Verification of Locally Tight Programs

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submitted 1 January 2003; revised 1 January 2003; accepted 1 January 2003

Abstract

ANTHEM is a proof assistant that can be used for verifying the correctness of tight programs in the input language of the answer set grounder GRINGO with respect to specifications expressed by first-order formulas. We define the concept of a locally tight program and prove that the verification process used by ANTHEM is applicable in this more general setting. Unlike tightness, the local tightness condition allows some forms of recursion. In particular, some programs describing effects of actions are locally tight.

1 Introduction

ANTHEM (Fandinno et al. 2020) is a proof assistant that can be used for verifying the correctness of a program written in the input language of the answer set grounder GRINGO (Gebser et al. 2019) with respect to a specification expressed by first-order formulas.

In addition to the rules of the program, the user of ANTHEM is expected to provide some information on the intended use of the rules: which symbols are meant to serve as placeholders, which predicates are meant to be defined in the input, and which predicates are considered part of the output. The predicates that do not belong to either category are auxiliary, or “private,” and they are not allowed in a specification. A set of rules along with this additional information is termed a “program with input and output,” or an “io-program.” For example, in the program encoding the exact cover problem\(^1\)

\[
\{\text{in\_cover}(1..n)\}.
\text{:- } I \neq J, \text{ in\_cover}(I), \text{ in\_cover}(J), s(X,I), s(X,J).
\text{covered}(X) \text{:- } \text{in\_cover}(I), s(X,I).
\text{:- } s(X,I), \text{ not covered}(X).
\]

the symbol \(n\) is a placeholder (for the number of sets); \(s/2\) is an input predicate (\(s(X,I)\) means that that \(X\) belongs to the \(I\)-th set); \text{in\_cover}/1 is an output predicate; and \text{covered}/1 is private.

A specification given to ANTHEM consists of two groups of first-order sentences. One group, “assumptions,” is the list of conditions on the values of placeholders and input predicates that are satisfied when the input is considered valid. In the exact cover example,

\(^1\) An exact cover of a collection \(S\) of sets is a subcollection \(S'\) of \(S\) such that each element of the union of all sets in \(S\) belongs to exactly one set in \(S'\).
there are two assumptions: one is that \( n \) is a nonnegative integer; the other is expressed by the formula

\[
\forall Y (\exists X s(X,Y) \rightarrow 1 \leq Y \text{ and } Y \leq n).
\]  

(1)

The second group, “specs,” is the list of conditions that characterize correct outputs. In the exact cover example, the specs encode the definition of an exact cover:

\[
\begin{align*}
\forall Y & (\text{in_cover}(Y) \rightarrow 1 \leq Y \text{ and } Y \leq n), \\
\forall X & (\exists Y s(X,Y) \rightarrow \exists Y (s(X,Y) \text{ and } \text{in_cover}(Y))), \\
\forall Y, Z & (\exists X (s(X,Y) \text{ and } s(X,Z)) \\
& \rightarrow Y = Z).
\end{align*}
\]

(2)

To bridge the gap between the language of io-programs and the language of specifications, ANTHEM employs a transformation similar to program completion (Clark 1978). The completion \( \text{COMP}[\Omega] \) of an io-program \( \Omega \) faithfully represents the meaning of \( \Omega \) whenever \( \Omega \) satisfies the syntactic condition called “tightness.” Tightness of io-programs is a generalization of a condition identified in early work on the relationship between stable models and completion (Fages 1994). The operation of ANTHEM is based on the fact that the correctness of a tight io-program \( \Omega \) with respect to a specification can be described by the sentence

\[
A \rightarrow (\text{COMP}[\Omega] \leftrightarrow S),
\]

(3)

where \( A \) is the conjunction of the assumptions, and \( S \) is the conjunction of the specs (Fandinno et al. 2020, Section 6.3).

A program is considered tight if its predicate dependency graph has no cycles consisting of positive edges. Thus the tightness requirement eliminates practically all uses of recursion in the program. For example, the program

\[
\begin{align*}
p(0). \\
p(X+1) & : \ p(X), \ X < 5.
\end{align*}
\]

(4)

is not tight. Its only stable model consists of the atoms \( p(0), \ldots, p(5) \), so that it correctly implements the specification

\[
\forall X \ (p(X) \leftrightarrow 0 \leq X \text{ and } X \leq 5),
\]

but the verification method employed by ANTHEM cannot establish this fact.

In this paper we extend the main result by Fandinno et al. (2020) by showing that the above-mentioned property of tight io-programs—the possibility of characterizing correctness by formula (3)—holds, more generally, for “locally tight” io-programs, which allow some forms of recursion.

For example, program (4) will become locally tight if we replace its second rule by

\[
\begin{align*}
p(X+1) & : \ p(X), \ 0 \leq X, \ X < 5.
\end{align*}
\]

or by

\[
\begin{align*}
p(X+1) & : \ p(X), \ X = 0..4.
\end{align*}
\]

The io-programs obtained from (4) by replacing the second rule with

\[
\begin{align*}
p(X+1) & : \ p(X), \ X = 0..n.
\end{align*}
\]

(5)
or with the choice rule
\[ \{p(X+1)\} :- p(X), X = 0..n. \]  
(6)
where \(n\) is a placeholder, are locally tight as well. Yet another example is given by the rule
\[ p(X+1) :- p(X), q(X). \]  
(7)
where \(q/1\) is an input predicate, in place of the second rule of (4).

The idea of this generalization of tightness is to look at dependencies between ground atoms rather than dependencies between predicate symbols. Locally tight io-programs are somewhat similar to atomic-tight sentences studied by Lee and Meng (2011).

Rules resembling (6) and (7) are found “in the wild.” For example, the rule
\[ \text{blocked}(D-1,P,T) :- \text{blocked}(D,P,T), \text{disk}(D). \]
from a Towers of Hanoi program (Gebser et al. 2019, Section 2.2) is similar to rule (7), and the inertia rule
\[ \{\text{holds}(\text{loc}(B,L),T+1)\} :- \text{holds}(\text{loc}(B,L),T), T = 0..h-1. \]  
(Lifschitz 2019, Listing 8.1) is similar ro rule (6).

The next section is a review of definitions and notation introduced in earlier publications on ANTHEM (Lifschitz et al. 2019; Fandinno et al. 2020). In Section 3 we define locally tight io-programs and state a Fages-style theorem for them—a theorem that characterizes their semantics in terms of completion. The definition of a specification adopted in the ANTHEM project is reviewed in Section 4, and the use of ANTHEM for verifying locally tight programs is discussed in Section 5.

Proofs in this paper rely on the approach to completion, logic of here-and-there, and stable models that allows the underlying first-order language to be many-sorted and distinguishes between intensional and extensional predicate symbols. This is discussed in Appendix A. Appendix B contains most of the proofs. Main Lemma, stated in Section B.1 and proved in Section B.2, may be of more general interest than the rest of the paper. It generalizes two Fages-style theorems for first-order formulas published earlier (Ferraris et al. 2011, Theorem 11; and Lee and Meng 2011, Corollary 4). Appendix A and Sections B.1, B.2 can be understood without reading the rest of the paper.

An extended abstract of this paper is included in the Technical Communications of the 37th International Conference on Logic Programming (Fandinno and Lifschitz 2021).

2 Background
The review in this section follows earlier publications on ANTHEM (Lifschitz et al. 2019; Fandinno et al. 2020).

2.1 Programs
The programming language mini-GRINGO, defined in this section, is a subset of the input language of the grounder GRINGO. The description of mini-GRINGO programs below uses “abstract syntax,” which disregards some details related to representing programs by strings of ASCII characters. We assume that three countably infinite sets of symbols are
selected: numerals, symbolic constants, and variables. We assume that a 1-1 correspondence between numerals and integers is chosen; the numeral corresponding to an integer \( n \) will be denoted by \( \pi n \). Precomputed terms are numerals, symbolic constants, and the symbols \( \inf, \sup \). Terms allowed in a mini-GRINGO program are formed from precomputed terms and variables using the six operation names

\[
+ \quad - \quad \times \quad / \quad \backslash \quad .
\]

An \textit{atom} is a symbolic constant optionally followed by a tuple of terms in parentheses. For instance, \textit{blocked}\((D - \mathit{\overline{1}}, B, T)\) is an atom. A \textit{literal} is an atom possibly preceded by one or two occurrences of \textit{not}. A \textit{comparison} is an expression of the form \( t_1 \prec t_2 \), where \( t_1, t_2 \) are mini-GRINGO terms and \( \prec \) is one of the six comparison symbols

\[
= \quad \neq \quad < \quad > \quad \leq \quad \geq
\]

A \textit{rule} is an expression of the form

\[
\text{Head} \leftarrow \text{Body},
\]

where

- \textit{Body} is a conjunction (possibly empty) of literals and comparisons, and
- \textit{Head} is either an atom (then (9) is a \textit{basic rule}), or an atom in braces (then (9) is a \textit{choice rule}), or empty (then (9) is a \textit{constraint}).

We identify an atom \( A \) with the basic rule \( A \leftarrow \).

A \textit{program} is a finite set of rules.

A \textit{predicate symbol} is a pair \( p/n \), where \( p \) is a symbolic constant and \( n \) is a nonnegative integer. About a program or another syntactic expression we say that it \textit{contains} a predicate symbol \( p/n \), or that \( p/n \) \textit{occurs} in it, if it contains an atom of the form \( p(t_1, \ldots, t_n) \).

The semantics of ground terms is defined by assigning to every ground term \( t \) the finite set \([t]\) of its \textit{values} (Lifschitz et al. 2019, Section 3). Values of a ground term are precomputed terms. For instance,

\[
[2/2] = \{1\}, \quad [2/0] = \emptyset, \quad [0..2] = \{0, \mathit{\overline{1}}, 2\}.
\]

Stable models of a program are defined as stable models of the set of propositional formulas obtained from it by the syntactic transformation \( \tau \) (Lifschitz et al. 2019, Section 3). Atomic parts of these formulas are \textit{precomputed atoms}—atoms \( p(t) \) such that the members of \( t \) are precomputed terms. Thus, every stable model is a set of precomputed atoms.

Most constructs included in this language are available also in the input language of \textsc{dlv} (Alviano et al. 2017). The latter does not include choice rules, and it does not allow intervals in bodies of rules.
2.2 Two-sorted formulas

The language of assumptions and specs is a first-order language with two sorts: the sort \textit{general} and its subsort \textit{integer}. General variables are meant to range over arbitrary precomputed terms, and we will identify them with variables used in mini-GRINGO rules. Integer of the second sort are meant to range over numerals (or, equivalently, integers).

The signature $\sigma_0$ of the language includes

- all precomputed terms as object constants; an object constant is assigned the sort \textit{integer} iff it is a numeral;
- the symbols $+$, $-$ and $\times$ as binary function constants; their arguments and values have the sort \textit{integer};
- predicate symbols\, $p/n$ as \(n\)-ary predicate constants;
- comparison symbols (8) as binary predicate constants.

The symbols $/$ and $\setminus$ are nor included in this signature because division is not a total function on integers. The symbol $..$ is not included either, because intervals are not meant to be among values of variables of this first-order language.

An atomic formula \((p/n)(t)\) can be abbreviated as \(p(t)\). An atomic formula \(\prec(t_1,t_2)\), where $\prec$ is a comparison symbol, can be written as \(t_1 \prec t_2\).

In the exact cover example discussed in the Introduction, \(X, Y, Z\) are general variables. The assumption that \(n\) is a nonnegative integer can be expressed by formula

$$\exists N \ (N = n \text{ and } N \geq 0),$$

where \(N\) is an integer variable.

2.3 Translating literals and comparisons

We define, for every mini-GRINGO term \(t\), a formula \(val_t(Z)\) over the signature $\sigma_0$, where \(Z\) is a general variable that does not occur in \(t\). That formula expresses, informally speaking, that \(Z\) is one of the values of \(t\). The definition is recursive:

- if \(t\) is a precomputed term or a variable then \(val_t(Z)\) is \(Z = t\);
- if \(t\) is \((t_1 \ op \ t_2)\), where \(op\) is $+$, $-$, or $\times$ then \(val_t(Z)\) is
  $$\exists IJ(Z = I \ op \ J \land val_{t_1}(I) \land val_{t_2}(J))$$
  where \(I, J\) are fresh (that is, unused) integer variables;
- if \(t\) is \((t_1 / t_2)\) then \(val_t(Z)\) is
  $$\exists IJQR(I = J \times Q + R \land val_{t_1}(I) \land val_{t_2}(J)$$
  $$\land J \neq 0 \land R \geq 0 \land R < Q \land Z = Q),$$
  where \(I, J, Q, R\) are fresh integer variables;
- if \(t\) is \((t_1 \setminus t_2)\) then \(val_t(Z)\) is
  $$\exists IJQR(I = J \times Q + R \land val_{t_1}(I) \land val_{t_2}(J)$$
  $$\land J \neq 0 \land R \geq 0 \land R < Q \land Z = R),$$

\(^2\) The syntax and semantics of many-sorted formulas are reviewed in Section A.1. The need to use a language with two sorts is explained by the fact that function symbols in a first-order language are supposed to represent total functions, and arithmetic operations are not defined on symbolic constants.
where $I$, $J$, $Q$, $R$ are fresh integer variables;

- if $t$ is $(t_1..t_2)$ then $\text{val}_t(Z)$ is
  $$\exists IJK (\text{val}_{t_1}(I) \land \text{val}_{t_2}(J) \land I \leq K \land K \leq J \land Z = K),$$
  where $I$, $J$, $K$ are fresh integer variables.

If $t$ is a tuple $t_1,\ldots,t_n$ of mini-GRINGO terms, and $Z$ is a tuple $Z_1,\ldots,Z_n$ of distinct general variables, then $\text{val}_t(Z)$ stands for the conjunction $\text{val}_{t_1}(Z_1) \land \cdots \land \text{val}_{t_n}(Z_n)$.

The translation $\tau^b$, described below, transforms literals and comparisons into formulas over the signature $\sigma_0$. (The superscript $b$ reflects the fact that this translation is close to the meaning of expressions in bodies of rules.)

- $\tau^b(p(t)) = \tau^b(\neg \neg p(t)) = \exists Z (\text{val}_t(Z) \land p(Z));$
- $\tau^b(\neg p(t)) = \exists Z (\text{val}_t(Z) \land \neg p(Z));$
- $\tau^b(t_1 \prec t_2) = \exists Z_1 Z_2 (\text{val}_{t_1}(Z_1) \land \text{val}_{t_2}(Z_2) \land Z_1 \prec Z_2).$

If $\text{Body}$ is a conjunction $B_1 \land B_2 \land \cdots$ of literals and comparisons, then $\tau^b(\text{Body})$ stands for the conjunction $\tau^b(B_1) \land \tau^b(B_2) \land \cdots$.

### 2.4 Programs with input and output

A program with input and output, or an io-program, is a quadruple

$$(\Pi, \text{PH}, \text{In}, \text{Out}),$$

where

- $\Pi$ is a program,
- $\text{PH}$ is a finite set of symbolic constants;
- $\text{In}$ is a finite set of predicate symbols that do not occur in the heads of rules of $\Pi$;
- $\text{Out}$ is a finite set of predicate symbols that is disjoint from $\text{In}$.

Members of $\text{PH}$ are the placeholders of (11); members of $\text{In}$ are the input symbols of (11); members of $\text{Out}$ are the output symbols of (11). The input symbols and output symbols of an io-program are collectively called its public symbols. The other predicate symbols occurring in the rules are private. An atom $p(t_1,\ldots,t_n)$ is public if $p/n$ is public, and private if $p/n$ is private.

A valuation on a set $\text{PH}$ of symbolic constants is a function that maps elements of $\text{PH}$ to precomputed terms that do not belong to $\text{PH}$. For a valuation $v$ and a rule $R$, by $v(R)$ we denote the rule obtained from $R$ by replacing every occurrence of every symbol $t$ in the domain of $v$ by the term $v(t)$. Applying valuations to programs, terms and formulas is understood in a similar way.

An input for an io-program (11) is a pair $(v, I)$, where

- $v$ is a valuation on the set $\text{PH}$ of its placeholders, and
- $I$ is a set of precomputed atoms that do not contain placeholders, such that the predicate symbol of each atom in $I$ is an input symbol.

About a set of precomputed atoms we say that it is an io-model of an io-program (11) for an input $(v, I)$ if it is the set of all public atoms of some stable model of the program $v(\Pi) \cup I$. 
Take, for instance, the encoding of exact covers discussed in the Introduction. The input \((v, I)\) for that io-program that represents the set \(\{\{a, b\}, \{b, c\}, \{c\}\}\) is defined by the conditions
\[
v(n) = 3 \quad \text{and} \quad I = \{s(a, 1), s(b, 1), s(b, 2), s(c, 2), s(c, 3)\}.
\]
The program \(v(\Pi)\), written in the notation of Section 2.1, consists of the rules
\[
\{ \text{in_cover}(1..3) \},
\]
\[
\leftarrow I \neq J \land \text{in_cover}(I) \land \text{in_cover}(J) \land s(X, I) \land s(X, J),
\]
\[
\text{covered}(X) \leftarrow \text{in_cover}(I) \land s(X, I),
\]
\[
\leftarrow s(X, I) \land \text{not covered}(X).
\]
The program \(v(\Pi) \cup I\) has one stable model,
\[
\{ s(a, 1), \ldots, s(c, 3), \text{in_cover}(1), \text{in_cover}(3), \text{covered}(a), \text{covered}(b), \text{covered}(c) \},
\]
so that the only io-model of \(\Omega\) for the input \((v, I)\) is
\[
\{ s(a, 1), \ldots, s(c, 3), \text{in_cover}(1), \text{in_cover}(3) \}. \quad (12)
\]

### 2.5 Completion of a program with input and output

The definition of the completion of an io-program (Fandinno et al. 2020, Section 6) generalizes the completion process described by Keith Clark (1978). The completion \(\text{COMP}[\Omega]\) of an io-program \(\Omega\) is a formula over the two-sorted signature \(\sigma_0\), and there are bound integer variables in it if rules of \(\Omega\) contain arithmetic operations. It is, generally, a second-order formula: the private symbols of \(\Omega\) are replaced in it by existentially quantified predicate variables.

Let \(p/n\) be an output symbol or a private symbol of \(\Omega\). The definition of \(p/n\) in \(\Omega\) is the set of all rules of \(\Omega\) that have the form
\[
p(t) \leftarrow \text{Body} \quad (13)
\]
or
\[
\{ p(t) \} \leftarrow \text{Body}, \quad (14)
\]
where \(t\) is an \(n\)-tuple of terms. The completed definition of \(p/n\) is obtained from the definition \(\{R_1, \ldots, R_k\}\) of \(p/n\) as follows. Take an \(n\)-tuple \(V\) of fresh general variables. If \(R_i\) is (13), then let \(B_i\) be the formula
\[
\tau^b(\text{Body}) \land \text{val}_t(V).
\]
If \(R_i\) is (14), then let \(B_i\) be the formula
\[
\tau^b(\text{Body}) \land \text{val}_t(V) \land p(V).
\]
The completed definition of \(p/n\) in \(\Omega\) is the formula
\[
\forall V \left( p(V) \leftrightarrow \bigvee_{i=1}^k \exists U_i B_i \right), \quad (15)
\]
where \(U_i\) is the list of all variables occurring in rule \(R_i\).

The first-order completion \(\text{COMP}^1[\Omega]\) of an io-program \(\Omega\) is the conjunction of
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- the completed definitions of all output symbols and private symbols of \( \Omega \), and
- the sentences

\[ \forall \neg \tau (\text{Body}) \quad (16) \]

for all constraints \( \leftarrow \text{Body of } \Omega \).

The completion \( \text{COMP}[\Omega] \) of \( \Omega \) is the sentence

\[ \exists P_1 \cdots P_l F, \]

where \( F \) is obtained from \( \text{COMP}^1[\Omega] \) by replacing the private symbols \( p_1/n_1, \ldots, p_l/n_l \) of \( \Omega \) by distinct predicate variables \( P_1, \ldots, P_l \).

Consider, for example, the io-program with the rules

\[ p(a), \]
\[ p(b), \]
\[ q(X,Y) \leftarrow p(X) \land p(Y), \]

without input symbols and with the output symbol \( q/2 \). The definition of \( p/1 \) consists of the first two rules, and the corresponding formulas \( B_1, B_2 \) are \( V = a \) and \( V = b \). The completed definition of \( p/1 \) is

\[ \forall V (p(V) \leftrightarrow V = a \lor V = b). \]

The third rule of the program is the definition of \( q/2 \), and the corresponding formula \( B \) is

\[ V_1 = X \land V_2 = Y \land \exists Z (Z = X \land p(Z)) \land \exists Z (Z = Y \land p(Z)). \]

The completed definition of \( q/2 \) is

\[ \forall V_1 V_2 (q(V_1, V_2) \leftrightarrow \exists XY (V_1 = X \land V_2 = Y \land \exists Z (Z = X \land p(Z)) \land \exists Z (Z = Y \land p(Z)))). \]

The first-order completion of the program is

\[ \forall V (p(V) \leftrightarrow V = a \lor V = b) \land \forall V_1 V_2 (q(V_1, V_2) \leftrightarrow \exists XY (V_1 = X \land V_2 = Y \land \exists Z (Z = X \land p(Z)) \land \exists Z (Z = Y \land p(Z)))). \]

and its completion is

\[ \exists P (\forall V (p(V) \leftrightarrow V = a \lor V = b) \land \forall V_1 V_2 (q(V_1, V_2) \leftrightarrow P(V_1) \land P(V_2))). \]

The last formula can be equivalently rewritten as

\[ \exists P (\forall V (p(V) \leftrightarrow V = a \lor V = b) \land \forall V_1 V_2 (q(V_1, V_2) \leftrightarrow P(V_1) \land P(V_2))). \]

Second-order quantifiers in completion formulas can be often eliminated (Fandinno et al. 2020, Section 6.4). For instance, formula (17) is equivalent to the first-order formula

\[ \forall V_1 V_2 (q(V_1, V_2) \leftrightarrow (V_1 = a \lor V_1 = b) \land (V_2 = a \lor V_2 = b)). \]

3 The symbol \( \forall \) denotes universal closure.
3 Locally tight programs

For any tuple $t_1, \ldots, t_n$ of ground terms, $[t_1, \ldots, t_n]$ stands for the set of tuples $r_1, \ldots, r_n$ of precomputed terms such that $r_1 \in [t_1], \ldots, r_n \in [t_n]$.

The positive dependency graph of an io-program $\Omega$ for an input $(v, I)$ is the directed graph defined as follows. Its vertices are the ground atoms $p(r_1, \ldots, r_n)$ such that $p/n$ is an output symbol or a private symbol of $\Omega$, and each $r_i$ is a precomputed term different from the placeholders of $\Omega$. It has an edge from $p(r)$ to $p'(r')$ iff there exists a ground instance $R$ of one of the rules of $v(\Pi)$ such that

(a) the head of $R$ has the form $p(t)$ or $\{p(t)\}$, where $t$ is a tuple of terms such that $r \in [t]$;
(b) the body of $R$ has a conjunctive term of the form $p'(t)$, where $t$ is a tuple of terms such that $r' \in [t]$;
(c) for every conjunctive term of the body of $R$ that contains an input symbol of $\Omega$ and has the form $q(t)$ or $\text{not not } q(t)$, the set $[t]$ contains a tuple $r$ such that $q(r) \in I$;
(d) for every conjunctive term of the body of $R$ that contains an input symbol of $\Omega$ and has the form $\text{not } q(t)$, the set $[t]$ contains a tuple $r$ such that $q(r) \notin I$;
(e) for every comparison $t_1 \prec t_2$ in the body of $R$, there exist terms $r_1, r_2$ such that $r_1 \in [t_1], r_2 \in [t_2]$, and the relation $\prec$ holds for the pair $(r_1, r_2)$.

Recall that an infinite walk in a directed graph is an infinite sequence of edges which joins a sequence of vertices. If the positive dependency graph of $\Omega$ for an input $(v, I)$ has no infinite walks, then we say that $\Omega$ is locally tight on that input. The examples below illustrate the intuitive meaning of this condition: the absence of “nonterminating recursion.”

Example 1
The rules of the io-program $\Omega_1$ are

$$p(\overline{0}),$$
$$p(X + \overline{1}) \leftarrow p(X) \land X = \overline{0} \ldots n$$

(18)

(the first of rules (4) and rule (5) in the notation of Section 2.1); $n$ is its placeholder, and $p/1$ is its output symbol. Since $\Omega_1$ has no input symbols, an input for it is just a valuation $v$ that assigns a precomputed term to the symbol $n$. The vertices of the positive dependency graph of $\Omega_1$ for an input $v$ are atoms $p(r)$ for all precomputed terms $r$ different from $n$. Ground instances of the rules of $\Omega_1(v)$ are

$$p(\overline{0}),$$
$$p(r + \overline{1}) \leftarrow p(r) \land r = \overline{0} \ldots v(n)$$

for all precomputed terms $r$. The first rule does not contribute edges to the dependency graph because its body is empty. The rule in the second line contributes an edge only if $r$ belongs to the set $[\overline{0} \ldots v(n)]$. This set is $\{\overline{0}, \overline{1}, \ldots, v(n)\}$ if $v(n)$ is a nonnegative numeral, and it is empty otherwise. The edges of the graph are

$$(p(\overline{1}), p(\overline{0})), (p(\overline{2}), p(\overline{1})), \ldots, (p(\overline{n_0 + 1}), p(\overline{n_0}))$$

if $v(n)$ is a nonnegative numeral $\overline{n_0}$. Otherwise, the graph has no edges. In either case, there are no infinite walks in the graph. We conclude that $\Omega_1$ is locally tight on all inputs.
Program $\Omega_1$ can be viewed as an implementation of the following specification: given a nonnegative integer $n$, find the set $\{0, \ldots, n + 1\}$. In Section 5 we show how this claim can be verified using ANTHEM.

Example 2
Consider the io-program $\Omega_2$, obtained from $\Omega_1$ by replacing its second rule with the rule
$$p(X + 1) \leftarrow p(X) \land X < n.$$ If $v(n)$ is a numeral $n_0$ then the edges of the dependency graph are $(p(i + 1), p(i))$ for all $i$ that are smaller than $n_0$. There is an infinite walk in this graph, $p(n_0), p(n_0 - 1), \ldots$, so that $\Omega_2$ is not locally tight on any such input $v$.

Example 3
The io-program $\Omega_3$ below describes the effect of an action—walking from one room to another—on the location of a person. It has the rules
$$\text{in}(P, R, 0) \leftarrow \text{in}_0(P, R),$$ $$\text{in}(P, R, T + 1) \leftarrow \text{goto}(P, R, T),$$ $$\{\text{in}(P, R, T + 1)\} \leftarrow \text{in}(P, R, T) \land T = \overline{0}..h - 1,$$ $$\leftarrow \text{in}(P, R_1, T) \land \text{in}(P, R_2, T) \land R_1 \neq R_2,$$ $$\text{in}_{\text{building}}(P, T) \leftarrow \text{in}(P, R, T),$$ $$\leftarrow \text{not in}_{\text{building}}(P, T) \land \text{person}(P) \land T = \overline{0}..h,$$ with the placeholder $h$, input symbols $\text{person}/1$, $\text{in}_0/2$, and $\text{goto}/3$, and the output symbol $\text{in}/3$. We can read $\text{in}(P, R, T)$ as “person $P$ is in room $R$ at time $T$,” and $\text{goto}(P, R, T)$ as “person $P$ goes to room $R$ between times $T$ and $T + 1$.” The symbol $h$ represents the “horizon”—the length of the scenarios under consideration. The third rule of the program encodes the commonsense law of inertia for this domain (Lifschitz 2019, Section 8.4), and this rule makes the program nontight. The vertices of its positive dependency graph for an input $(v, I)$ are atoms $\text{in}(p, r, t)$ and $\text{in}_{\text{building}}(p, t)$, where $p, r, t$ are precomputed terms different from $h$. Its edges go from $\text{in}_{\text{building}}(p, t)$ to $\text{in}(p, r, t)$ and, if $v(h)$ is a numeral $h_0$, from $\text{in}(p, r, i + 1)$ to $\text{in}(p, r, i)$ for $i = 0, \ldots, h_0 - 1$. It is clear that this graph has no infinite walks, so that $\Omega_3$ is locally tight on all inputs.

Theorem 1 below describes the relationship between the completion of a locally tight io-program and its io-models. To state the theorem, we need to relate interpretations of the signature $\sigma_0$ to sets of precomputed atoms.

An interpretation $I$ of $\sigma_0$ is standard if
(i) $|I|_{\text{general}}$ is the set of all precomputed terms;
(ii) $|I|_{\text{integer}}$ is the set of all numerals;
(iii) for every precomputed term $t$, $t^I = t$;
(iv) for every pair $t_1$, $t_2$ of ground integer terms, if $t_1^I = m$ and $t_2^I = n$ then $$(t_1 + t_2)^I = m + n,$$
and similarly for subtraction and multiplication;
(v) for every pair $t_1$, $t_2$ of ground terms, $(t_1 \prec t_2)^I = \text{true}$ iff the relation $\prec$ holds for the pair $t_1^I$, $t_2^I$. 

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For any set $J$ of precomputed atoms there exists a unique standard interpretation $I$ of $\sigma_0$ such that for every precomputed atom $p(t)$, $p(t)^I = true$ iff $p(t) \in J$. This interpretation will be denoted by $J^\uparrow$.

In the theorem and the corollary below, $\Omega$ is an io-program, and $\mathcal{P}$ is a set of public precomputed atoms that do not contain placeholders of $\Omega$. By $\mathcal{P}^{in}$ we denote the set of all atoms in $\mathcal{P}$ that include an input symbol of $\Omega$.

**Theorem 1**
For any input $(v, I)$ on which $\Omega$ is locally tight, $\mathcal{P}$ is an io-model of $\Omega$ for the input $(v, I)$ iff $\mathcal{P}^\uparrow$ satisfies $v(\text{COMP}[\Omega])$ and $\mathcal{P}^{in} = I$.

**Corollary 1**
If $\Omega$ is locally tight on all inputs, then $\mathcal{P}$ is an io-model of $\Omega$ for the input $(v, \mathcal{P}^{in})$ iff $\mathcal{P}^\uparrow$ satisfies $v(\text{COMP}[\Omega])$.

Consider, for instance, the io-program $\Omega_3$ (Example 3 above). The corollary shows that a set $\mathcal{P}$ of public precomputed atoms that do not contain the placeholder $h$ is an io-model of $\Omega_3$ iff $\mathcal{P}^\uparrow$ satisfies the formula obtained from the completion of $\Omega_3$ by substituting the term $v(h)$ for all occurrences of $h$.

### 4 Review: Specifications

A *specification* (Fandinno et al. 2020, Section 5.3) is a quintuple

$$ (PH, In, Out, A, S) $$

(19)

where

- $PH$ is a finite set of symbolic constants,
- $In$ and $Out$ are disjoint finite sets of predicate symbols;
- $A$ is a finite set of sentences such that all their predicate symbols other than comparisons belong to $In$;
- $S$ is a finite set of sentences such that all their predicate symbols other than comparisons belong to $In \cup Out$.

Members of $PH$ are the *placeholders* of (19); members of $In$ are the *input symbols* of (19); members of $Out$ are the *output symbols* of (19). Members of $A$ are the *assumptions* of (19), and members of $S$ are the *specs* of (19). We identify each of the sets $A, S$ with the conjunction of its elements.

The semantics of specifications defines which io-programs implement a given specification. In the following definition, $\Sigma$ is a specification, and $\Omega$ is an io-program with the same placeholders, the same input symbols, and the same output symbols as $\Sigma$. We say that $\Omega$ *implements* $\Sigma$ if, for every valuation $v$ on the set of placeholders and every set $\mathcal{P}$ of precomputed public atoms without placeholders that satisfies the assumptions of $\Sigma$ for $v$,

$$ \mathcal{P} \text{ is an io-model of } \Omega \text{ for the input } (v, \mathcal{P}^{in}) \text{ iff } \mathcal{P} \text{ satisfies the specs of } \Sigma \text{ for } v. $$
5 Verifying locally tight programs

An interpretation of $\sigma_0$ is standard for a set $PH$ of symbolic constants if it satisfies conditions (i), (ii), (iv), (v) from the definition of a standard interpretation in Section 3 and the following weak form of condition (iii):

(iii)' for every precomputed term $t$ that belongs to $PH$, $t^I$ does not belong to $PH$;

(iii)'' for every precomputed term $t$ that does not belong to $PH$, $t^I = t$.

For any valuation $v$ on $PH$ and any set $J$ of precomputed atoms that do not contain symbols from $PH$, there exists a unique interpretation $I$ of $\sigma_0$ such that

- $I$ is standard for $PH$,
- for every symbol $t$ from $PH$, $t^I = v(t)$, and
- for every precomputed atom $p(t^I) = true$ iff $p(t) \in J$.

This interpretation will be denoted by $J^v$. If $J^v$ satisfies a sentence $F$ over $\sigma_0$ then we say that $J$ satisfies $F$ for the valuation $v$.

In the definition and the theorem below, $\Sigma$ is a specification, and $\Omega$ is an io-program with the same placeholders, the same input symbols, and the same output symbols as $\Sigma$. We say that $\Omega$ is locally tight for $\Sigma$ if $\Omega$ is locally tight on every input $(v, I)$ such that $I$ satisfies all assumptions of $\Sigma$ for the valuation $v$.

**Theorem 2**

If $\Omega$ is locally tight for $\Sigma$ then the following conditions are equivalent:

(i) $\Omega$ implements $\Sigma$;

(ii) the sentence

$$A \rightarrow (\text{COMP}[\Omega] \leftrightarrow S),$$

where $A$ is the conjunction of the assumptions of $\Sigma$, and $S$ is the conjunction of the specs of $\Omega$, is satisfied by all interpretations that are standard for the set of placeholders of $\Omega$.

This theorem can be derived from Theorem 1 as follows. Assume that $\Omega$ is locally tight for $\Sigma$. Condition (i) means that for every valuation $v$ on the set of placeholders and every set $P$ of precomputed public atoms without placeholders such that $P^v$ satisfies $A$,

$$P \text{ is an io-model of } \Omega \text{ for the input } (v, P^\text{in}) \text{iff } P^v \text{ satisfies } S.$$  

Consider a valuation $v$ on the set of placeholders and a set $P$ of precomputed public atoms without placeholders such that $P^v$ satisfies $A$. Since every predicate symbol occurring in $A$ is an input symbol, $(P^\text{in})^v$ satisfies $A$ as well, so that $P^\text{in}$ satisfies all assumptions of $\Omega$ for the valuation $v$. Since $\Omega$ is locally tight for $\Sigma$, it follows that $\Omega$ is locally tight on the input $(v, P^\text{in})$. By Theorem 1, we can conclude that

$$P \text{ is an io-model of } \Omega \text{ for the input } (v, P^\text{in}) \text{ iff } P^\uparrow \text{ satisfies } v(\text{COMP}[\Omega]),$$

or, equivalently,

$$P \text{ is an io-model of } \Omega \text{ for the input } (v, P^\text{in}) \text{ iff } P^v \text{ satisfies } \text{COMP}[\Omega].$$

It follows that condition (21) is equivalent to the condition

$$P^v \text{ satisfies } \text{COMP}[\Omega] \text{ iff } P^v \text{ satisfies } S,$$
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and consequently to the condition

\[ P^v \text{ satisfies } \text{COMP}[\Omega] \leftrightarrow S. \]

We conclude that \( \Omega \) implements \( \Sigma \) iff for all \( v \) and \( P \), \( P^v \) satisfies (20). It remains to observe that an interpretation of \( \sigma_0 \) is standard for \( PH \) iff it can be represented in the form \( P^v \).

According to Theorem 2, we establish that an io-program \( \Omega \) implements a specification \( \Sigma \) for which it is locally tight if we derive formula (20) from a set of axioms that are satisfied by all interpretations that are standard for the set of placeholders. This is how ANTHEM operates. To find a proof of formula (20), it eliminates the second-order quantifiers from it as in the example at the end of Section 2.5, and then calls the resolution theorem prover VAMPIRE (Kovačs and Voronkov 2013). The prover is actually called twice: first to derive \( S \) from \( A \) and \( \text{COMP}[\Omega] \), and then to derive \( \text{COMP}[\Omega] \) from \( A \) and \( S \).

In some cases, such as the verification of the exact cover program shown in the Introduction, the ANTHEM/VAMPIRE tandem works as a stand-alone system: the user gives ANTHEM a specification and a program, and ANTHEM comes back with a success message. But more often, the user has to help VAMPIRE find a proof. One way to do that is to provide a series of lemmas leading from axioms to the goal; these helper lemmas are derived by VAMPIRE one by one before proving the goal.

The user can also help by stating facts about integers that VAMPIRE cannot (easily) prove; they may be used as additional axioms. In one experiment, for instance, an io-program was verified using the induction axiom for the predicate symbol \( p/1 \):

\[
p(0) \text{ and } \forall N \ (N \geq 0 \text{ and } p(N) \rightarrow p(N+1)) \rightarrow \forall N \ (N \geq 0 \rightarrow p(N))
\]

(Fandinno et al. 2020, Section 7).

The claim that the locally tight io-program \( \Omega_1 \) (Example 1 above) implements the specification

\[
\begin{align*}
\text{assume: } & n \geq 0. \\
\text{spec: } & \forall X \ (p(X) \leftrightarrow 0 \leq X \text{ and } X \leq n+1).
\end{align*}
\]

has been verified by ANTHEM and VAMPIRE using the induction axiom for the property \( N \leq n+1 \rightarrow p(N) \):

\[
(0 \leq n+1 \rightarrow p(0)) \quad \text{and} \quad \forall N \ (N \geq 0 \text{ and } (N \leq n+1 \rightarrow p(N))) \rightarrow (N+1 \leq n+1 \rightarrow p(N+1)) \rightarrow \forall N \ (N \geq 0 \rightarrow (N \leq n+1 \rightarrow p(N))).
\]

The implication from the completion to the spec was derived by VAMPIRE in 7.58 seconds. In the process of deriving the converse we helped VAMPIRE by suggesting the lemma

\[
\forall M \ (p(M) \leftrightarrow M = 0 \text{ or } (p(M-1) \text{ and } 0 \leq M-1 \text{ and } M \leq n+1)).
\]

This helper lemma was derived in 0.08 seconds, and then the converse was derived in 90 seconds.
6 Conclusion

The main result of this paper, Theorem 2 (see Section 5), shows that the verification process implemented in ANTHEM is applicable more widely than envisioned in the original publication (Fandinno et al. 2020). The local tightness condition allows some forms of recursion. In particular, some programs describing effects of actions are locally tight. Using ANTHEM to verify locally tight programs does not require any enhancement of existing software.

The class of tight programs is defined in terms of a finite graph, and membership in this class is easy to check; ANTHEM does that as part of processing the program that it is instructed to verify. On the other hand, membership in the class of locally tight programs is undecidable, even for one-rule programs without placeholders and input symbols. Indeed, a rule of the form

\[ p \leftarrow p \land t = 0, \]

where \( t \) is a polynomial, is locally tight if and only if the Diophantine equation in the body has no solutions. Establishing that a program is locally tight involves proving a property of an infinite graph, and it may be possible to use the reasoning capabilities of VAMPIRE to assist the user of ANTHEM in this process. This is a topic for future work.

The designers of ANTHEM chose to use VAMPIRE as its automated reasoning companion because VAMPIRE supports both typed first-order reasoning and integer arithmetics, and because its performance outclassed other provers in preliminary experiments. But it seems that the absence of induction in the toolbox of VAMPIRE will often cause difficulties when ANTHEM is used to verify programs that are locally tight but not tight. Experiments with ANTHEM may become a useful source of challenge problems for developers of automated reasoning systems.

Acknowledgements

We thank Yuliya Lierler, Nathan Temple, and the anonymous referees for their valuable comments.

Competing interests: The authors declare none.

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In Technical Communications of the 37th International Conference on Logic Programming (ICLP), 68–69.


Appendix A Completion and stable models in many-sorted languages

A.1 Formulas

A many-sorted signature consists of symbols of three kinds—sorts, function constants, and predicate constants. A reflexive and transitive subsort relation is defined on the set of sorts. A tuple $s_1, \ldots, s_n$ ($n \geq 0$) of argument sorts is assigned to every function constant and to every predicate constant; in addition, a value sort is assigned to every function constant. Function constants with $n = 0$ are called object constants.

We assume that for every sort, an infinite sequence of object variables of that sort is chosen. Terms over a (many-sorted) signature $\sigma$ are defined recursively:

- object constants and object variables of a sort $s$ are terms of sort $s$;
- if $f$ is a function constant with argument sorts $s_1, \ldots, s_n$ ($n > 0$) and value sort $s$, and $t_1, \ldots, t_n$ are terms such that the sort of $t_i$ is a subsort of $s_i$ ($i = 1, \ldots, n$), then $f(t_1, \ldots, t_n)$ is a term of sort $s$.

Atomic formulas over $\sigma$ are

- expressions of the form $p(t_1, \ldots, t_n)$, where $p$ is a predicate constant and $t_1, \ldots, t_n$ are terms such that their sorts are subsorts of the argument sorts $s_1, \ldots, s_n$ of $p$, and
- expressions of the form $t_1 = t_2$, where $t_1$ and $t_2$ are terms such that their sorts have a common supersort.

(First-order) formulas over $\sigma$ are formed from atomic formulas and the 0-place connective $\bot$ (falsity) using the binary connectives $\land$, $\lor$, $\rightarrow$ and the quantifiers $\forall$, $\exists$. The other connectives are treated as abbreviations: $\neg F$ stands for $F \rightarrow \bot$ and $F \leftrightarrow G$ stands for $(F \rightarrow G) \land (G \rightarrow F)$.

An interpretation $I$ of a signature $\sigma$ assigns

- a non-empty domain $|I|^s$ to every sort $s$ of $I$, so that $|I|^s_1 \subseteq |I|^s_2$ whenever $s_1$ is a subsort of $s_2$,
- a function $f^I$ from $|I|^s_1 \times \cdots \times |I|^s_n$ to $|I|^s$ to every function constant $f$ with argument sorts $s_1, \ldots, s_n$ and value sort $s$, and
- a Boolean-valued function $p^I$ on $|I|^s_1 \times \cdots \times |I|^s_n$ to every predicate constant $p$ with argument sorts $s_1, \ldots, s_n$.

If $I$ is an interpretation of a signature $\sigma$ then by $\sigma^I$ we denote the signature obtained from $\sigma$ by adding, for every element $d$ of a domain $|I|^s$, its name $d^*$ as an object constant of sort $s$. The interpretation $I$ is extended to $\sigma^I$ by defining $(d^*)^I = d$. The value $t^I$ assigned by an interpretation $I$ of $\sigma$ to a ground term $t$ over $\sigma^I$ and the satisfaction relation between an interpretation of $\sigma$ and a sentence over $\sigma^I$ are defined recursively, in the usual way.

If $d$ is a tuple $d_1, \ldots, d_n$ of elements of domains of $I$ then $d^*$ stands for the tuple $d_1^*, \ldots, d_n^*$ of their names. If $t$ is a tuple $t_1, \ldots, t_n$ of ground terms then $t^I$ stands for the tuple $t_1^I, \ldots, t_n^I$ of values assigned to them by $I$. 
A.2 Grounding

Miroslaw Truszczynski (2012) defined the process of grounding with respect to an interpretation, which transforms first-order sentences into infinitary propositional formulas. We generalize that construction in two ways. First, it is applied here to formulas over a many-sorted signature. Second, we assume that the set of predicate constants of a signature $\sigma$ is partitioned into two (possibly empty) subsets, intensional symbols and extensional symbols. In the process of grounding, symbols of these two types will be handled in different ways.

For any interpretation $I$ of a signature $\sigma$ and any sentence $F$ over $\sigma^I$, the grounding $gr_I(F)$ of $F$ with respect to $I$ is the infinitary propositional formula defined as follows:

- $gr_I(\bot) = \bot$;
- for intensional $p$, $gr_I(p(t)) = p(t^I)$;
- for extensional $p$, $gr_I(p(t)) = \top$ if $I$ satisfies $p(t)$, and $gr_I(p(t)) = \bot$ otherwise;
- $gr_I(t_1 = t_2) = \top$ if $t^I_1 = t^I_2$, and $gr_I(t_1 = t_2) = \bot$ otherwise;
- $gr_I(F \land G) = gr_I(F) \land gr_I(G)$, and similarly for $\lor$ and $\rightarrow$;
- $gr_I(\forall X F(X)) = \{gr_I(F(d^s)) : d \in |I|^s \land\}$, where $s$ is the sort of $X$;
- $gr_I(\exists X F(X)) = \{gr_I(F(d^s)) : d \in |I|^s \lor\}$, where $s$ is the sort of $X$.

Thus, the formula $gr_I(F)$ is formed from atoms of the form $p(d^I)$, where $p$ is an intensional symbol and $d$ is a tuple of elements of domains of $I$. By $I^{int}$ we denote the set of atoms of this form that are satisfied by $I$.

The following proposition relates the meaning of a sentence to the meaning of its grounding. It is a generalization of Proposition 2 from Truszczynski’s to the many-sorted case paper, and it can be proven by induction.

**Proposition 1**

An interpretation $I$ satisfies a sentence $F$ over $\sigma^I$ iff $I^{int}$ satisfies $gr_I(F)$.

A.3 Completion

In this section, the definition of completion due to Lloyd and Topor (1984) is generalized by allowing the underlying signature to be many-sorted, and by allowing some predicate constants to be designated as extensional, as in Section A.2.

A nondisjunctive rule over $\sigma$ is a formula over $\sigma$ that has the form $\forall(F \rightarrow G)$, where $G$ is atomic or does not contain intensional symbols.

Given a nondisjunctive rule $\forall(F \rightarrow G)$, we say that it *defines* an intensional symbol $p$ if $G$ is an atomic formula that begins with $p$. We say that $\forall(F \rightarrow G)$ is a *constraint* if $G$ does not contain intensional symbols.

Assume that the set of predicate symbols of $\sigma$ designated as intensional is finite, and let $\Gamma$ be a finite set of nondisjunctive rules over $\sigma$. If the argument sorts of an intensional symbol $p$ are $s_1, \ldots, s_n$, and the members of $\Gamma$ defining $p$ are

$$\forall(F_i \rightarrow p(t_i)) \quad i = 1, \ldots, k,$$

then the completed definition of $p$ in $\Gamma$ is the sentence

$$\forall \mathbf{V} \left( p(\mathbf{V}) \leftrightarrow \bigvee_{i=1}^k \exists \mathbf{U}_i (F_i \land \mathbf{V} = t_i) \right), \quad \text{(A1)}$$
where $V$ is an $n$-tuple of fresh variables of sorts $s_1, \ldots, s_n$, and $U_i$ is the list of all variables that are free in $F_i \rightarrow p(t_i)$. The expression $V = t_i$ stands here for the conjunction of $n$ equalities between the corresponding members of the tuples $V$ and $t_i$.

The completion $\text{COMP}[\Gamma]$ of $\Gamma$ is the conjunction of all completed definitions of intensional symbols in $\Gamma$ and all constraints of $\Gamma$.

### A.4 Logic of here-and-there

First-order logic of here-and-there was introduced by Pearce and Valverde (2004). The definitions below extend the version defined by Ferraris et al. (2011) to many-sorted formulas.

An HT-interpretation of $\sigma$ is a pair $\langle H, I \rangle$, where $I$ is an interpretation of $\sigma$, and $H$ is a subset of $I^{int}$. (In terms of Kripke models with two sorts, $I$ is the there-world, and $H$ describes the intensional predicates in the here-world). The satisfaction relation $\models_{ht}$ between HT-interpretation $\langle H, I \rangle$ of $\sigma$ and a sentence $F$ over $\sigma^t$ is defined recursively as follows:

- $\langle H, I \rangle \models_{ht} p(t)$, where $p$ is intensional, if $p(t^I) \in H$;
- $\langle H, I \rangle \models_{ht} p(t)$, where $p$ is extensional, if $I \models p(t)$;
- $\langle H, I \rangle \models_{ht} t_1 = t_2$ if $t_1^I = t_2^I$;
- $\langle H, I \rangle \not\models_{ht} \bot$;
- $\langle H, I \rangle \models_{ht} F \land G$ if $\langle H, I \rangle \models_{ht} F$ and $\langle H, I \rangle \models_{ht} G$;
- $\langle H, I \rangle \models_{ht} F \lor G$ if $\langle H, I \rangle \models_{ht} F$ or $\langle H, I \rangle \models_{ht} G$;
- $\langle H, I \rangle \models_{ht} F \rightarrow G$ if
  - (i) $\langle H, I \rangle \not\models_{ht} F$ or $\langle H, I \rangle \models_{ht} G$, and
  - (ii) $I \models F \rightarrow G$;
- $\langle H, I \rangle \models_{ht} \forall X F(X)$ if $\langle H, I \rangle \models_{ht} F(d^s)$ for each $d \in |I|^s$, where $s$ is the sort of $X$;
- $\langle H, I \rangle \models_{ht} \exists X F(X)$ if $\langle H, I \rangle \models_{ht} F(d^s)$ for some $d \in |I|^s$, where $s$ is the sort of $X$.

If $\langle H, I \rangle \models_{ht} F$ holds, we say that $\langle H, I \rangle$ satisfies $F$ and that $\langle H, I \rangle$ is an HT-model of $F$. If two formulas have the same HT-models then we say that they are HT-equivalent.

In the following proposition, we collected some properties of this satisfaction relation that can be proved by induction.

**Proposition 2**

(a) If $\langle H, I \rangle \models_{ht} F$ then $I \models F$.

(b) For any sentence $F$ that does not contain intensional symbols, $\langle H, I \rangle \models_{ht} F$ iff $I \models F$.

(c) For any subset $S$ of $H$ such that the predicate symbols of its members do not occur in $F$, $\langle H \setminus S, I \rangle \models_{ht} F$ iff $\langle H, I \rangle \models_{ht} F$.

The proposition below relates the meaning of a sentence in the logic of here-and-there to the meaning of its grounding in the infinitary version of that logic (Truszczyński 2012, Section 2). It is a generalization of Proposition 4 from that paper and can be proved by induction in a similar way.

**Proposition 3**

An HT-interpretation $\langle H, I \rangle$ satisfies a sentence $F$ over $\sigma^t$ iff $\langle H, I^{int} \rangle$ satisfies $gr_1(F)$.
A.5 Stable models

The definition of a stable model in this section is based on the relationship between stable models and equilibrium logic described by Pearce (1997); in application to finite sets of formulas with a single sort, it is equivalent to the definition in terms of the operator SM (Ferraris et al. 2011).

About a model $I$ of a set $\Gamma$ of sentences over $\sigma$ we say it is stable if, for every proper subset $H$ of $I^{int}$, HT-interpretation $\langle H, I \rangle$ does not satisfy $\Gamma$. We say that $I$ is pointwise stable if, for every element $M$ of $I^{int}$, $\langle I^{int} \setminus \{M\}, I \rangle$ does not satisfy $\Gamma$.

The proposition below relates stable models as defined above to stable models of infinitary propositional formulas (Truszczynski 2012, Section 2). It is a generalization of Theorem 5 from that paper.

Proposition 4

An interpretation $I$ is a stable model of a set $\Gamma$ of sentences over $\sigma$ iff $I^{int}$ is a stable model of $\{gr_I(F) : F \in \Gamma\}$.

Proof

By Proposition 1, an interpretation $I$ is a model of $\Gamma$ iff $I^{int}$ is a model of $\{gr_I(F) : F \in \Gamma\}$.

By Proposition 3, for any proper subset $H$ of $I^{int}$, $\langle H, I \rangle$ satisfies $\Gamma$ iff $\langle H, I^{int} \setminus \{M\}, I \rangle$ satisfies $\{gr_I(F) : F \in \Gamma\}$.

Appendix B Proofs

B.1 Main Lemma

Consider a signature $\sigma$ (possibly many-sorted), with its predicate constants classified into intensional and extensional, as in Sections A.2–A.5. We define, for a sentence $F$ over $\sigma$ and an interpretation $I$ of $\sigma$, the set $\text{Pos}(F, I)$ of (strictly) positive atoms of $F$ with respect to $I$. Elements of this set are formulas of the signature $\sigma$ that have the form $p(d^*)$. This set is defined recursively, as follows. If $F$ does not contain intensional symbols or is not satisfied by $I$ then $\text{Pos}(F, I) = \emptyset$. Otherwise,

(i) $\text{Pos}(p(t), I) = \{p((t^I)^*)\}$;
(ii) $\text{Pos}(F_1 \land F_2, I) = \text{Pos}(F_1 \lor F_2, I) = \text{Pos}(F_1, I) \cup \text{Pos}(F_2, I)$;
(iii) $\text{Pos}(F_1 \rightarrow F_2, I) = \text{Pos}(F_2, I)$;
(iv) $\text{Pos}(\forall XF(X), I) = \text{Pos}(\exists XF(X), I) = \bigcup_{d \in |I|} \text{Pos}(F(d^*), I)$ if $X$ is a variable of sort $s$.

As defined in Section A.2, $I^{int}$ stands for the set of formulas over $\sigma^I$ that have the form $p(d^*)$ and are satisfied by $I$, where $p$ is intensional. It is easy to check by induction on $F$ that $\text{Pos}(F, I)$ is a subset of $I^{int}$.

As defined in Section A.3, a nondisjunctive rule over $\sigma$ is a formula over $\sigma$ that has the form $\forall(F \rightarrow G)$, where $G$ is atomic or does not contain intensional symbols. An instance of a nondisjunctive rule $\forall(F \rightarrow G)$ for an interpretation $I$ of $\sigma$ is a sentence over $\sigma^I$ that is obtained from $F \rightarrow G$ by substituting names $d^*$ for its free variables.

Footnote: For the definitions of the signature $\sigma^I$ and names $d^*$, see Section A.1.
For any interpretation $I$ of $\sigma$ and any set $\Gamma$ of nondisjunctive rules over $\sigma^I$, the positive dependency graph $G^p_I(\Gamma)$ is the directed graph defined as follows. Its vertices are elements of $I^{\text{int}}$. It has an edge from $A$ to $B$ iff, for some instance $F \to G$ of a member of $\Gamma$, $A \in \text{Pos}(G, I)$ and $B \in \text{Pos}(F, I)$.

**Main Lemma**

For any interpretation $I$ of a signature $\sigma$ and any finite set $\Gamma$ of nondisjunctive rules over $\sigma^I$ such that the graph $G^p_I(\Gamma)$ has no infinite walks, $I$ is a stable model of $\Gamma$ iff $I \models \text{COMP}[\Gamma]$.

**Example 4**

The signature consists of a single sort, the object constants $a$, $b$, and the intensional unary predicate constant $p$. The set $\Gamma$ is \{⊤→$p(a)$, $p(a) \land p(b)$ → $p(b)$\}, and the interpretation $I$ is defined by the conditions

$$|I| = \{0, 1, \ldots\}; a^I = 0; b^I = 1; p^I(n) = \text{true} \text{ iff } n \in \{0, 1\}.$$  

Then, the set $I^{\text{int}}$ of vertices of the graph $G^p_I(\Gamma)$ is \{\(p(0^*)\),\(p(1^*)\),\(\ldots\)\}. The edges are \((p(1^*), p(0^*))\) and \((p(1^*), p(1^*))\), because

\[\text{Pos}(\top, I) = \emptyset, \text{Pos}(p(a) \land p(b), I) = \{p(0^*), p(1^*)\}, \text{Pos}(p(b), I) = \{p(1^*)\}.\]

The graph has an infinite walk, \((p(1^*), p(1^*)\ldots\), so that the assumptions of Main Lemma are not satisfied. On the other hand, if we replace the first rule of $\Gamma$ by $\top \to p(b)$, and replace the definition of $p^I$ by

$$p^I(n) = \text{true} \text{ iff } n = 1,$$

then the set of edges of the graph will become empty, because the modified interpretation does not satisfy $p(a) \land p(b)$. Main Lemma asserts in this case that $I$ is a stable model of $\Gamma$ (which is true) iff $I$ satisfies the completion

\[\forall V(p(V) \leftrightarrow (\top \land V = b) \lor (p(a) \land p(b) \land V = b))\]

of $\Gamma$ (which is true as well).

Main Lemma is similar to two results published earlier (Ferraris et al. 2011, Theorem 11; Lee and Meng 2011, Corollary 4), but those are less general: the former because it refers to dependencies between predicate constants rather than ground atoms; the latter because it is only applicable to nondisjunctive rules of a special syntactic form.

About an interpretation $I$ of $\sigma$ we say that it is *semi-Herbrand* if

- all elements of its domains are object constants of $\sigma$, and
- $d^I = d$ for every such element $d$.

If $I$ is semi-Herbrand then the graph $G^p_I(\Gamma)$ can be modified by replacing each vertex $p(d^*)$ by the atom $p(d)$ of the signature $\sigma$. This modified positive dependency graph is isomorphic to $G^p_I(\Gamma)$ as defined above. It can be defined independently, by replacing clauses (i) and (iv) in the definition of $\text{Pos}(F, I)$ above by

\[\text{This is weaker than the condition defining Herbrand interpretations; some ground terms over } \sigma \text{ may be different from elements of the domains } |I|^i.\]
(i)' \( \text{Pos}(p(t), I) = \{p(t^i)\} \);
(iv)' \( \text{Pos}(\forall X F(X), I) = \text{Pos}(\exists X F(X), I) = \bigcup_{d \in [I]} \text{Pos}(F(d), I) \) if \( X \) is a variable of sort \( s \).

### B.2 Proof of Main Lemma

Main Lemma expresses a property of the completion operator, and its proof below consists of two parts. We first prove a similar property of pointwise stable models, defined in Section A.5 (Lemma 3); then we relate pointwise stable models to completion.

**Lemma 1**

For any HT-interpretation \( \langle H, I \rangle \) and any sentence \( F \) over \( \sigma^I \), if \( I \models F \) and \( \text{Pos}(F, I) \subseteq H \) then \( \langle H, I \rangle \models_{ht} F \).

**Proof**

By induction on the size of \( F \). Case 1: \( F \) does not contain intensional symbols. The assertion of the lemma follows from Proposition 2(b). Case 2: \( F \) contains an intensional symbol. Since \( I \models F \), the set \( \text{Pos}(F, I) \) is determined by the recursive clauses in the definition of \( \text{Pos} \). Case 2.1: \( F \) is \( p(t) \), where \( p \) is intensional. Then, the assumption \( \text{Pos}(F, I) \subseteq H \) and the claim \( \langle H, I \rangle \models_{ht} F \) turn into the condition \( p((t^i)^*) \in H \). Case 2.2: \( F \) is \( F_1 \land F_2 \). Then, from the assumption \( I \models F \) we conclude that \( I \models F_i \) for \( i = 1, 2 \). On the other hand,

\[
\text{Pos}(F_1, I) \subseteq \text{Pos}(F, I) \subseteq H.
\]

By the induction hypothesis, it follows that \( \langle H, I \rangle \models_{ht} F_1 \), and consequently \( \langle H, I \rangle \models_{ht} F \). Case 2.3: \( F \) is \( F_1 \lor F_2 \). Similar to Case 2.2. Case 2.4: \( F \) is \( F_1 \rightarrow F_2 \). Since \( I \models F_1 \rightarrow F_2 \), we only need to check that \( \langle H, I \rangle \models_{ht} F_1 \) or \( \langle H, I \rangle \models_{ht} F_2 \). Case 2.4.1: \( I \models F_1 \). Since \( I \models F_1 \rightarrow F_2 \), it follows that \( I \models F_2 \). On the other hand,

\[
\text{Pos}(F_2, I) = \text{Pos}(F, I) \subseteq H.
\]

By the induction hypothesis, it follows that \( \langle H, I \rangle \models_{ht} F_2 \). Case 2.4.2: \( I \not\models F_1 \). By Proposition 2(a) in the appendix, it follows that \( \langle H, I \rangle \not\models_{ht} F_1 \). Case 2.5: \( F \) is \( \forall X G(X) \), where \( X \) is a variable of sort \( s \). Then, for every element \( d \) of \( |I|^s \), \( I \models G(d^*) \) and

\[
\text{Pos}(G(d^*), I) \subseteq \text{Pos}(F, I) \subseteq H.
\]

By the induction hypothesis, it follows that \( \langle H, I \rangle \models_{ht} G(d^*) \). Consequently \( \langle H, I \rangle \models_{ht} \forall X G(X) \). Case 2.6: \( F \) is \( \exists X G(X) \). Similar to Case 2.5. \( \square \)

The definition of \( G^{sp}_I(\Gamma) \) in Section B.1 is restricted to the case when \( \Gamma \) is a set of nondisjunctive rules. It can be generalized to arbitrary sets of sentences as follows. A rule subformula of a formula \( F \) is an occurrence of an implication in \( F \) that does not belong to the antecedent of any implication (Ferraris et al. 2011, Section 7.3; Lee and Meng 2011, Section 3.3). Let \( I \) be an interpretation of \( \sigma \), and let \( \Gamma \) be a set of sentences over \( \sigma^I \). The vertices of \( G^{sp}_I(\Gamma) \) are elements of \( I^{int} \). It has an edge from \( A \) to \( B \) iff, for some sentence \( F_1 \rightarrow F_2 \) obtained from a rule subformula of a member of \( \Gamma \) by substituting names \( d^* \) for its free variables, \( A \in \text{Pos}(F_2, I) \) and \( B \in \text{Pos}(F_1, I) \).

For any sentence \( F \), \( G^{sp}_I(F) \) stands for \( G^{sp}_I(\{F\}) \).
Lemma 2
For any HT-interpretation $\langle \mathcal{H}, I \rangle$, any atom $M$ in $I^{int}\setminus\mathcal{H}$, and any sentence $F$ over $\sigma^I$, if

(i) for every edge $(M, B)$ of $G^{Ip}_I(F)$, $B \in \mathcal{H}$,
(ii) $M \in \text{Pos}(F, I)$, and
(iii) $\langle \mathcal{H}, I \rangle \models_{ht} F$,

then $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F$.

Proof
By induction on the size of $F$. Sentence $F$ is neither atomic nor $\bot$. Indeed, in that case $F$ would be an atomic sentence of the form $p(t)$, where $p$ is intensional, because, by (ii), $\text{Pos}(F, I)$ is non-empty. Then, from (iii), $p((t^I)^*) \in \mathcal{H}$. On the other hand, $\text{Pos}(F, I)$ is $\{p((t^I)^*)\}$, and from (ii) we conclude that $M = p((t^I)^*)$. This contradicts the assumption that $M \in I^{int}\setminus\mathcal{H}$. Thus five cases are possible.

Case 1: $F$ is $F_1 \land F_2$. From (iii) we can conclude that $\langle \mathcal{H}, I \rangle \models_{ht} F_i$ for $i = 1, 2$. It is sufficient to show that $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F_i$; then the conclusion $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F$ will follow. Case 1.1: $M \in \text{Pos}(F_i, I)$. Since $\text{Pos}(F_i, I)$ is a subset of $I^{int}$, $\text{Pos}(F_i, I) \subseteq I^{int}\setminus\{M\}$. On the other hand, from the fact that $\langle \mathcal{H}, I \rangle \models_{ht} F_i$, we can conclude, by Proposition 2(a), that $I \models F_i$. By Lemma 1, it follows that $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F_i$.

Case 2: $F$ is $F_1 \lor F_2$. From (iii) we can conclude that $\langle \mathcal{H}, I \rangle \models_{ht} F_i$ for $i = 1$ or $i = 2$. It is sufficient to show that $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F_i$; then the conclusion that $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F$ follow. The reasoning is the same as in Case 1.

Case 3: $F$ is $F_1 \to F_2$. Then, $\text{Pos}(F, I) = \text{Pos}(F_2, I)$. By (ii), it follows that $M \in \text{Pos}(F_2, I)$.

On the other hand, $F$ is a rule subformula of itself, so that for every atom $B$ in $\text{Pos}(F_1, I)$, $(M, B)$ is an edge of the graph $G^{Ip}_I(F)$. By (i), it follows that every such atom $B$ belongs to $\mathcal{H}$. Consequently

$$\text{Pos}(F_1, I) \subseteq \mathcal{H}.$$  \hfill (B2)

Case 3.1: $\langle \mathcal{H}, I \rangle \models_{ht} F_2$, so that $F_2$ satisfies condition (iii). Since $G^{Ip}_I(F_2)$ is a subgraph of $G^{Ip}_I(F)$, $F_2$ satisfies condition (i) as well. By (B1), $F_2$ satisfies condition (ii). Then, by the induction hypothesis, $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F_2$. Consequently $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F$.

Case 3.2: $\langle \mathcal{H}, I \rangle \nmodels_{ht} F_2$. Then, in view of (iii), $\langle \mathcal{H}, I \rangle \nmodels_{ht} F_1$. From this fact and formula (B2) we can conclude, by Lemma 1, that $I \nmodels F_1$. By Proposition 2(a), it follows that $\langle I^{int}\setminus\{M\}, I \rangle \nmodels_{ht} F_1$, which implies $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F$.

Case 4: $F$ is $\forall X G(X)$. From (iii) we can conclude that for every $d^*$ in the domain $|I|^*$, where $s$ is the sort of $X$, $\langle \mathcal{H}, I \rangle \models_{ht} G(d^*)$. It is sufficient to show that $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} G(d^*)$; then the conclusion that $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} F$ will follow. Case 4.1: $M \in \text{Pos}(G(d^*), I)$, so that formula $G(d^*)$ satisfies condition (ii). That formula satisfies condition (i) as well, because $G^{Ip}_I(G(d^*))$ is a subgraph of $G^{Ip}_I(F)$, and it satisfies condition (iii). So the conclusion $\langle I^{int}\setminus\{M\}, I \rangle \models_{ht} G(d^*)$ follows by the induction hypothesis. Case 4.2: $M \nmodels \text{Pos}(G(d^*), I)$. Since $\text{Pos}(G(d^*), I)$ is a subset of $I^{int}$, we
can conclude that \( \text{Pos}(G(d^*), I) \subseteq I^{\text{int}} \setminus \{ M \} \). On the other hand, from the fact that \( \langle \mathcal{H}, I \rangle \models_{ht} G(d^*) \) we conclude, by Proposition 2(a), that \( I \models G(d^*) \). By Lemma 1, it follows that \( \langle I^{\text{int}} \setminus \{ M \}, I \rangle \models_{ht} G(d^*) \).

**Case 5:** \( F \) is \( \exists X G(X) \). From (iii) we can conclude that for some \( d \) in the domain \( |I|^s \), where \( s \) is the sort of \( X \), \( \langle \mathcal{H}, I \rangle \models_{ht} G(d^*) \). It is sufficient to show that \( \langle I^{\text{int}} \setminus \{ M \}, I \rangle \models_{ht} G(d^*) \); then the conclusion that \( \langle I^{\text{int}} \setminus \{ M \}, I \rangle \models_{ht} F \) will follow. The reasoning is the same as in Case 4. \( \square \)

**Lemma 3**
For any interpretation \( I \) of a signature \( \sigma \) and any set \( \Gamma \) of sentences over \( \sigma^I \) such that the graph \( G_{\rho}^p(\Gamma) \) has no infinite walks, \( I \) is a stable model of \( \Gamma \) iff \( I \) is pointwise stable.

**Proof**
We need to show that if \( I \) is a model of \( \Gamma \) such that the graph \( G_{\rho}^p(\Gamma) \) has no infinite walks, and there exists a proper subset \( \mathcal{H} \) of \( I^{\text{int}} \) such that \( \langle \mathcal{H}, I \rangle \) satisfies \( \Gamma \), then a subset with this property can be obtained from \( I^{\text{int}} \) by removing a single element.

The set \( I^{\text{int}} \setminus \mathcal{H} \) contains an atom \( M \) such that for every edge \( (M, B) \) of the graph \( G_{\rho}^p(\Gamma) \), \( B \notin I^{\text{int}} \setminus \mathcal{H} \). Indeed, otherwise this graph would have an infinite walk consisting of elements of \( I^{\text{int}} \setminus \mathcal{H} \). On the other hand, for every such edge, \( B \in I^{\text{int}} \). Indeed, from the definition of the graph \( G_{\rho}^p(\Gamma) \) we see that for every edge \( (M, B) \) of that graph, \( B \) belongs to the set \( \text{Pos}(F, I) \) for some sentence \( F \), and that set is contained in \( I^{\text{int}} \). Consequently for every edge \( (M, B) \) of \( G_{\rho}^p(\Gamma) \), \( B \in \mathcal{H} \).

We will show that \( \langle I^{\text{int}} \setminus \{ M \}, I \rangle \) satisfies \( \Gamma \). Take a sentence \( F \) from \( \Gamma \). **Case 1:** \( M \in \text{Pos}(F, I) \). Then, condition (ii) of Lemma 2 is satisfied for the HT-interpretation \( \langle \mathcal{H}, I \rangle \).

Condition (i) is satisfied for this HT-interpretation as well, because \( G_{\rho}^p(F) \) is a subgraph of \( G_{\rho}^p(\Gamma) \); furthermore, condition (iii) is satisfied because \( \langle \mathcal{H}, I \rangle \) satisfies \( \Gamma \). Consequently \( \langle I^{\text{int}} \setminus \{ M \}, I \rangle \models_{ht} F \) by Lemma 2. **Case 2:** \( M \notin \text{Pos}(F, I) \). Then, \( \text{Pos}(F, I) \subseteq I^{\text{int}} \setminus \{ M \} \).

Since \( I \models F \), we can conclude that \( \langle I^{\text{int}} \setminus \{ M \}, I \rangle \models_{ht} F \) by Lemma 1. \( \square \)

A model \( I \) of a set \( \Gamma \) of nondisjunctive rules over \( \sigma \) is **supported** if for every atom \( p(d^*) \) in \( I^{\text{int}} \) there exists an instance \( F \rightarrow p(t) \) of a member of \( \Gamma \) such that \( t^I = d \) and \( I \models F \).

**Lemma 4**
Every pointwise stable model of a set of nondisjunctive rules is supported.

**Proof**
Let \( I \) be a pointwise stable model of a set \( \Gamma \) of nondisjunctive rules. Take an atom \( p(d^*) \) from \( I^{\text{int}} \). We need to find an instance \( F \rightarrow p(t) \) of a nondisjunctive rule from \( \Gamma \) such that \( t^I = d \) and \( I \models F \).

By the definition of a pointwise stable model, \( \langle I^{\text{int}} \setminus \{ p(d^*) \}, I \rangle \) does not satisfy \( \Gamma \). Then, one of the nondisjunctive rules from \( \Gamma \) has an instance \( F \rightarrow G \) such that
\[
\langle I^{\text{int}} \setminus \{ p(d^*) \}, I \rangle \not\models_{ht} F \rightarrow G.
\]
(\text{B3})

We will show that this instance has the required properties. Since \( I \) is a model of \( \Gamma \),
\[
I \models F \rightarrow G.
\]
(\text{B4)}
From (B3) and (B4) we conclude that
\[ \langle I^{int} \setminus \{ p(d^*) \}, I \rangle \models_{ht} F \] (B5)
and
\[ \langle I^{int} \setminus \{ p(d^*) \}, I \rangle \not\models_{ht} G. \] (B6)

From (B5) and Proposition 2(a), \( I \models F. \) Then, in view of (B4),
\[ I \models G \] (B7)
From (B6), (B7) and Proposition 2(c) we can conclude that formula \( G \) contains \( p \), so that it has the form \( p(t) \). Then, from (B7), \( p((t^I)^*) \in I^{int} \), and from (B6), \( p((t^I)^*) \not\in I^{int} \setminus \{ p(d^*) \} \). Consequently \( p((t^I)^*) = p(d^*) \), so that \( t^I = d \). \( \square \)

Lemma 5
For any interpretation \( I \) of a signature \( \sigma \) and any set \( \Gamma \) of nondisjunctive rules over \( \sigma \) such that the graph \( G^p_I(\Gamma) \) has no infinite walks, \( I \) is a supported model of \( \Gamma \) iff \( I \) is stable.

Proof
The if part follows from Lemmas 3 and 4. For the only if part, consider a supported model \( I \) of a set \( \Gamma \) of nondisjunctive rules such that the graph \( G^p_I(\Gamma) \) has no infinite walks; we need to prove that \( I \) is stable. According to Lemma 3, it is sufficient to check that \( I \) is pointwise stable.

Take any atom \( M \) in \( I^{int} \); we will show that \( \langle I^{int} \setminus \{ M \}, I \rangle \) is not an HT-model of \( \Gamma \). Since \( I \) is supported, one of the nondisjunctive rules in \( \Gamma \) has an instance \( F \rightarrow M \) such that \( I \models F. \) Atom \( M \) does not belong to \( Pos(F, I) \), because otherwise \( M, M, \ldots \) would be an infinite walk in \( G^p_I(\Gamma) \). Since the set \( Pos(F, I) \) is a subset of \( I^{int} \), we can conclude that it is a subset of \( I^{int} \setminus \{ M \} \). By Lemma 1, it follows that \( \langle I^{int} \setminus \{ M \}, I \rangle \models_{ht} F \). Therefore \( \langle I^{int} \setminus \{ M \}, I \rangle \not\models_{ht} F \rightarrow M \). \( \square \)

Lemma 6
For any interpretation \( I \) of a signature \( \sigma \) and any finite set \( \Gamma \) of nondisjunctive rules over \( \sigma \), \( I \) is a supported model of \( \Gamma \) iff \( I \models \text{COMP}[\Gamma] \).

Proof
In the rules of \( \Gamma \) defining \( p \), the variables that follow the outermost universal quantifier will be shown explicitly:
\[ \forall U_i(F_i(U_i) \rightarrow p(t_i(U_i))) \quad (i = 1, \ldots, k). \] (B8)

Accordingly, the completed definition (A1) of \( p \) is written as
\[ \forall V \left( p(V) \leftrightarrow \bigvee_{i=1}^k \exists U_i (F_i(U_i) \land V = t_i(U_i)) \right). \]

An interpretation \( I \) of \( \sigma \) satisfies \( \text{COMP}[\Gamma] \) iff
\begin{enumerate}
  \item for every intensional \( p \), \( I \) satisfies the sentence
  \[ \forall V \left( p(V) \rightarrow \bigvee_{i=1}^k \exists U_i (F_i(U_i) \land V = t_i(U_i)) \right); \]
\end{enumerate}
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(b) for every intensional \( p \) and for every \( i \), \( I \) satisfies the sentence
\[
\forall V (\exists U_i (F_i(U_i) \land V = t_i(U_i)) \rightarrow p(V)); \tag{B9}
\]

(c) \( I \) satisfies all constraints of \( \Gamma \).

Formula (B9) is equivalent to nondisjunctive rule (B8). Consequently \( I \) is a model of \( \Gamma \) if and only if conditions (b) and (c) hold. It remains to be checked that condition (a) holds if and only if the model \( I \) is supported.

Condition (a) can be expressed by saying that for every atom \( p(d^*) \) in \( I^{int} \) there exists \( i \) such that
\[
I \models \exists U_i (F_i(U_i) \land d^* = t_i(U_i)),
\]
or, equivalently, that

for every atom \( p(d^*) \) in \( I^{int} \) there exist \( i \) and a tuple \( d_i \) of domain elements such that \( I \models F_i(d_i) \) and \( d = t_i(d_i)^I \).

Since the members of \( \Gamma \) defining \( p \) are nondisjunctive rules (B8), the last condition expresses that \( I \) is a supported model of \( \Gamma \).

The assertion of Main Lemma follows from Lemmas 5 and 6.

B.3 Translation \( \tau^* \)

The translation \( \tau^B \) (Lifschitz et al. 2019) is similar to the translation \( \tau^b \) defined in Section 2.3, except that it handles the combination not not differently:

- \( \tau^B(p(t)) = \exists Z (val_t(Z) \land p(Z)) \);
- \( \tau^B(\text{not } p(t)) = \exists Z (val_t(Z) \land \neg p(Z)) \);
- \( \tau^B(\text{not } \text{not } p(t)) = \exists Z (val_t(Z) \land \neg \neg p(Z)) \);
- \( \tau^B(t_1 \prec t_2) = \exists Z_1 Z_2 (val_{t_1}(Z_1) \land val_{t_2}(Z_2) \land Z_1 \prec Z_2) \).

If \( Body \) is a conjunction \( B_1 \land B_2 \land \cdots \) of literals and comparisons then \( \tau^B(\text{Body}) \) stands for the conjunction \( \tau^B(B_1) \land \tau^B(B_2) \land \cdots \). It is clear that \( \tau^b(\text{Body}) \) can be obtained from \( \tau^B(\text{Body}) \) by dropping some double negations, so that these two formulas are equivalent.

The operator \( \tau^* \) (Lifschitz et al. 2019)\(^6\) transforms mini-GRINGO rules into first-order rules over the signature \( \sigma_0 \). It converts a basic rule
\[
p(t) \leftarrow Body \tag{B10}
\]
into the formula
\[
\tilde{\varphi}(val_t(Z) \land \tau^B(\text{Body}) \rightarrow p(Z)),
\]
where \( Z \) is a tuple of fresh general variables. A choice rule
\[
\{p(t)\} \leftarrow Body
\]

\(^6\) The definition of \( \tau^* \) in that publication is a little different from the definition below, but the two versions are intuitionistically equivalent to each other.
is converted into a formula of the form

\[ \bar{\varphi}(\text{val}_t(Z) \land \tau^B(\text{Body}) \land \neg \neg p(Z) \rightarrow p(Z)), \]

and a constraint of the form \( \leftarrow \text{Body} \) becomes formula \( \bar{\nu} \tau^B(\text{Body}) \).

For any program \( \Pi, \tau^*\Pi \) stands for the set of first-order rules \( \tau^* R \) for all rules \( R \) of \( \Pi \).

This translation will be useful to us because of its properties expressed by Lemmas 7 and 8 below. The first of them relates stable models of mini-GRINGO programs to stable models in the general framework of Section A.5 applied to the special case when the underlying signature is \( \sigma_0 \) (Section 2.2), with all comparison symbols viewed as extensional.

**Lemma 7**

A set \( J \) of precomputed atoms is a stable model of a mini-GRINGO program \( \Pi \) iff \( J^\uparrow \) is a stable model of \( \tau^* \Pi \).

**Proof**

Since \( (J^\uparrow)^{\text{int}} = J \), the special case of Proposition 4 (Section A.5) with \( \tau^* \Pi \) as \( \Gamma \) can be expressed as follows: \( J^\uparrow \) is a stable model of \( \tau^* \Pi \) iff \( J \) is a stable model of the set of formulas \( \text{gr}_{J^\uparrow}(\tau^* R) \) for all rules \( R \) of \( \Pi \). That set has the same stable models as \( \Pi \), because \( \text{gr}_{J^\uparrow}(\tau^* R) \) is strongly equivalent\(^7\) to \( \tau R \) (Lifschitz et al. 2019, Proposition 3).

Lemma 8 relates the first-order completion of an io-program \( \Omega \) to the completion in the general sense of Section A.3 applied to the case when the underlying signature is \( \sigma_0 \), with all comparison symbols and all input symbols of \( \Omega \) viewed as extensional.

**Lemma 8**

If \( \Pi \) is the set of rules of an io-program \( \Omega \) then \( \text{COMP}^1[\Omega] \) is equivalent to \( \text{COMP}[\tau^* \Pi] \).

**Proof**

The first-order completion \( \text{COMP}^1[\Omega] \) is the conjunction of completed definitions (15) and the sentences (16) corresponding to the constraints of \( \Omega \). Formula (16) is equivalent to the constraint \( \bar{\nu} \tau^B(\text{Body}) \) of \( \tau^* \Pi \).

We check that the completed definition of a predicate symbol in \( \Omega \) is equivalent to the completed definition of the same symbol in \( \tau^* \Pi \). If the definition of \( p/n \) in \( \Omega \) consists of rules \( R_i \) then the definition of that symbol in \( \tau^* \Pi \) consists of rules of the form

\[ \forall U_i, Z_i (F_i \rightarrow p(Z_i)), \]

where \( U_i \) is the list of variables occurring in \( R_i \), \( Z_i \) is a list of distinct general variables disjoint from \( U_i \), and \( F_i \) is

\[
\begin{align*}
\text{val}_t(Z_i) \land \tau^B(\text{Body}) & \quad \text{if } R_i \text{ is } p(t) \leftarrow \text{Body}, \\
\text{val}_t(Z_i) \land \tau^B(\text{Body}) \land \neg \neg p(Z_i) & \quad \text{if } R_i \text{ is } \{p(t)\} \leftarrow \text{Body}.
\end{align*}
\]

The completed definition of \( p/n \) in \( \tau^* \Pi \) is

\[ \forall V (p(V) \leftrightarrow \bigvee \exists U_i Z_i (F_i \land V = Z_i)) \].

\(^7\) Strong equivalence of infinitary propositional formulas was defined and studied by Harrison et al. (2017).
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That definition is equivalent to

\[ \forall V \left( p(V) \leftrightarrow \bigvee_i \exists U_i F'_i \right), \]

(B11)

where \( F'_i \) is

\[
\begin{align*}
val_t(V) & \land \tau^B(Body) \quad \text{if } R_i \text{ is } p(t) \leftarrow Body, \text{ and} \\
val_t(V) & \land \tau^B(Body) \land \lnot \lnot p(V) \quad \text{if } R_i \text{ is } \{p(t)\} \leftarrow Body.
\end{align*}
\]

It remains to observe that formula (B11) is equivalent to (15). \( \square \)

B.4 Positive dependency graphs

To prove Theorem 1, we need to relate the positive dependency graph of an io-program to the positive dependency graph of the corresponding set of nondisjunctive rules with respect to a standard interpretation of \( \sigma_0 \). Such a relationship is described by Lemma 13 below.

Lemma 9

For any tuple \( t \) of ground terms in the language of mini-GRINGO and for any tuple \( r \) of precomputed terms of the same length, the formula \( val_t(r) \) is equivalent to \( \top \) if \( r \in [t] \), and to \( \bot \) otherwise.

Proof

The special case when \( t \) is a single term is Lemma 1 by Lifschitz et al. (2020). The general case easily follows. \( \square \)

Lemma 10

Let \( J \) be a set of precomputed atoms, and let \( L \) be a ground literal such that \( J^+ \models \tau^B(L) \).

(a) If \( L \) is \( p(t) \) or \( \text{not not } p(t) \) then, for some tuple \( r \) in \([t]\), \( p(r) \in J \).

(b) If \( L \) is \( \text{not } p(t) \) then, for some tuple \( r \) in \([t]\), \( p(r) \notin J \).

Proof

Consider the case when \( L \) is \( p(t) \). Then, \( \tau^B(L) \) is \( \exists Z (val_t(Z) \land p(Z)) \). Since \( J^+ \) satisfies this formula, there exists a tuple \( r \) of precomputed terms such that \( J^+ \) satisfies \( val_t(r) \) and \( p(r) \). Since \( val_t(r) \) is satisfiable, we can conclude by Lemma 9 that \( r \in [t] \). The other two cases are analogous. \( \square \)

Lemma 11

If \( U \) is a tuple of general variables, and \( u \) is a tuple of precomputed terms of the same length, then

(a) for any term \( t(U) \) and any precomputed term \( r \), the result of substituting \( u \) for \( U \) in \( val_t(U)(r) \) is \( val_{t(U)}(r) \);

(b) for any conjunction \( Body(U) \) of literals and comparisons, the result of substituting \( u \) for \( U \) in \( \tau^*(Body(U)) \) is \( \tau^*(Body(u)) \).
Proof
Part (i) is easy to prove by induction. Part (ii) immediately follows.

In the rest of this section, \( v \) is a valuation on the set of placeholders of \( \Omega \), and \( \mathcal{J} \) is a set of precomputed atoms that do not contain placeholders of \( \Omega \). Standard interpretations of \( \sigma_0 \) are semi-Herbrand, and the references to \( \text{Pos} \) and \( G^{sp} \) below refer to the definitions modified as described at the end of Section B.1.

**Lemma 12**
Let \( U \) be a list of distinct general variables, and let \( p(t(U)) \leftarrow \text{Body}(U) \) be a basic rule of \( \Omega \) with all variables explicitly shown. For any tuple \( u \) of precomputed terms of the same length as \( U \), any tuple \( r \) from \([v(t(u))]\), and any precomputed atom \( A \) from \( \text{Pos}(\tau^B(v(\text{Body}(u))), \mathcal{J}^\uparrow) \), the positive dependency graph of \( \Omega \) for the input \((v, \mathcal{J}^{in})\) has an edge from \( p(r) \) to \( A \).

**Proof**
We will show that the instance

\[
p(v(t(u))) \leftarrow v(\text{Body}(u))
\]

of the rule

\[
p(v(t(U))) \leftarrow v(\text{Body}(U))
\]
satisfies conditions (a)–(e) imposed on \( R \) in the definition of the positive dependency graph of an io-program (see Section 3).

Verification of condition (a): the argument of \( p \) in the head of \( R \) is \( v(t(u)) \), and \( r \in [v(t(u))] \).

Verification of condition (b): since \( A \in \text{Pos}(\tau^B(v(\text{Body}(u))), \mathcal{J}^\uparrow) \), the body \( v(\text{Body}(u)) \) of \( R \) has a conjunctive term \( p'(r') \) such that

\[
A \in \text{Pos}(\tau^B(p'(r'), \mathcal{J}^\uparrow)) = \text{Pos}(\exists Z(r' = Z \land p'(Z)), \mathcal{J}^\uparrow) = \text{Pos}(p'(r'), \mathcal{J}^\uparrow) \subseteq \{p'(r')\}.
\]

It follows that \( A = p'(r') \), so that \( A \) is a conjunctive term of the body required by condition (b).

Verification of conditions (c)–(e): let \( L \) be a conjunctive term of the body \( v(\text{Body}(u)) \) of \( R \) that contains an input symbol of \( \Omega \) or is a comparison. Since \( \text{Pos}(\tau^B(v(\text{Body}(u))), \mathcal{J}^\uparrow) \) is non-empty, \( \tau^B(v(\text{Body}(u))) \) is satisfied by \( \mathcal{J}^\uparrow \). Therefore, the conjunctive term \( \tau^B(L) \) of that formula is satisfied by \( \mathcal{J}^\uparrow \) as well. If \( L \) has the form \( q(t) \) or \( \text{not not } q(t) \) then, by Lemma 10(a), there exists a tuple \( r' \) in \([t]\) such that \( q(r') \in \mathcal{J} \), and consequently \( q(r') \in \mathcal{J}^{in} \). If \( L \) has the form \( \text{not not } q(t) \) then, by Lemma 10(b), there exists a tuple \( r' \) in \([t]\) such that \( q(r') \notin \mathcal{J} \), and consequently \( q(r') \notin \mathcal{J}^{in} \). If \( L \) is a comparison \( t_1 \prec t_2 \) then \( \tau^B(L) \) is

\[
\exists Z_1 Z_2 (\text{val}_{t_1}(Z_1) \land \text{val}_{t_2}(Z_2) \land Z_1 \prec Z_2).
\]

Since this formula is satisfied by \( \mathcal{J}^\uparrow \), the relation \( \prec \) holds for a pair \((r_1, r_2)\) of precomputed terms such that \( \text{val}_{t_1}(r_1) \) and \( \text{val}_{t_2}(r_2) \) are satisfied by \( \mathcal{J}^\uparrow \). By Lemma 9, \( r_1 \in [t_1] \) and \( r_2 \in [t_2] \). \( \square \)
Lemma 13
Let \( \Pi \) be the set of rules of \( \Omega \).

(i) The positive dependency graph of \( \Omega \) for the input \((v, J^\text{in})\) is a supergraph of the positive dependency graph of \( \tau^*(v(\Pi)) \).

(ii) If \( \Omega \) is locally tight on the input \((v, J^\text{in})\) then the graph \( G^\text{sp}_{J^\uparrow}(\tau^*(v(\Pi))) \) has no infinite walks.

Proof
(i) Replacing a choice rule \( \{p(t)\} \leftarrow \text{Body} \) in \( \Omega \) by the basic rule \( p(t) \leftarrow \text{Body} \) affects neither the positive dependency graph of \( \Omega \) nor the positive dependency graph of \( \tau^*(v(\Pi)) \). Consequently we can assume, without loss of generality, that \( \Omega \) is an io-program without choice rules.

Pick any edge \((p(r), A)\) of the (modified) graph \( G^\text{sp}_{J^\uparrow}(\tau^*(v(\Pi))) \). Then, there is an instance \( F \rightarrow G \) of the nondisjunctive rule obtained by applying \( \tau^* \) to a basic rule of \( v(\Pi) \) such that \( p(r) \in Pos(G, J^\uparrow) \) and \( A \in Pos(F, J^\uparrow) \). That rule of \( v(\Pi) \) can be written as

\[
p(v(t(U))) \leftarrow v(Body(U)),
\]

where \( p(t(U)) \leftarrow Body(U) \)

is a rule of \( \Pi \), and \( U \) is the list of its variables. The result of applying \( \tau^* \) to (B12) is

\[
\forall UZ(val_v(t(U))(Z) \land \tau^B(v(Body(U))) \rightarrow p(Z)),
\]

where \( Z \) is a tuple of general variables. Let \( u, z \) be tuples of precomputed terms that are substituted for \( U, Z \) in the process of forming the instance \( F \rightarrow G \). By Lemma 11, that instance can be written as

\[
val_v(t(u))(z) \land \tau^B(v(Body(u))) \rightarrow p(z).
\]

Consequently \( p(r) \in Pos(p(z), J^\uparrow) \) and

\[
A \in Pos(val_v(t(u))(z) \land \tau^B(v(Body(u))), J^\uparrow).
\]

The first of these conditions implies \( z = r \), so that the second can be rewritten as

\[
A \in Pos(val_v(t(u))(r) \land \tau^B(v(Body(u))), J^\uparrow).
\]

(B13)

It follows that \( J^\uparrow \) satisfies \( val_v(t(u))(r) \). By Lemma 9, we can conclude that \( r \in v(t(u)) \). On the other hand, the set \( Pos(val_v(t(u))(r), J^\uparrow) \) is empty, because the formula \( val_v(t(u))(r) \) does not contain intensional symbols. Consequently (B13) implies that

\[
A \in Pos(\tau^B(v(Body(u))), J^\uparrow).
\]

By Lemma 12, it follows that \((p(r), A)\) is an edge of the positive dependency graph of \( \Omega \) for the input \((v, J^\text{in})\).

(ii) Immediate from part (i).

B.5 Proof of Theorem 1

One part of Theorem 1 is easy to check: the assumption that \( \mathcal{P} \) is an io-model of \( \Omega \) for an input \((v, I)\) implies that \( \mathcal{P}^\text{in} = I \). Indeed, that assumption means that \( \mathcal{P} \) is the set
of public atoms of some stable model $J$ of the program $v(\Pi) \cup \mathcal{I}$, where $\Pi$ is the set of rules of $\Omega$. In that program, the facts $\mathcal{I}$ are the only rules containing input symbols in the head, so that $J^{in} = \mathcal{I}$. Since all input symbols are public, it follows that $P^{in} = \mathcal{I}$.

To complete the proof of the theorem, we need to show that

\[ P \text{ is an io-model of } \Omega \text{ for } (v, P^{in}) \text{ iff } P^\uparrow \text{ satisfies } v(\text{COMP}[\Omega]) \quad \text{(B14)} \]

under the assumption that $\Omega$ is locally tight on $(v, P^{in})$.

The condition in the left-hand side of (B14) means that $P$ is the set of public atoms of some stable model of $v(\Pi) \cup P^{in}$. By Lemma 7, this condition can be equivalently reformulated as follows: for some set $J$ obtained from $P$ by adding private precomputed atoms, $J^\uparrow$ is a stable model of $v(\Pi) \cup P^{in}$. We can further reformulate this condition using Main Lemma (Section B.1) with

- the signature $\sigma_0$ as $\sigma$,
- all comparison symbols viewed as extensional,
- $J^\uparrow$ as $I$,
- $\tau^*(v(\Pi) \cup P^{in})$ as $\Gamma$.

The graph $G_{sp}^{(v(\Pi) \cup P^{in})}$ has no infinite walks, because it is identical to the graph $G_{sp}^{(v(\Pi))}$, which has no infinite walks by Lemma 13(ii). Consequently the left-hand side of (B14) is equivalent to the condition: for some set $J$ obtained from $P$ by adding private precomputed atoms, $J^\uparrow$ satisfies $\text{COMP}[\tau^*(v(\Pi) \cup P^{in})]$.

The formula $\text{COMP}[\tau^*(v(\Pi) \cup P^{in})]$ is a conjunction that includes the completed definitions of all input symbols among its conjunctive terms. The interpretation $J^\uparrow$ satisfies these completed definitions, because $J^{in} = P^{in}$. The remaining conjunctive terms of $\text{COMP}[\tau^*(v(\Pi) \cup P^{in})]$ form the completion $\text{COMP}[\tau^*(v(\Pi))]$ under the assumption that both comparison symbols and the input symbols are considered extensional. This formula can be also written as $v(\text{COMP}[\tau^*(\Pi)])$. By Lemma 8, it is equivalent to $v(\text{COMP}^1[\Omega])$. Consequently the left-hand side of (B14) is equivalent to the condition:

for some set $J$ obtained from $P$ by adding private precomputed atoms,
\[ J^\uparrow \text{ satisfies } v(\text{COMP}^1[\Omega]). \]

Since $\text{COMP}[\Omega]$ is obtained from $\text{COMP}^1[\Omega]$ by replacing private symbols with existentially quantified variables, this condition is equivalent to the right-hand side of (B14) as claimed.