CS313K: Logic, Sets and Functions
Fall 2010

Problem Set 6: Functions

In Problem Set 3 we defined a function as a rule that applies to a number and produces a number. From now on, we will use the term "function" in a more general sense. For any sets A and B, a *function from A to B* is a rule f that can be applied to any element x of A and produces an element $f(x)$ of B. The set A is called the *domain* of f. The subset of B consisting of the values $f(x)$ for all $x \in A$ is called the *range* of f. If the range of f is the whole set B then we say that f is a function *onto* B.

This definition of a function is more general because it does not assume that the domain and the range consist of numbers. In the following examples, the domain of each function is the set $\mathbf{S}$ of all bit strings:

$$\mathbf{S} = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}.$$

1. Function l from $\mathbf{S}$ to $\mathbf{N}$: $l(x)$ is the length of x. For instance, $l(00110) = 5$.

2. Function z from $\mathbf{S}$ to $\mathbf{N}$: $z(x)$ is the number zeroes in x. For instance, $z(00110) = 3$.

3. Function n from $\mathbf{S}$ to $\mathbf{N}$: $n(x)$ is the number represented by x in binary notation. For instance, $n(00110) = 6$.

4. Function e from $\mathbf{S}$ to $\mathbf{S}$: $e(x)$ is the string $1x$. For instance, $t(00110) = 100110$.

5. Function r from $\mathbf{S}$ to $\mathbf{S}$: $r(x)$ is the string x reversed. For instance, $r(00110) = 01100$.

6. Function p from $\mathbf{S}$ to $\mathcal{P}(\mathbf{S})$: $p(x)$ is the set of prefixes of x. For instance, $p(00110) = \{\epsilon, 0, 00, 001, 0011, 00110\}$.

6.1. Which of the functions l, z, n, e, r, p defined above are onto?

A function f can be characterized by the set of the ordered pairs $\langle x, f(x)\rangle$ for all x in the domain of f. It is customary to talk about a function as if it were the same thing as the corresponding set of ordered pairs. For instance, if the function f from $\mathbf{N}$ to $\mathbf{N}$ is defined by the formula $f(n) = 2n + 1$ then we can write

$$f = \{\langle 0, 1\rangle,\ \langle 1, 3\rangle,\ \langle 2, 5\rangle,\ \ldots\}.$$

We can say that a function from A to B is a subset f of $A \times B$ satisfying the following condition: for every element x of A there exists a unique element y of B such that $\langle x, y\rangle \in f$.

6.2. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$. What is the number of subsets of $A \times B$? How many of them are functions from A to B?

A function f from A to B is called *one-to-one* if, for any pair of different elements x, y of A, $f(x)$ is different from $f(y)$. If a function f is both onto and one-to-one then we say that f is a *bijection*. A *permutation* of a set A is a bijection from A to A.

6.3. (a) Which of the functions l, z, n, e, r, p defined above are one-to-one? Which of them are bijections? (b) Find all permutations of the set $\{1, 2, 3\}$.

If f is a function from A to B, and g is a function from B to C, then the *composition* of these functions is the function h from A to C defined by the formula $h(x) = g(f(x))$. This function is denoted by $g \circ f$.

6.4. For the functions l, z, n, e and r defined above, which of the following formulas are true?

- $l \circ r = l$,
- $z \circ r = z$,
- $n \circ r = n$,
- $e \circ r = r \circ e$.

If f is a bijection from A to B then the *inverse* of f is the function g from B to A such that, for every $x \in A$, $g(f(x)) = x$. This function is denoted by f^{-1}.

6.5. Find all permutations f of $\{1, 2, 3\}$ such that $f^{-1} \neq f$.