'^-(t,a_n))) \\ &\quad = V'^-(h, f_r(a_1, \ldots, a_n, W(a_1), \ldots, W(a_n))) \end{aligned}$$

Hence if $V'$ is a fuzzy $N5$ model of $P^W$ then $V'$ is also a fuzzy $N5$ model of $P$ and vice versa. This means that $V$ is not $h$-minimal w.r.t. $P^W$ iff $V$ is not $h$-minimal w.r.t. $P$. Since $V$ is assumed to be a fuzzy equilibrium model of $P$, we thus find that $V$ is a fuzzy equilibrium model of $P^W$, and given the first part of this proof, that $W$ is an answer set of $P^W$.

(3) Finally assume that $P$ includes a set of constraints $C$. We may assume that each rule $r$ from $P$ is of the form $c \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)$ with $f_r$ as before

and either $c \in At$ or $c \in [0,1]$. Since $V$ is a fuzzy equilibrium model of $P$, we also have that $W$ is a model of $P$.
It remains to be shown that $W$ is an answer set of $(P \setminus C)^W$. Given the second part of this proof, it is sufficient to show that $V$ is a fuzzy equilibrium model of $(P \setminus C)^W$. Assume that this were not the case, and that there exists a fuzzy $N5$ model $V'$ of $(P \setminus C)^W$ such that $V'(t,a) = V(t,a)$ for all $a$ in $At$, and such that $V'(h,a) \supset V(h,a)$ for at least one $a$ in $At$. Because $V$ is a fuzzy equilibrium model of $P$, $V'$ cannot be a fuzzy $N5$ model of $P$, hence there needs to be a constraint $c$ in $C$ of the form ($k \in [0,1]$)

$$k \leftarrow f_c(a_1, \ldots, a_n, a_1, \ldots, a_n)$$

which is not satisfied by $V'$, which means that $V'(h, f_c(a_1, \ldots, a_n, a_1, \ldots, a_n)) > k$. However, since $V'^-(h,a) \le V^-(h,a)$, we have

$$\begin{aligned}
&V'(h, f_c(a_1, \ldots, a_n, a_1, \ldots, a_n)) \\
&\quad = f_c(V'(h,a_1), \ldots, V'(h,a_n), V'(t,a_1), \ldots, V'(t,a_n)) \\
&\quad = f_c(V'(h,a_1), \ldots, V'(h,a_n), V(t,a_1), \ldots, V(t,a_n)) \\
&\quad \le f_c(V(h,a_1), \ldots, V(h,a_n), V(t,a_1), \ldots, V(t,a_n)) \\
&\quad = V(h, f_c(a_1, \ldots, a_n, a_1, \ldots, a_n))
\end{aligned}$$

which would mean that the constraint $c$ is also violated by $V$, a contradiction.

### Proof of Proposition 5

Let $At$ be given by $\{a_1, a_2, \ldots, a_n\}$. Without loss of generality, we may assume that each rule $r$ from $P$ is of the form $c \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)$ where $f_r$ is a monotonically increasing function in the first $n$ arguments, and monotonically decreasing in the last $n$ arguments, and either $c \in At$ or $c \in [0,1]$. First note that $V$ is a fuzzy $N5$ model of $P$. Indeed, since $W$ is a model of $P$, we find

$$\begin{aligned}
&V^-(h, c \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)) \\
&= \min(V^-(h,c) \leftarrow f(V^-(h,a_1), \ldots, V^-(h,a_n), V^-(t,a_1), \ldots, V^-(t,a_n)), \\
&\qquad V^-(t,c) \leftarrow f(V^-(t,a_1), \ldots, V^-(t,a_n), V^-(t,a_1), \ldots, V^-(t,a_n))) \\
&= W(c) \leftarrow f(W(a_1), \ldots, W(a_n), W(a_1), \ldots, W(a_n)) \\
&= 1
\end{aligned}$$

To see that $V$ is $h$-minimal, assume that there exists a fuzzy $N5$ model $V'$ such that $V'(t,a) = V(t,a)$ for all $a \in At$. It holds that

$$\begin{aligned}
&V'^-(h, c \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)) \\
&= \min(V'^-(h,c) \leftarrow f(V'^-(h,a_1), \ldots, V'^-(h,a_n), V'^-(t,a_1), \ldots, V'^-(t,a_n)), \\
&\qquad V'^-(t,c) \leftarrow f(V'^-(t,a_1), \ldots, V'^-(t,a_n), V'^-(t,a_1), \ldots, V'^-(t,a_n)))
\end{aligned}$$

Since $V'$ is assumed to be a fuzzy $N5$ model, we find in particular that for each rule $r$ of the form $c \leftarrow f_r(a_1, \ldots, a_n, a_1, \ldots, a_n)$, it holds that

$$\big(V'^-(h,c) \leftarrow f(V'^-(h,a_1), \ldots, V'^-(h,a_n), V'^-(t,a_1), \ldots, V'^-(t,a_n))\big) = 1$$

in other words

$$\big(V'^{-}(h,c) \leftarrow f(V'^{-}(h,a_1),\ldots,V'^{-}(h,a_n),W(a_1),\ldots,W(a_n))\big) = 1$$

This means that the interpretation $W'$ defined by $W'(a) = V'^{-}(h,a)$ is a model of $(P \setminus C)^W$, with $C$ the set of constraints in $P$. Now if $V'$ is such that $V'(h,a) \supset V(h,a)$ for at least one $a$, we have that $W$ is not a minimal model of $(P \setminus C)^W$. This would mean that $W$ is not an answer set of $P$, a contradiction.

Proof of Proposition 7

(1) Assume that $(\lambda_1,...,\lambda_n)$ is a strong Nash equilibrium. Then we show that the fuzzy $N5$ valuation $V$, defined as follows, is a fuzzy equilibrium model:

$$V(t,a_i) = [\lambda_i,\lambda_i] \quad V(t,b_i^-) = [1,1] \quad V(t,b_i^+) = [1,1] \quad V(t,c_i^-) = [1,1] \tag{80}$$

$$V(t,c_i^+) = [1,1] \quad V(t,d_i^-) = [1,1] \quad V(t,d_i^+) = [1,1] \quad V(t,e_i^-) = [1,1] \tag{81}$$

$$V(t,e_i^+) = [1,1] \quad V(t,w) = [1,1] \tag{82}$$

and $V(h,x) = V(t,x)$ for all atoms $x$. Note that clearly $V$ is a fuzzy $N5$ model of $\Theta$. Thus, it only remains to be shown that $V$ is $h$-minimal. Assume that this were not the case, and that $V' \preccurlyeq V$ with $V' \neq V$. Due to (45), it must clearly hold that $V(h,a_i) = V(t,a_i) = V'(h,a_i) = V'(t,a_i)$. Furthermore, given (55)–(57) and (53), $V'$ can only differ from $V$ if $V'(h,w) = [0,1]$. From (51) we then obtain $V'^{-}(h,e_i^-) \leq 1 - V'^{-}(h,e_i^+)$, and from (46), (52) and (53) that for all $i$ either $V'^{-}(h,c_i^-) = 0$ and $V'^{-}(h,c_i^+) = 1$, or $V'^{-}(h,c_i^-) = 1$ and $V'^{-}(h,c_i^+) = 0$.
Let $K$ be defined as $K = \{i | V'^{-}(h,c_i^+) = 1\}$. Then we show that $K$ defines a coalition which can improve the global strategy $(\lambda_1,...,\lambda_n)$. Since $V'$ satisfies (50) with $V'^{-}(h,w) = 0$ it must hold that $V'^{-}(h, c_1^- \otimes_m ... \otimes_m c_n^-) = 0$ and $V'^{-}(h,\alpha_1 \otimes_m ... \otimes_m \alpha_n) = 1$. From the former equality, we derive that $K \neq \emptyset$, while the latter means that for every $i$, $V'^{-}(h,\alpha_i) = 1$, and thus either $V'^{-}(h, U_i(e_1^+,...,e_n^+; e_1^-,...,e_n^-) > V'^{-}(t, U_i(a_1^+,...,a_n^+; {\sim}a_1^-,...,{\sim}a_n^-))$ or $V'^{-}(h,c_i^-) = 1$. This entails that either $V'^{-}(h, U_i(e_1^+,...,e_n^+; 1-e_1^+,...,1-e_n^+) > V'^{-}(t, U_i(a_1^+,...,a_n^+; {\sim}a_1^-,...,{\sim}a_n^-))$, since $V'^{-}(h,e_i^-) \leq 1 - V'^{-}(h,e_i^+)$ and the partial mappings of $U_i$ are increasing, or $V'^{-}(h,c_i^+) = 0$. In other words, $(V'^{-}(h,e_1^+),...,V'^{-}(h,e_n^+))$ is a global strategy which constitutes a counterexample for the claim that $(\lambda_1,...,\lambda_n)$ is a strong Nash equilibrium. Thus, by contraposition, we obtain that $V$ must be $h$-minimal, and therefore a fuzzy equilibrium model.

(2) Now assume conversely that the fuzzy $N5$ valuation $V$ is a fuzzy equilibrium model. From (57), we derive that $V^-(t,w) = 1$. This means that in any fuzzy equilibrium model $V$, we must also have $V^-(h,w) = 1$. Without loss of generality, we can thus assume that $V$ is as defined as in (80)–(82). Assume that there exists a non-empty coalition $K$ and values $e_1^*,...,e_n^*$ such that for all $i \in K$, $U_i(e_1^*,...,e_n^*; 1-e_1^*,...,1-e_n^*) < U_i(\lambda_1,...,\lambda_n; 1-\lambda_1,...,1-\lambda_n)$ and such that $e_j^* = \lambda_j$ for all $j \notin K$. Now define $V'$ as follows

$$V'(h,a_i) = V'(t,a_i) = [\lambda_i,\lambda_i] \qquad V'(h,c_i^-) = \begin{cases} [0,1] & \text{if } i \in K \\ [1,1] & \text{otherwise} \end{cases}$$

$$V'(h, c_i^+) = \begin{cases} [0,1] & \text{if } i \notin K \\ [1,1] & \text{otherwise} \end{cases} \qquad V'(h, w) = [0,1]$$
$$V'(h, d_i^-) = [1 - e_i^*, 1] \qquad V'(h, d_i^+) = [e_i^*, 1]$$
$$V'(h, e_i^-) = [1 - e_i^*, 1] \qquad V'(h, e_i^+) = [e_i^*, 1]$$

and for all literals $l$, $V'(t,l) = V(t,l)$. Then it is straightforward to verify that $V'$ is a fuzzy $N5$ model of $\Theta$, which means that $V$ cannot be a fuzzy equilibrium model since $V$ and $V'$ agree on their valuations in the *there*-world.

Proof of Proposition 8

We proceed by reducing the decision problems for classical logic to the corresponding decision problems for Łukasiewicz logic. Let $T$ be a classical propositional theory over the set of atoms $At$, and let $H$, $M$ and $O$ be the sets of possible hypotheses, possible manifestations, and observations as before. For a classical propositional formula $\alpha$ let us write $\phi(\alpha)$ for the corresponding formula in Łukasiewicz logic, obtained by replacing conjunction, disjunction, implication and negation by their counterparts $\otimes_l$, $\oplus_l$, $\rightarrow_l$ and $\neg$. Now we construct a set of Łukasiewicz logic formulas $T'$ as follows:

$$T' = \{\phi(\alpha) | \alpha \in T\} \cup \{a \vee \neg a | a \in At\}$$

The set of formulas $\{a \vee \neg a | a \in At\}$ ensures that in every model $I'$ of $T'$, $I'(a) = 0$ or $I'(a) = 1$ for $a \in At$ (recall that the maximum $\vee$ is definable in terms of the Łukasiewicz connectives). Hence there is a one-on-one correspondence between the models of $T$ and those of $T'$, as for any model $I$ of $T$, it holds that $I'$ defined by $I'(a) = 1$ if $a \in I$ and $I'(a) = 0$ otherwise is a model of $T'$, and vice versa.

Now let $O'$ be the fuzzy set in $M$ defined by $O'(m) = 1$ if $m \in O$ and $O'(m) = 0$ otherwise. Now it is easy to see that when $S'$ is an abductive explanation for $O'$ (w.r.t. $T'$) then $S = \{e | e \in H \wedge S'(e) = 1\}$ is an abductive explanation for $O$ (w.r.t. $T$), and vice versa. Hence there is a one-on-one correspondence between the abductive explanations of $O$ and those of $O'$, and thus the hardness results of the decision problems in Łukasiewicz logic immediately follow from the hardness results of the corresponding tasks in propositional logic.

Proof of Lemma 1

Clearly, if $S \cup T$ has a model $W$ then the fuzzy $N5$ valuation $V$ defined by $V(h, c_a) = V(t, c_a) = [W(a), W(a)]$ for all $a$ in $At$ is a model of $\Theta_1 \cup \Theta_2$. Conversely, let $V$ be a fuzzy $N5$ model of $\Theta_1 \cup \Theta_2$ and let $V'$ be the refinement of $V$ defined by $V'(h, c_a) = V'(t, c_a) = [V^-(t, c_a), V^-(t, c_a)]$, and $V(w, x) = V'(w, x)$ for all other atoms $x$ and $w \in \{h, t\}$. Clearly, $V'$ is a model of $\Theta_1$. The fact that $V'$ is also a model of $\Theta_2$ follows from the observation that the only connectives occurring in $\Theta_2$ are $\oplus_l$, $\otimes_l$ and $\sim$. In such a case, any refinement of a model needs to be a model as well. From the fact that $V'$ is a model of $\Theta_1 \cup \Theta_2$, we immediately find that the interpretation $W$ defined by $W(a) = V'^-(h, c_a)$ is a model of $T$ and moreover, satisfies $W(a) \geq S(a)$ for all $a \in H$.

Proof of Proposition 9

(1) Suppose that $S$ is an abductive explanation for $O$. By Lemma 1, we know that there exists a fuzzy $N5$ model $V$ of $\Theta_1 \cup \Theta_2$, in which $V(h, s_e) = V(t, s_e) = [S(e), S(e)]$ for all $e \in H$. Furthermore, we may assume without lack of generality that $V$ is minimal in the sense that decreasing the value of $V^-(h, c_a)$ or increasing the value of $V^+(h, c_a)$ for any $a \in At$ would yield a valuation which is no longer a model of $\Theta_1 \cup \Theta_2$, and furthermore, since $\Theta_1 \cup \Theta_2$ does not contain negation-as-failure, that $V(h, c_a) = V(t, c_a)$ for all $a$ in $At$. Furthermore, we may assume that $V$ is defined for the atoms $w$, $v_a$ and $\overline{v_a}$ ($a \in At$) as:

$$V(h, v_a) = V(h, \overline{v_a}) = V(h, w) = V(t, v_a) = V(t, \overline{v_a}) = V(t, w) = [1, 1]$$

We show that $V$ is a fuzzy equilibrium model of $\Theta_1 \cup \Theta_2 \cup \Theta_3$. By construction $V$ is a fuzzy $N5$ model of $\Theta_1 \cup \Theta_2$. It is moreover trivial to verify that $V$ is a fuzzy $N5$ model of $\Theta_3$. Since, moreover, $V$ is identical in worlds $h$ and $t$, it only remains to be shown that $V$ is $h$-minimal.

Assume that $V' \preccurlyeq V$ with $V' \neq V$. Clearly, due to (67)–(69) this is only possible when $V'(h, w) = [0, 1]$. Assume that $V'$ were a fuzzy $N5$ model of $\Theta_1 \cup \Theta_2 \cup \Theta_3$. For any $a \in At$, we find from (66) and the fact that $V'^-(h, w) = 0$ that

$$\begin{aligned}
& && V'^-(h, 0 \leftarrow_l v_a \otimes \overline{v_a}) = 1\\
&\Rightarrow && 0 \leftarrow_l V'^-(h, v_a \otimes \overline{v_a}) = 1\\
&\Leftrightarrow && V'^-(h, v_a \otimes \overline{v_a}) = 0\\
&\Leftrightarrow && V'^-(h, v_a) + V'^-(h, \overline{v_a}) \leq 1
\end{aligned}$$

From Lemma 2, we find that there is a model $W$ of $T$ such that $W(a) = V'^-(h, v_a)$ for all $a \in At$. Since $S$ is an abductive explanation for $O$, and since $V'^-(h, v_e) \geq V'^-(h, s_e) = S(e)$ for each $e \in H$, due to (60), this means that $W(a) = V'^-(h, v_a) \geq O(a)$ for all $a \in M$. On the other hand, since $V'$ is a model of (65), we find

$$\begin{aligned}
& && V'^-(h, w \leftarrow_l \bigwedge_{a \in M} (v_a \geq O(a))) = 1\\
&\Rightarrow && 0 \leftarrow_l \bigwedge_{a \in M} V'^-(h, v_a \geq O(a)) = 1\\
&\Leftrightarrow && \exists a \in M \,.\, V'^-(h, v_a \geq O(a)) = 0\\
&\Leftrightarrow && \exists a \in M \,.\, V'^-(h, v_a) < O(a)
\end{aligned}$$

a contradiction, since we have already established that $V'^-(h, a) \geq O(a)$ for all $a \in M$.

(2) Suppose that $V$ is a fuzzy equilibrium model of $\Theta$. Note that this entails, using (70), that $V(h, w) = [1, 1]$. Let us define the fuzzy set $S$ by $S(e) = V^-(h, s_e)$ for all $e \in H$. We show that $S$ is an abductive explanation for $O$.

The fact that $V$ is a model of (58) means that $V(h, s_e) = V(t, s_e) = [S(e), S(e)]$ for all $e \in H$. Using Lemma 1 we already obtain that $S$ is consistent with $T$. It remains to be shown that $S \cup T \models O$. Assume that this were not the case, and that there were a model $W$ of $T$ such that $W(e) \geq S(e)$ for all $e \in H$, while

$O(m) > W(m)$ for some $m \in M$. Let $V'$ be the fuzzy $N5$ valuation which coincides with $V$ in world $t$, and is defined in world $h$ by

$$V'(h, v_a) = [W(a), 1] \qquad V'(h, \overline{v_a}) = [1 - W(a), 1] \qquad V'(h, w) = [0, 1]$$

and $V(h, a) = V'(h, a)$ for all remaining atoms $a$. Clearly, $V'$ is a fuzzy $N5$ model of $\Theta$, implying that $V$ would not be $h$-minimal, a contradiction.