1. Without a calculator, determine which of the numbers

\[ 10^{30}, \ 10^{50}, \ 10^{70} \]

gives the best approximation to the value of the fraction \( \frac{100!}{(50!)^2} \). Justify your answer.

*Solution.* Using the approximation

\[ n! \approx \left( \frac{n}{e} \right)^n \]

we calculate:

\[
\frac{100!}{(50!)^2} \approx \frac{\left( \frac{100}{e} \right)^{100}}{\left( \frac{50}{e} \right)^{100}} = \left( \frac{100}{50} \right)^{100} = 2^{100} = (2^{10})^{10} \approx 1000^{10} = (10^3)^{10} = 10^{30}.
\]

2. Prove that

(i) \( n^2 + n + 1 = O(n^2) \),
(ii) \( 3 \cdot 2^n + 100 = O(2^n) \),
(iii) \( e^n + e^{n+1} = O(e^n) \).

*Solution.*

(i) Take \( C = 3 \) and \( N = 1 \). We now claim that for all \( n \geq N \),

\[ n^2 + n + 1 \leq 3n^2. \]

*Proof:*

\[
\begin{align*}
    n^2 + n + 1 & \leq n^2 + n^2 + 1 & \text{Since } n \leq n^2 \text{ when } n \geq 1 \\
    & \leq n^2 + n^2 + n^2 & \text{Since } 1 \leq n^2 \text{ when } n \geq 1 \\
    & = 3n^2.
\end{align*}
\]

(ii) Take \( C = 103 \) and \( N = 1 \). We now claim that for all \( n \geq N \),

\[ 3 \cdot 2^n \leq 103 \cdot 2^n. \]
Proof:

\[ 3 \cdot 2^n + 100 \leq 3 \cdot 2^n + 100 \cdot 2^n \quad \text{Since } 2^n > 1 \text{ when } n \geq 1 \]
\[ = 103 \cdot 2^n \]

(iii) Take \( C = 4 \) and \( N = 1 \). We now claim that for all \( n \geq N \),

\[ e^n + e^{n+1} \leq 4 \cdot e^n. \]

Proof:

\[ e^n + e^{n+1} = (e + 1) \cdot e^n \leq 4 \cdot e^n \quad \text{Since } e + 1 \leq 4 \]

3. Let \( A \) be the set \( \{\{1\}, \{2\}, \{3\}\} \).

(i) How many elements does \( A \) have?

(ii) Does \( A \) have a pair of different elements \( x, y \) such that \( x \subseteq y \)?

(iii) How many subsets does \( A \) have?

(iv) Does \( A \) have a pair of different subsets \( x, y \) such that \( x \subseteq y \)?

Justify your answers.

Solution.

(i) Set \( A \) has 3 elements: \( \{1\}, \{2\}, \) and \( \{3\} \).

(ii) No. Neither element of \( A \) is contained in another element of \( A \).

(iii) A set containing \( n \) elements has \( 2^n \) subsets. Since \( A \) has 3 elements, it has 8 subsets.

(iv) Yes. For example, \( \{\{1\}\} \subseteq \{\{1\}, \{2\}\} \).