1. Prove that the assertion
\[ m = F_i \land n = F_{i+1} \land i \leq 10 \]
where \( F_i \) is the \( i \)-th Fibonacci number, is a loop invariant for the loop

\[
\text{while } i < 10 \text{ do}
\begin{align*}
& i \leftarrow i + 1; \\
& k \leftarrow m + n; \\
& m \leftarrow n; \\
& n \leftarrow k
\end{align*}
\text{enddo}
\]

Solution:
\[
\begin{align*}
\{ m = F_i \land n = F_{i+1} \land i \leq 10 \land i < 10 \} \\
\{ m = F_i \land n = F_{i+1} \land i < 10 \} \\
\{ n = F_{i+1} \land m + n = F_{i+2} \land i + 1 \leq 10 \} \\
\{ n = F_i \land m + n = F_{i+1} \land i \leq 10 \} \\
\{ k \leftarrow m + n; \\
\{ n = F_i \land k = F_{i+1} \land i \leq 10 \} \\
\{ m \leftarrow n; \\
\{ m = F_i \land k = F_{i+1} \land i \leq 10 \} \\
\{ n \leftarrow k \\
\{ m = F_i \land n = F_{i+1} \land i \leq 10 \}
\end{align*}
\]

2. Determine which of the assertions
\[ n \geq 10, \quad n \mid 10, \quad 10 \mid n \]
are loop invariants for the loop

\[
\text{while } n < 10 \text{ do}
\begin{align*}
& n \leftarrow n \times 2 \\
\end{align*}
\text{enddo}
\]

Solution. Assertion \( n \geq 10 \) is a loop invariant:
\[
\begin{align*}
\{ n \geq 10 \land n < 10 \} \\
\{ \text{false} \} \\
\{ n \times 2 \geq 10 \} \\
\{ n \leftarrow n \times 2 \\
\{ n \geq 10 \}
\end{align*}
\]
Assertion $n \mid 10$ is not a loop invariant; counterexample: $n = 10$.

Assertion $10 \mid n$ is a loop invariant:

\[
\{10 \mid n \land n < 10\}
\{10 \mid n\}
\{10 \mid n \times 2\}
\]
\[
n \leftarrow n \times 2
\]
\[
\{10 \mid n\}
\]

3. The sequence $A_0, A_1, \ldots$ is defined by the formulas

\[
A_n = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even,} \\
\frac{n+1}{2}, & \text{otherwise.} 
\end{cases}
\]

(a) Prove that all members of this sequence are integers.

Solution. Case 1: $n$ is even. Then $A_n = \frac{n}{2}$, which is an integer, Case 2: $n$ is odd. Then $A_n = \frac{n+1}{2}$, since $n+1$ is even, this is an integer,

(b) Prove that for every $n$, $A_{n+2} = A_n + 1$.

Solution. Case 1: $n$ is even. Then $n+2$ is even too, and

\[
A_{n+2} = \frac{n+2}{2} = \frac{n}{2} + 1 = A_n + 1.
\]

Case 2: $n$ is odd. Then $n+2$ is odd too, and

\[
A_{n+2} = \frac{(n+2) + 1}{2} = \frac{n+3}{2} = \frac{n+1}{2} + 1 = A_n + 1.
\]

4. The sequence $B_0, B_1, \ldots$ is defined by the formulas

\[
B_0 = 0,
B_1 = 1,
B_{n+2} = 4B_{n+1} - B_n.
\]

(i) Find an explicit formula for $B_n$. (ii) Determine how the formula will change if we replace the first two equations by

\[
B_0 = 1,
B_1 = 2.
\]

Solution. First we need to find the values of $c$ for which the sequence $1, c, c^2, \ldots$ satisfies the condition $B_{n+2} = 4B_{n+1} - B_n$. If we take $n = 0$ in this formula and replace $B_n$ with $c^n$, we get the equation $c^2 = 4c - 1$. From this equation we find:

\[
c_1 = 2 + \sqrt{3}, \quad c_2 = 2 - \sqrt{3}.
\]
It follows that any sequence $B_n$ defined by a formula of the form

$$B_n = a \left(2 + \sqrt{3}\right)^n + b \left(2 - \sqrt{3}\right)^n$$

satisfies the condition $B_{n+2} = 4B_{n+1} - B_n$. It remains to find the coefficients $a$ and $b$.

(i) From the initial conditions $B_0 = 0$, $B_1 = 1$ we find:

$$a + b = 0,$$

$$a \left(2 + \sqrt{3}\right) + b \left(2 - \sqrt{3}\right) = 1.$$ 

From these equations,

$$a = \frac{1}{2\sqrt{3}}, \quad b = -\frac{1}{2\sqrt{3}}.$$ 

So the formula for $B_n$ in this case is

$$B_n = \frac{1}{2\sqrt{3}} \left(2 + \sqrt{3}\right)^n - \frac{1}{2\sqrt{3}} \left(2 - \sqrt{3}\right)^n.$$

(ii) From the initial conditions $B_0 = 1$, $B_1 = 2$ we find:

$$a + b = 1,$$

$$a \left(2 + \sqrt{3}\right) + b \left(2 - \sqrt{3}\right) = 2.$$ 

From these equations,

$$a = b = \frac{1}{2}.$$ 

So the formula for $B_n$ in this case is

$$B_n = \frac{1}{2} \left(2 + \sqrt{3}\right)^n + \frac{1}{2} \left(2 - \sqrt{3}\right)^n.$$

5. Draw a graph with the adjacency matrix

$$\begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 1 & 0 
\end{bmatrix}$$

Answer: