Introduction to Mathematical Logic, Handout 7
Predicate Formulas: Syntax

A *predicate signature* is a set of symbols of two kinds—*object constants* and *predicate constants*—with a positive integer, called the *arity*, assigned to every predicate constant. A predicate constant is said to be *unary* if its arity is 1, *binary* if its arity is 2, and *ternary* if its arity is 3.

Take a predicate signature σ that contains at least one predicate constant and does not contain any of the following symbols:

- the *object variables* $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \ldots$,
- the propositional connectives,
- the *universal quantifier* $\forall$ and the *existential quantifier* $\exists$,
- the parentheses and the comma.

The alphabet of predicate logic consists of the elements of σ and of the four groups of additional symbols listed above. A *string* is a finite string of symbols in this alphabet.

A *term* is an object constant or an object variable. A string is called an *atomic formula* if it has the form

$$P(t_1, \ldots, t_n)$$

where P is a predicate constant of arity n and $t_1, \ldots, t_n$ are terms.

We define when a string is a *(predicate) formula* recursively, as follows:

- every atomic formula is a formula,
- if F is a formula then $\neg F$ is a formula,
- for any binary connective $\odot$, if F and G are formulas then $(F \odot G)$ is a formula,
- for any quantifier K and any variable v, if F is a formula then KvF is a formula.

When we write predicate formulas, we will use the abbreviations introduced at the end of Handout 1. A string of the form $\forall v_1 \cdots \forall v_n$ will be written as $\forall v_1 \cdots v_n$, and similarly for the existential quantifier.

An occurrence of a variable v in a formula F is *bound* if it belongs to a part of F that has the form KvG; otherwise it is *free*. We say that a variable v is *free (bound)* in F if at least one occurrence of v in F is free (bound). A formula without free variables is called a *closed formula*, or a *sentence*.

In the following problems, represent the given conditions by formulas of the signature consisting of the object constant a and the ternary predicate constants *Sum* and *Prod*. Think of the object variables as ranging over all nonnegative integers, and interpret the object and predicate constants as follows:

- a represents 0,
- $Sum(x, y, z)$ represents the condition $x + y = z$,
- $Prod(x, y, z)$ represents the condition $xy = z$.

Problem 7.1 (a) $x = y$; (b) $x = 0$; (c) $x = 1$.

Problem 7.2 (a) $x \leq y$; (b) $x < y$; (c) $x + 1 < y$.

Problem 7.3 (a) x is an even number; (b) the sum of any two odd numbers is even; (c) addition is commutative.

Problem 7.4 Number x can be represented as the sum of two complete squares.

Problem 7.5 Number x is prime.