CS313K: Logic, Sets and Functions
Fall 2010

Problem Set 7: Propositional Formulas

Truth values are the symbols **f** ("false") and **t** ("true"). *Propositional connectives* are functions that are applied to truth values and return truth values. We will use these propositional connectives:

$\neg$	"not" (negation)
$\wedge$	"and" (conjunction)
$\vee$	"or" (disjunction)
$\rightarrow$	"implies" (implication)
$\leftrightarrow$	"is equivalent to" (equivalence)

Negation is a unary connective (takes one argument); the others are binary (take two arguments). Here are the tables of values for these connectives:

p	$\neg p$
f	**t**
t	**f**

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
f	**f**	**f**	**f**	**t**	**t**
f	**t**	**f**	**t**	**t**	**f**
t	**f**	**f**	**t**	**f**	**f**
t	**t**	**t**	**t**	**t**	**t**

Propositional variables are variables for truth values, such as p and q in the tables above. *Propositional formulas* are built from truth values and propositional variables using propositional connectives. For instance,

$$\neg(p \vee q) \rightarrow (r \leftrightarrow \mathbf{f})$$

is a propositional formula.

7.1. (a) Find the value of this formula for $p = \mathbf{f}$, $q = \mathbf{t}$, $r = \mathbf{t}$. (b) Find all combinations of values of p, q, r that make this formula false. (c) Complete the table showing all values of this formula:

p	q	r	$\neg(p \vee q) \rightarrow (r \leftrightarrow \mathbf{f})$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	

The table showing the values of a propositional formula for all combinations of values of its variables is called the *truth table* of that formula. A propositional formula is called a *tautology* if each of its truth values is $\mathbf{t}$. For instance, $p \rightarrow p$ and $p \vee \neg p$ are tautologies.

7.2. Make truth tables for these formulas:

(a) $p \rightarrow \mathbf{f}$

(b) $\neg p \vee q$

(c) $\neg p \rightarrow q$

7.3. Make truth tables for these formulas:

(a) $\neg(\neg p \vee \neg q)$

(b) $\neg(\neg p \wedge \neg q)$

(c) $(p \wedge q) \vee (\neg p \wedge \neg q)$

7.4. Determine which of these formulas are tautologies:

(a) $(p \rightarrow q) \vee (q \rightarrow p)$

(b) $((p \rightarrow q) \rightarrow p) \rightarrow p$

(c) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

Propositional formulas F and G are said to be *equivalent* to each other if, for every combination of values of their variables, the value of F is the same as the value of G. For instance, every tautology is equivalent to $\mathbf{t}$. The formula $p \wedge \neg p$ is equivalent to $\mathbf{f}$. Other examples:

$$p \wedge (q \vee r) \text{ is equivalent to } (p \wedge q) \vee (p \wedge r)$$

and

$$p \vee (q \wedge r) \text{ is equivalent to } (p \vee q) \wedge (p \vee r)$$

(distributive laws for conjunction and disjunction);

$$\neg(p \wedge q) \text{ is equivalent to } \neg p \vee \neg q$$

and

$$\neg(p \vee q) \text{ is equivalent to } \neg p \wedge \neg q$$

(De Morgan's laws).

Propositional variables are also called *atoms*. Atoms and negated atoms, such as p and $\neg p$, are also called *literals*. A propositional formula is said to be in *negation normal form* if it is formed from literals using conjunctions ($\wedge$) and disjunctions ($\vee$). Any propositional formula can be equivalently rewritten in negation normal form by

- eliminating all connectives other than $\neg$, $\wedge$, $\vee$, and
- using De Morgan's laws to push negations inside.

For instance, to convert the formula

$$\neg((p \wedge \neg q) \to (r \wedge s))$$

to negation normal form, we rewrite it as follows:

$$\begin{gathered} \neg(\neg(p \wedge \neg q) \vee (r \wedge s)), \\ p \wedge \neg q \wedge \neg(r \wedge s), \\ p \wedge \neg q \wedge (\neg r \vee \neg s). \end{gathered}$$

7.5. Convert the formulas

$$(p \to q) \to r$$

and

$$p \leftrightarrow (q \wedge r)$$

to negation normal form.

A propositional formula is said to be in *conjunctive normal form (CNF)* if it is a conjunction of disjunctions of literals. A propositional formula is said to be in *disjunctive normal form (DNF)* if it is a disjunction of conjunctions of literals. A formula in negation normal form can be turned into CNF or DNF using distributive laws. For instance, the formula

$$p \wedge \neg q \wedge (\neg r \vee \neg s)$$

is in CNF; we can turn it into DNF by rewriting it as

$$(p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg s).$$

7.6. Convert the formulas from Problem 7.5 to CNF and DNF.