CS313K: Logic, Sets and Functions
Fall 2010

Problem Set 4: Sets

A *set* is a collection of objects. We write $x \in A$ if object x is an element of set A, and $x \notin A$ otherwise.

The set whose elements are $x_1, \ldots, x_n$ is denoted by $\{x_1, \ldots, x_n\}$. The set $\{\,\}$ is called *empty* and denoted also by $\emptyset$. The set of nonnegative integers is denoted by $\mathbf{N}$:

$$\mathbf{N} = \{0, 1, 2, \ldots\}.$$

When we specify which objects belong to a set, this defines the set completely; there is no such thing as the order of elements in a set or the number of repetitions of an element in a set. For instance,

$$\{2, 3\} = \{3, 2\} = \{2, 2, 3\}.$$

If C is a condition, then by $\{x \,:\, C\}$ we denote the set of all objects x satisfying this condition. For instance,

$$\{x \,:\, x = 2 \;\text{ or }\; x = 3\}$$

is the same set as $\{2, 3\}$.

If A is a set and C is a condition, then by $\{x \in A \,:\, C\}$ we denote the set of all elements of A satisfying condition C. For instance, $\{2, 3\}$ can be also written as

$$\{x \in \mathbf{N} \,:\, 1 < x < 4\}.$$

If a set A is finite then the number of elements of A is also called the *cardinality* of A and denoted by $|A|$. For instance,

$$|\emptyset| = 0, \;\; |\{2, 3\}| = 2.$$

We say that a set A is a *subset* of a set B, and write $A \subseteq B$, if every element of A is an element of B. For instance,

$$\emptyset \subseteq \mathbf{N}, \;\; \{2, 3\} \subseteq \mathbf{N}.$$

4.1. (a) Find all subsets of $\{2, 3, 5\}$. (b) If the cardinality of A is n then how many subsets does A have? (c) Consider the set $\{x \in \mathbf{N} \,:\, 10 \le x \le 20\}$. How many subsets does it have?

For any sets A and B, by $A \cup B$ we denote the set

$$\{x \ : \ x \in A \text{ or } x \in B\},$$

called the *union* of A and B. By $A \cap B$ we denote the set

$$\{x \ : \ x \in A \text{ and } x \in B\},$$

called the *intersection* of A and B. For instance,

$$\begin{aligned} \{2,3\} \cup \{3,5\} &= \{2,3,5\}, \\ \{2,3\} \cap \{3,5\} &= \{3\}. \end{aligned}$$

4.2. Let $|A| = 3$, $|B| = 4$. What can you say about the cardinalities of $A \cup B$ and $A \cap B$?

4.3. Simplify the expressions $(A \cap B) \cup A$ and $(A \cup B) \cap A$.

4.4. For any sets A, B, C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

True or false? Answer the same question about the formula

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

4.5. For any sets A, B, C, if

$$A \cap B \neq \emptyset, \ A \cap C \neq \emptyset, \ B \cap C \neq \emptyset$$

then $A \cap B \cap C \neq \emptyset$. True or false?

By $A \setminus B$ we denote the set

$$\{x \ : \ x \in A \text{ and } x \notin B\},$$

called the *difference* of A and B. For instance,

$$\{2,3\} \setminus \{3,5\} = \{2\}.$$

4.6. Let $|A| = 3$, $|B| = 4$. What can you say about the cardinalities of $A \setminus B$ and $B \setminus A$?

4.7. Simplify the expressions

$$(A \cup B) \setminus A,$$
$$(A \cap B) \setminus A,$$
$$(A \cap B) \cup (A \setminus B).$$

The *Cartesian product* of sets A and B is the set of ordered pairs $\langle x, y\rangle$ such that $x \in A$ and $y \in B$:

$$A \times B = \{\langle x, y\rangle \mid x \in A \text{ and } \ y \in B\}.$$

For instance,

$$\begin{array}{rcl} \{1,2\} \times \{2,3,4,5,6\} & = & \{\langle 1,2\rangle,\ \langle 1,3\rangle,\ \langle 1,4\rangle,\ \langle 1,5\rangle,\ \langle 1,6\rangle, \\ & & \ \langle 2,2\rangle,\ \langle 2,3\rangle,\ \langle 2,4\rangle,\ \langle 2,5\rangle,\ \langle 2,6\rangle\}. \end{array}$$

4.8. Determine whether the following assertions are true. (a) For any sets A and B, $A \times B = B \times A$. (b) For any sets A and B, if A is infinite then $A \times B$ is infinite too. (c) For any sets A and B, if $|A \times B| = 91$ then one of the sets A, B is a singleton.

4.9. For any sets A, B, C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

True or false?

An element of a set can itself be a set. For instance, $\{\emptyset, \{2,3\}\}$ is a set whose elements are $\emptyset$ and $\{2,3\}$.

By $\mathcal{P}(A)$ we denote the *power set* of a set A, that is, the set of all subsets of A:

$$\mathcal{P}(A) = \{B \ : \ B \subseteq A\}.$$

For instance,

$$\mathcal{P}(\{2,3\}) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}.$$

4.10. For any sets A and B,

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B).$$

True or false? Answer the same question about the formula

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$