CS313K: Logic, Sets and Functions
Fall 2010

Problem Set 5: Binary Relations

Any condition on a pair of elements of a set A defines a *binary relation* (or simply *relation*) on A. For instance, the condition $x < y$ defines a relation on the set $\mathbf{N}$ of nonnegative integers (or on any other set of numbers). If R is a relation, the formula xRy expresses that R holds for the pair x, y.

A relation R can be characterized by the set of all ordered pairs $\langle x, y\rangle$ such that xRy. In mathematics, it is customary to talk about a relation as it were the same thing as the corresponding set of ordered pairs. For instance, we can say that the relation $<$ on the set $\{1, 2, 3, 4\}$ is the set

$$\{\langle 1,2\rangle,\ \langle 1,3\rangle,\ \langle 1,4\rangle,\ \langle 2,3\rangle,\ \langle 2,4\rangle,\ \langle 3,4\rangle\}.$$

A relation R on a set A is said to be *reflexive* if, for all $x \in A$, xRx. For instance, the relations $=$ and $\leq$ on the set $\mathbf{N}$ (or on any set of numbers) are reflexive, and the relations $\neq$ and $<$ are not.

A relation R on a set A is said to be *symmetric* if, for all $x, y \in A$, xRy implies yRx. For instance, the relations $=$ and $\neq$ on $\mathbf{N}$ are symmetric, and the relations $<$ and $\leq$ are not.

5.1. Find the number of binary relations on the set $\{1, 2, 3\}$. How many of them are reflexive? How many of them are symmetric?

A relation R on a set A is said to be *transitive* if, for all $x, y, z \in A$, xRy and yRz imply xRz. For instance, the relations $=$, $<$ and $\leq$ on the set $\mathbf{N}$ are transitive, and the relation $\neq$ is not.

5.2. (a) Consider the relation R on the set $\mathbf{N}$ defined by the condition: xRy if $x \geq y + 5$. Is it reflexive? Is it symmetric? Is it transitive? (b) Answer the same questions for the relation: xRy if $|x - y| < 5$.

A *partition* of a set A is a collection P of non-empty subsets of A such that every element of A belongs to exactly one of these subsets. For instance, here are some partitions of $\mathbf{N}$:

$$\begin{aligned}
P_1 &= \{\{0,1\},\{2,3,4,5,\dots\}\},\\
P_2 &= \{\{0,2,4,\dots\},\{1,3,5,\dots\}\},\\
P_3 &= \{\{0,1\},\{2,3\},\{4,5\},\dots\},\\
P_4 &= \{\{0\},\{1\},\{2\},\{3\},\dots\}.
\end{aligned}$$

5.3. Find all partitions of the set $\{1, 2, 3\}$.

If P is a partition of a set A then the relation "x and y belong to the same element of P" is reflexive, symmetric and transitive. A binary relation that has all three properties is called an *equivalence relation*. For instance, the equivalence relation corresponding to partition P_4 is equality.

5.4. (a) Let R be the relation on the set of positive integers defined by the condition: xRy if the last digit of x in binary notation is the same as the last digit of y. Check that this is an equivalence relation, and find the corresponding partition of $\mathbf{N}$. (b) Do the same for the relation: xRy if the number of digits of x in binary notation is the same as the number of digits of y.