

## Introduction to Mathematical Logic, Handout 3

### Adequate Sets of Connectives, Normal Forms, and Clausification

**Problem 3.1** For any formula, there exists an equivalent formula that contains no connectives other than (i) conjunction and negation; (ii) disjunction and negation; (iii) implication and negation.

**Problem 3.2** Any propositional formula equivalent to  $\neg p$  contains negation.

A *literal* is an atom or the negation of an atom. A propositional formula is said to be in *negation normal form* if

- it contains no connectives other than conjunction, disjunction, and negation, and
- every negation in it is part of a literal.

**Problem 3.3** Any formula is equivalent to a formula in negation normal form.

A *simple conjunction* is a formula of the form  $L_1 \wedge \cdots \wedge L_n$  ( $n \geq 1$ ), where  $L_1, \dots, L_n$  are literals. A formula is in *disjunctive normal form (DNF)* if it has the form  $C_1 \vee \cdots \vee C_m$  ( $m \geq 1$ ), where  $C_1, \dots, C_m$  are simple conjunctions.

**Problem 3.4** Any formula is equivalent to a formula in disjunctive normal form.

A *simple disjunction* is a formula of the form  $L_1 \vee \cdots \vee L_n$  ( $n \geq 1$ ), where  $L_1, \dots, L_n$  are literals. (Simple disjunctions are also called *clauses*.) A formula is in *conjunctive normal form (CNF)* if it has the form  $D_1 \wedge \cdots \wedge D_m$  ( $m \geq 1$ ), where  $D_1, \dots, D_m$  are simple disjunctions.

**Problem 3.5** Let  $F$  be a formula in disjunctive normal form. Show that  $\neg F$  is equivalent to a formula in conjunctive normal form.

**Problem 3.6** Any formula is equivalent to a formula in conjunctive normal form.

To *clausify* a formula  $F$  of a signature  $\sigma$  means to find a formula  $F'$  of some signature  $\sigma'$  containing  $\sigma$  such that

- $F'$  is in conjunctive normal form,
- any interpretation  $I$  of  $\sigma$  satisfying  $F$  can be extended to an interpretation  $I'$  of  $\sigma'$  that satisfies  $F'$ ,
- for any interpretation  $I'$  of  $\sigma'$  satisfying  $F'$ , the restriction of  $I'$  to  $\sigma$  satisfies  $F$ .

Here is an algorithm for clausifying a propositional formula:

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CLAUSIFY( $F$ )
begin
   $\Gamma \leftarrow \emptyset$ 
  while  $F$  is not CNF do
     $A \leftarrow$  a new atom
     $G \leftarrow$  a minimal non-literal subformula of  $F$ 
     $F \leftarrow$  the result of replacing  $G$  in  $F$  by  $A$ 
     $\Delta \leftarrow$  the set of clauses of the CNF of  $A \leftrightarrow G$ 
     $\Gamma \leftarrow \Gamma \cup \Delta$ 
  return the conjunction of  $F$  with clauses  $\Gamma$ 
end

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**Problem 3.7** (a) Apply the algorithm CLAUSIFY to the formula

$$p \vee \neg(q \rightarrow r). \tag{1}$$

(b) Determine whether (1), viewed as a formula of the extended signature, is equivalent to the result of its clausification.