1. Determine whether the given formula is true or false. Justify your answers.

(a) \( \forall m \exists n (2 \mid m + n) \). **Solution:** True. Choose \( n = -m \). Then \( m + n = 0 \); 0 is even.

(b) \( \exists m \forall n (m - 5 \mid n) \). **Solution:** True. Choose \( m = 6 \). Then \( m - 5 = 1 \); every integer is a multiple of 1.

2. Function \( f \) is defined by the formulas

\[
f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

Find numbers \( a, b \) such that for all values of \( x \)

\[
f(x) = ax + b|x|.
\]

Prove that your formula is correct.

**Solution.** If \( f(x) = ax + b|x| \) for all \( x \) then \( 0 = f(-1) = -a + b \) and \( 1 = f(1) = a + b \). Solving gives us \( a = \frac{1}{2} \) and \( b = \frac{1}{2} \). To prove the formula

\[
f(x) = \frac{x}{2} + \frac{|x|}{2},
\]

consider two cases. Case 1: \( x < 0 \). Then \( |x| = -x \), so that

\[
\frac{x}{2} + \frac{|x|}{2} = \frac{1}{2}x - \frac{1}{2}x = 0 = f(x).
\]

Case 2: \( x \geq 0 \). Then \( |x| = x \), so that

\[
\frac{x}{2} + \frac{|x|}{2} = \frac{1}{2}x + \frac{1}{2}x = x = f(x).
\]

3. Prove that for every positive integer \( n \)

\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1)(2n + 1)} = \frac{n}{2n + 1}.
\]

**Solution:** We will prove the formula by induction. **Basis:** \( n = 1 \). The given formula turns into the correct equality \( \frac{1}{13} = \frac{1}{21} \). **Induction step:** Assuming that the given formula is true for \( n \), we can prove that

\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1)(2n + 1)} + \frac{1}{(2n + 1)(2n + 3)} = \frac{n + 1}{2n + 3}
\]
as follows:

\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)}
\]

\[= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}
\]

\[= \frac{n(2n+3) + 1}{(2n+1)(2n+3)}
\]

\[= \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)}
\]

\[= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)}
\]

\[= \frac{n + 1}{2n + 3}.
\]

4. Prove that for every positive integer \(n\), 43 divides \(6^{n+1} + 7^{2n-1}\).

Solution. We will prove this assertion by induction. Basis: \(n = 1\). Then

\[6^{n+1} + 7^{2n-1} = 6^2 + 7^1 = 43.\]

Induction step: Suppose that 43 divides \(6^{n+1} + 7^{2n-1}\). To prove that 43 divides \(6^{n+2} + 7^{2n+1}\), we rewrite this expression as follows:

\[6^{n+2} + 7^{2n+1} = 6^{n+1} \cdot 6 + 7^{2n-1} \cdot 49 = (6^{n+1} + 7^{2n-1}) \cdot 6 + 7^{2n-1} \cdot 43.\]

In the last expression, each of the two summands is a multiple of 43.

5. The numbers \(A_0, A_1, A_2, \ldots\) are defined by the formulas

\[A_0 = 0,\]

\[A_{n+1} = n \cdot (A_n + 1).\]

Prove that \(A_n < n!\).

Solution. Consider two cases. Case 1: \(n = 0\). Then \(A_n = A_0 = 0 < 1 = 0! = n!\).

Case 2: \(n > 0\). We will prove the inequality \(A_n < n!\) by induction. Basis: \(n = 1\). Then \(A_n = A_1 = 0 < 1 = 1! = n!\). Induction step: Assuming that \(A_n < n!\) for some positive \(n\), we can prove that \(A_{n+1} < (n + 1)!\) as follows:

\[A_{n+1} = n \cdot (A_n + 1) < n \cdot (n! + 1) = n \cdot n! + n \leq n \cdot n! + n! = (n + 1)n! = (n + 1)!.\]