CS313K: Logic, Sets and Functions
Fall 2010

Problem Set 1: Infinite Sequences

The infinite sequence of numbers $A_1, A_2, \ldots$ is defined by the condition:

$$A_n = \begin{cases} n+1, & \text{if } n \text{ is even,} \\ n-1, & \text{otherwise.} \end{cases}$$

1.1. Calculate the first 6 members of this sequence:

n	1	2	3	4	5	6
A_n						

Find a single formula for A_n that works for all numbers, both even and odd.

1.2. Find all values of n for which $A_n < 5$.

1.3. Sequence A contains every integer that is greater than 1. True or false?

1.4. Number 10 occurs in sequence A two times. True or false?

By B_n we denote the sum of all numbers from 1 to n:

$$B_n = 1 + 2 + \cdots + n.$$

C_n stands for the sum of the squares of these numbers, and D_n for the sum of their cubes:

$$C_n = 1^2 + 2^2 + \cdots + n^2,$$
$$D_n = 1^3 + 2^3 + \cdots + n^3.$$

Numbers B_n, C_n, D_n can be also described using "sigma notation." For instance, the formula

$$D_{100} = 1^3 + 2^3 + \ldots + 100^3,$$

written in sigma notation, looks like this:

$$D_{100} = \sum_{i=1}^{100} i^3.$$

The letter i is the *index variable* in this expression, and the numbers 1 and 100 are the *lower* and *upper bounds*. (We could have used any other variable instead of i, for instance j.) More generally,

$$B_n = \sum_{i=1}^{n} i, \qquad C_n = \sum_{i=1}^{n} i^2, \qquad D_n = \sum_{i=1}^{n} i^3.$$

1.5. Calculate the first 5 members of sequence B. Find a simple formula for B_n that uses neither dots nor sigma notation.

1.6. Calculate the first 5 members of sequence D. Can you guess what a simple formula for D_n may be like?

1.7. Calculate the first 5 members of sequence C. Can you guess what a simple formula for C_n may be like?

By E_n we denote the number of ways to choose two elements out of n. For instance, $E_4 = 6$, because a 4-element set $\{p, q, r, s\}$ has 6 subsets consisting of two elements each:

$$\{p, q\},\ \{p, r\},\ \{p, s\},\ \{q, r\},\ \{q, s\},\ \{r, s\}.$$

1.8. Calculate the first 5 members of sequence E. Find a formula for E_n.

By F_n we denote the number of parts into which a plane is divided by n lines taken at random. For instance, $F_3 = 7$, because 3 lines taken at random divide the plane into 7 parts: a triangle and 6 infinite regions.

1.9. Calculate the first 5 members of sequence F. Find a formula for F_n.