The Giant k-core of a Random Graph with a Specified Degree Sequence

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Abstract

The k-core of a graph is the maximal induced subgraph with minimum degree k. In this paper, we find conditions under which the k-core of a random graph with a specified degree sequence almost surely contains a constant fraction of the graph's vertices. This problem has been studied earlier by Pittel, Spencer and Wormald [10] for the special case of a random graph drawn from $\mathcal{G}_{n,m}$.

The existence of a giant k-core in a random graph is related to the probability that the genealogy tree of a certain branching process contains a perfect infinite $(k-1)$-ary tree. We provide solutions to both problems in terms of probability generating functions. Our results apply to random graphs with degree sequences that are *smooth* and *sparse* (terms that are defined in the paper), and for which the maximum nonzero degree is $o(n^{1/3})$, where n is the number of vertices in the graph.

We apply our main theorem to derive the following results:

- We show that any random graph whose degree sequence satisfies the requirements of our theorem has a giant 2-core almost surely if and only if it has a giant component.
- We derive some general conditions for distributions with all convergent moments, one consequence of which is an alternate derivation of a result of Pittel, Spencer, and Wormald [10] regarding the k-core thresholds for the random graph model $\mathcal{G}_{n,m}$.
- A degree sequence exhibits a power law if the number of vertices of degree i is proportional to $\frac{1}{i^\beta}$, for a suitable constant β. Random graphs with power law degree sequences are of particular interest, since graphs that occur in the real world, including the web graph, phone-call graphs, networks of molecules, and networks of social interaction, often exhibit a power law degree sequence. Our first result, in conjunction with a result of Aiello, Chung and Lu [1], establishes that a 2-core exists in a random power law graph if $\beta < 3.47875\ldots$. For $k \geq 3$ we show that if $2 < \beta < 3$, a random power law graph almost surely contains a giant k-core for every constant $k \geq 3$, and if $\beta \geq 3$, it almost surely does not contain a 3-core.

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1 Preliminaries

1.1 Random Graph Definitions

We begin by providing definitions for random graphs with fixed degree sequences (see Molloy and Reed [8, 9]). A sequence $D = \{d_1, d_2, \ldots, d_n\}$ is *graphical* if the set Ω_D of (labelled) graphs with degree sequence D (i.e. such that the degree of the i'th vertex is d_i) is nonempty. If D is a graphical sequence, let $G(D)$ denote a uniformly distributed random element of Ω_D. Thus $G(D)$ is a *random graph with degree sequence D.*

An *asymptotic degree sequence* $\mathcal{D}$ is an infinite sequence $D_1, D_2, \ldots$, where each $D_n = \{d_{n,1}, \ldots, d_{n,n}\}$ is a graphical sequence of length n. A *random graph with asymptotic degree sequence* $\mathcal{D}$, denoted by $G(\mathcal{D})$, is a sequence of random graphs $G(D_n)$. The random graph $G(\mathcal{D})$ has a property P *asymptotically almost surely* (*a.a.s.*) if the probability that $G(D_n)$ has property P converges to 1 as $n \to \infty$; $G(\mathcal{D})$ has property P *with exponentially high probability* (*w.e.h.p.*) if the probability that $G(D_n)$ has property P is $c^{-\Omega(n^\epsilon)}$, for some $c > 1$ and $\epsilon > 0$.

An asymptotic degree sequence $\mathcal{D}$ is *smooth* if there exists a sequence of real numbers $\lambda_0, \lambda_1, \ldots$ such that

$$\lim_{n\to\infty} \frac{|\{j : d_{n,j} \geq i\}|}{n} = \sum_{j \geq i} \lambda_j \tag{1}$$

for all i. Note that equation 1 implies that $\sum_i \lambda_i = 1$. The sequence λ_i is the *limiting degree distribution* of $\mathcal{D}$. If a degree sequence is smooth then clearly

$$\lim_{n\to\infty} \frac{|\{j : d_{n,j} = i\}|}{n} = \lambda_i. \tag{2}$$

In fact, Molloy and Reed [8] use equation 2 directly as a definition of smoothness.

A smooth asymptotic degree sequence is *sparse* if the limiting degree distribution satisfies

$$\sum_i i\lambda_i < \infty \tag{3}$$

and if

$$\lim_{n\to\infty} \frac{\sum_{j=1}^n d_{n,j}}{n} = \sum_{i=0}^{\infty} i\lambda_i. \tag{4}$$

Throughout this paper, whenever a property of random graph with degree sequence D is described asymptotically, it is assumed that D is part of a sparse smooth asymptotic degree sequence.

1.2 The Configuration Model.

It is difficult to directly examine random graphs with given degree sequences, so instead we use the *configuration model (or 'CM')* introduced by Bollobás [2]. For a degree sequence D, consider a set of n vertices and $\sum_i d_i$ *endpoints*, and assign d_i endpoints to the vertex v_i. Now choose a perfect matching of the endpoints uniformly at random, and for each pair of matched endpoints, draw an edge connecting the corresponding vertices.

This procedure generates a graph with degree sequence D; however, the graph may contain loops and/or multiple edges. We shall abuse notation and refer to such a random (multi-)graph as a *random graph with degree sequence D generated by the configuration model.* Definitions for asymptotic degree sequences generalize to the configuration model in the obvious way.

Under certain circumstances results about random graphs generated by the configuration model hold in general for random graphs with the same degree sequence [8, 9]. It is easy to see that every simple graph with degree sequence D occurs with the same probability using the configuration model. A result of McKay and Wormald [7] implies if the maximum degree of a degree sequence is $o(n^{1/3})$ and the average degree is $O(1)$, then a random configuration produces a simple graph with probability $\Theta(1)$. In this case, a.a.s. and w.e.h.p. results for the configuration model clearly generalize to random graphs in general.

2 The k-core of a Random Graph.

The *k-core* of a graph G is the unique maximal induced subgraph of G with minimum degree k. In this section we determine conditions under which a random graph with degree sequence D has a giant k-core (i.e., a k-core with $\Omega(n)$ vertices).

Let $\mathcal{D}$ be a sparse smooth asymptotic degree sequence with limiting degree distribution $\{\lambda_i\}$. We define the *residual degree distribution* $\{\mu_j\}$ of $\mathcal{D}$ by

$$\mu_j = \frac{(j+1)\lambda_{j+1}}{\sum_{i=1}^{\infty} i\lambda_i}. \tag{5}$$

Intuitively, if we choose a random endpoint e (working in the configuration model), then the number of additional endpoints assigned to the same vertex as e will be a random variable whose distribution converges to the residual degree distribution. Since $\mathcal{D}$ is smooth, $\sum_i i\lambda_i$ converges, and the residual degree distribution is well defined.

Let

$$g(q) = \sum_{i=0}^{\infty} \mu_i q^i$$

be the *probability generating function* [5] for the distribution μ_i. For $r \geq 0$, define

$$f_r(q) = \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q), \tag{6}$$

where $g^{(i)}$ is the i'th derivative of g, and note that $f_r(q)$ is the r'th order Taylor approximation of $g(1)$ about q, so $f_0(q) = g(q)$, $f_1(q) = g(q) + (1-q)g'(q)$, and so on.

We now state the main theorem of this paper:

Theorem 2.1 *Let $\mathcal{D}$ be a sparse smooth asymptotic degree sequence with maximum degree in D_n being $o(n^{1/3})$ and with residual degree distribution $\{\mu_i\}$. Then*

1. *If there exists a value q in the interval $[0,1)$ such that $f_{k-2}(q) < q$ then there exists a constant $C > 0$ such that the k-core of $G(\mathcal{D})$ contains at least Cn vertices w.e.h.p.*

2. *If $f_{k-2}(q) > q$ for all $q \in [0,1)$ then for every $C > 0$, then the k-core of $G(\mathcal{D})$ has less than Cn vertices w.e.h.p.*

The rest of this paper is organized as follows. In subsection 2.1 we present some intuition behind Theorem 2.1 by appealing to a connection to branching processes (as in [10]); in this subsection we also present a solution to the related branching process problem. In subsection 2.2 we present the proof of Theorem 2.1; this proof does not appeal to the result for the branching process, except for one identity used in that proof. Finally in section 3 we apply Theorem 2.1 to obtain results for certain types of degree sequences. Due to space constraints, most proofs are sketched and details are deferred to the Appendix.

2.1 Random Graphs and Branching Processes.

Intuitively, it is useful to note that a random graph with degree sequence D locally behaves like a branching process [4] based on the residual distribution $\{\mu_i\}$. Pittel, Spencer, and Wormald [10] noted, using the graph model $\mathcal{G}_{n,m}$, that a giant k-core in a random graph relates to the probability of finding an infinite $(k-1)$-ary subtree of a branching (or *genealogy*) tree. A similar argument can be applied to $G(D)$, and details are in the appendix. However, in both cases, this argument is incomplete and it is not clear that the link to the branching process can be made entirely rigorous.

Nevertheless, the following theorem (complete proof is in the appendix) answers the question of when a branching process generates a infinite complete r-ary tree and may be of independent interest. (This result is tangential to our main result, and is possibly known in the branching process literature.) In this theorem, the probability distribution $\{\mu_i\}$ is an arbitrary one, and μ_i is the probability that any given node in the branching tree has exactly i children.

Theorem 2.2 *Let q_r be the smallest fixed point of the function f_r (see equation 6) in the interval $[0,1]$. Then the probability that a branching process based on the probability distribution $\{\mu_i\}$ generates a genealogy tree which contains an infinite perfect $(r+1)$-ary tree is $1-q_r$.*

Proof. (Sketch) Let X be a random variable with distribution $\{\mu_i\}$. For any $0 \le q \le 1$, let Z_q be a random variable with distribution

$$P(Z_q = i) = \sum_{j=i}^{\infty} \mu_j \cdot \binom{j}{i} q^{j-i}(1-q)^i.$$

Note that $P(Z_q \le r) = \sum_{i=0}^{r}\sum_{j=i}^{\infty} \mu_j \cdot \binom{j}{i} q^{j-i}(1-q)^i \;=\; \sum_{i=0}^{r} E\left[\binom{X}{i} q^{X-i}(1-q)^i\right] \;=\; f_r(q).$

Now, Z_0 has the same distribution as X, hence $P(X > r) = 1 - f_r(0)$. From this we derive the probability of producing an $(r+1)$-ary tree of depth d as $1 - f_r^{[d]}(0)$, where $f_r^{[d]}$ is the d'th iterate of f_r. Since ${f'}_r(q) \ge 0$ in the interval $[0,1]$ and $f_r(1) = 1$, it follows that $f_r^{[d]}(0)$ approaches the lowest fixed point of f_r as $d \to \infty$. ∎

If we now use the μ_i as defined in equation 5, the informal connection between the existence of a k-core in G and the existence of an infinite $(k-1)$-ary subtree in the branching process gives an informal justification of Theorem 2.1.

2.2 Finding the k-core.

In this section we provide a different, more precise argument for the same result as in Theorem 2.2 for the existence of a giant k-core. Since we shall work with the configuration model, we cannot guarantee that a given configuration will produce a simple graph. Nevertheless, the k-core of a multigraph is well-defined as the maximal induced multigraph with minimum degree k.

Our approach is as follows. First, we give describe the *CM k-core algorithm*, which is a variant of an algorithm from [10], and which finds the k-core of a (multi)graph. Then, we determine the outcome of this algorithm w.e.h.p. under certain conditions. Finally, we conclude that if an asymptotic degree sequence $\mathcal{D}$ is such that the configuration model produces a simple graph with probability $\Theta(1)$, then our results will be valid for random simple graphs with degree sequence $\mathcal{D}$.

In order to find the k-core of a (multi)graph, the algorithm in [10] chooses a vertex v of degree less than k, removes all edges incident on v and repeats this procedure until there are no more vertices with degree less than k. The remaining edges and vertices will be the (possibly empty) k-core of the original graph.

We adapt the above algorithm to CM in a natural way. In particular, we shall choose the random matching used by the CM while the algorithm executes, exposing edges only as they are needed. When the algorithm terminates, the k-core will remain unexposed, and thus a corollary to Theorem 2.1 is that the k-core of a random graph with asymptotic degree sequence $\mathcal{D}$ is itself a random graph with a different asymptotic degree sequence and a limiting degree distribution which can be calculated from the limiting distribution of $\mathcal{D}$.

We first define the following random variables to describe the execution of our algorithm. In the following, t refers to the time step. In each time step, a certain number of endpoints are exposed and matched. The k-core is found at time step t_k. For technical reasons (which will become clear later) we extend the definitions of the random processes past t_k.

- $G(t)$ is the state of the graph at time t. The algorithm begins at time 0, and $G(0)$ is a set of vertices and endpoints according to the degree sequence D.
- An endpoint in $G(t)$ is *exposed* if it is chosen by the algorithm during a step $t' \leq t$. All of the endpoints in initial state $G(0)$ are *unexposed*. (If $t \leq t_k$, an endpoint is exposed if and only if its match has been revealed.)
- The *unexposed degree* of a vertex v in $G(t)$ is the number of unexposed endpoints associated with v. The unexposed degree of an endpoint is the unexposed degree of its associated vertex.
- $X_i(t)$ denotes the number of vertices of unexposed degree i at time t.
- $U(t) = U(0) - 2t$. (If $t \leq t_k$ then $U(t)$ is the total number of unexposed endpoints at time t.)
- A vertex is *k-light* at time t if it has unexposed degree at most $k-1$, otherwise it is *k-heavy*. An endpoint is k-light or k-heavy if its associated vertex is k-light or k-heavy, respectively.
- $H(t)$ denotes the number of unexposed k-heavy endpoints; $H(t) = \sum_{i \geq k} iX_i(t)$ for all t.
- $L(t) = U(t) - H(t)$. (t_k is the first time step in which $L(t_k)$ becomes zero; if $t < t_k$ then $L(t)$ denotes the number of unexposed k-light endpoints.)

We are now ready to describe our k-core algorithm whose analysis will prove Theorem 2.1.

CM k-core Algorithm

1. If $t < t_k$, we choose a single endpoint $e_1(t)$ uniformly at random from the set of all unexposed k-light endpoints in $G(t)$. We then choose a second endpoint $e_2(t)$ uniformly at random from the set of all unexposed endpoints in $G(t)$, we match $e_1(t)$ and $e_2(t)$, and we designate the endpoints $e_1(t)$ and $e_2(t)$ as *exposed*. If $L(t) = 0$ then the k-core has been found.
2. If $t \geq t_k$, we choose $\left\lfloor \frac{H(t)}{U(t)-1} \right\rfloor$ unexposed k-heavy endpoints uniformly at random and designate them as *exposed*. Then, with probability $\frac{H(t)}{U(t)-1} - \left\lfloor \frac{H(t)}{U(t)-1} \right\rfloor$, we choose another such endpoint and expose it as well. For simplicity of calculation, these endpoints are chosen *with* replacement (if an endpoint is chosen twice in the same time step, it is only exposed once). Note that endpoints that are exposed in this part of the algorithm are not actually matched, as they are in part one; once the k-core has been found, designating endpoints as exposed is merely an accounting tool.

We now explain the intuition behind the above CM k-core algorithm. The process in part 1 is clear: for each $t < t_k$, it exposes a match for a k-light endpoint until no k-light endpoint remains. While $t < t_k$, the following two properties follow directly from the definitions:

1. $U(t+1) = U(t) - 2$.
2. Any given unexposed k-heavy endpoint remains unexposed with probability $1 - \frac{1}{U(t)-1}$.

We have extended our random processes beyond time t_k in part 2 in such a way that these two properties continue to hold (approximately). To see this, let $t_k \leq t < U(0)/2$ (note that since two endpoints are exposed at each step of part 1, the algorithm can run for at most $U(0)/2$ steps). In order to satisfy property 1, we have defined $U(t) = U(0) - 2t$ for all t.

For property 2, we would like each k-heavy endpoint to have a $\frac{1}{U(t)-1}$ probability of becoming exposed at time t. If $H(t) \leq U_t - 1$ this is easy: with probability $\frac{H(t)}{U(t)-1}$, we choose a single k-heavy endpoint uniformly at random and expose it. However, for $t \geq t_k$ it is possible that $H(t) \geq U(t)$, in which case we obviously cannot perform any action with probability $\frac{H(t)}{U(t)-1} > 1$. However, by first exposing $\left\lfloor \frac{H(t)}{U(t)-1} \right\rfloor$ k-heavy endpoints, and then, with probability $\frac{H(t)}{U(t)-1} - \left\lfloor \frac{H(t)}{U(t)-1} \right\rfloor$ exposing one additional such endpoint, each k-heavy endpoint becomes exposed with probability approximately $\frac{1}{U(t)-1}$.

2.3 Proof of Theorem 2.1.

We prove Theorem 2.1 as follows. First, we calculate the expected values of the $X_i(t)$ and $H(t)$, and then we prove that $H(t)$ concentrates around its expectation w.e.h.p. Then, if there exists a t such that $E[H(t)] - E[U(t)]$ is sufficiently large, $H(t) > U(t)$ must occur w.e.h.p., and the k-core will have been found by time t (also w.e.h.p.). Finally, we show that these conditions are equivalent to the conditions we derived informally using the branching process argument.

First, we define
$$p(t) = \left(\frac{U(t)}{U(0)}\right)^{1/2}.$$

Intuitively, $p(t)$ might be considered the approximate probability that an endpoint remains unexposed at time t. However, this is not quite correct, since once an endpoint becomes k-light, it is much more likely to become exposed. The meaning of $p(t)$ is more clearly expressed in the following lemma.

Lemma 2.3 *For $i \geq k$, and for $t = O(U(t))$,*

$$E\left[\frac{X_i(t)}{n}\right] = \sum_{j=i}^{\infty} \lambda_j \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1). \tag{7}$$

Proof. (Sketch.) Let $P_{i,j}(t)$ denote the probability that a vertex of original degree j has exactly i unexposed endpoints at time t. Since vertices of original degree j are initially indistinguishable, $P_{i,j}$ is well defined, and

$$E[X_i(t)] = \sum_{j=i}^{\infty} X_j(0) P_{i,j}(t).$$

We will show that
$$P_{i,j}(t) = \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1) \tag{8}$$

for every j. Then,

$$\begin{aligned}
E\left[\frac{X_i(t)}{n}\right] &= \sum_{j=i}^{\infty}\frac{X_j(0)}{n}\left(\binom{j}{i}p(t)^i(1-p(t))^{j-i}\pm o(1)\right)\\
&= \sum_{j=i}^{h}(\lambda_j \pm o(1))\left(\binom{j}{i}p(t)^i(1-p(t))^{j-i}\pm o(1)\right) + O\left(\sum_{j=h}^{\infty}\frac{X_j(0)}{n}\right)\\
&= \sum_{j=i}^{\infty}\lambda_j\binom{j}{i}p(t)^i(1-p(t))^{j-i}\pm o(1) \text{ (see Appendix for details)}
\end{aligned}$$

To derive equation 8, choose a vertex v of original degree j, choose any i of v's endpoints, and assume that these i endpoints are unexposed at time t. We shall calculate the probability that they remain unexposed at time $t+1$.

First, suppose $t < t_k$, so we are in part 1 if the algorithm. Then, at time t, a single light endpoint is picked and matched to an unexposed endpoint chosen uniformly at random from the $U(t)-1$ remaining unexposed endpoints. Thus the probability that our set of i endpoints remain unexposed is $\frac{U(t)-1-i}{U(t)-1}$.

Next, suppose $t \geq t_k$. Let $r = \left\lfloor \frac{H(t)}{U(t)-1} \right\rfloor$ and $s = \frac{H(t)}{U(t)-1} - r$, and therefore $r + s = \frac{H(t)}{U(t)-1}$. Recall that in part 2 of the algorithm, we pick r unexposed k-heavy endpoints uniformly at random (with replacement) and designate them as exposed, and with probability s we expose an additional such endpoint. Also, note that since $H(t) \leq U(0)$ and $U(t) = \Theta(U(0))$, we have $r = O(1)$.

Now, in order for every one of our given set of i endpoints to remain unexposed, none of the r uniformly chosen endpoints, nor the additional endpoint chosen with probability s can be a member of our set. We sample with replacement, so the probability of this occurring is

$$\left(\frac{H(t)-i}{H(t)}\right)^r\left(\frac{H(t)-si}{H(t)}\right).$$

After suitable simplification (see appendix) this probability becomes $\left(\frac{U(t)-1-i}{U(t)-1}\right)\left(1 \pm O\left(\frac{1}{U(t)^2}\right)\right)$.

Thus, assuming $t = O(U(t))$, the probability that the i endpoints remain unexposed through t steps, written as a function of $U(0)$ and t, is

$$\begin{aligned}
f_i(U(0),t) &= \prod_{j=0}^{t-1}\left(\frac{U(j)-1-i}{U(j)-1}\right)\left(1 \pm O\left(\frac{1}{U(j)^2}\right)\right)\\
&= \frac{(U(0)-1-i)(U(0)-3-i)\cdots(U(0)-2t+1-i)}{(U(0)-1)(U(0)-3)\cdots(U(0)-2t+1)}\left(1 \pm O\left(\frac{t}{U(t)^2}\right)\right).
\end{aligned}$$

Note that

$$\begin{aligned}
f_i(U(0),t)\cdot f_i(U(0)-1,t) &= \frac{(U(0)-i)(U(0)-1-i)(U(0)-2-i))\cdots(U(0)-2t-i)}{(U(0))(U(0)-1)(U(0)-2)\cdots(U(0)-2t)}\cdot(1\pm o(1))\\
&= \left(\frac{U(t)}{U(0)}\right)^i \pm o(1) \text{ (since } i \text{ is constant and } U(0) \text{ grows with } n).
\end{aligned}$$

Therefore,

$$f_i(U(0),t) = \left(\frac{U(t)}{U(0)}\right)^{i/2} \pm o(1) = p(t)^i \pm o(1)$$

Using inclusion-exclusion, $P_{i,j} = \sum_{i \le i' \le j} (-1)^{i'-i} \binom{j}{i'} p(t)^{i'} \pm o(1) = \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1)$

This establishes equation 8 and hence the lemma is proved. ∎

Lemma 2.4 *Fix* $\epsilon > 0$. *For* $t < \frac{U(0)}{2}(1-\epsilon)$, $P\left(|H(t) - E[H(t)]| > \Theta(n^{2/3})\right) < e^{-\Theta(n^{1/3})}$.

Proof. (Sketch.) For t fixed we define the random process $Y(t') = E[H(t)|G(t')]$ for $0 \le t' \le t$. Y is a martingale by definition, and clearly $Y(t) = H(t)$. We show that Azuma's inequality is applicable, and the result follows. See appendix for details. ∎

Corollary 2.5 *For any* $\epsilon > 0$, *with probability* $1 - e^{-O(n^{1/3})}$, $|H(t) - E[H(t)]| = o(1)$ *for all* $1 \le t \le U(0)/2(1-\epsilon)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. (Sketch, see appendix for details.) Note that by Lemma 2.3

$$\frac{E[H(t)]}{U(0)} = \frac{1}{U(0)} \sum_{i=k}^{\infty} E[iX_i(t)] = \frac{n}{U(0)} \sum_{i=k}^{\infty} i \sum_{j=i}^{\infty} \lambda_j \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1)$$
$$= \frac{n}{U(0)} \sum_{j=k}^{\infty} \sum_{i=k}^{j} j\lambda_j \binom{j-1}{i-1} p(t)^i (1-p(t))^{j-i} \pm o(1).$$

Letting $B(p,i,j) = \binom{j}{i} p^i (1-p)^{j-i}$, we continue

$$\frac{E[H(t)]}{U(0)} = \frac{n}{U(0)} \sum_{j=k}^{\infty} j\lambda_j p(t) \sum_{i=k}^{j} B(p(t), i-1, j-1) \pm o(1)$$
$$= \frac{n}{U(0)} \sum_{j'=k-1}^{\infty} (j'+1)\lambda_{j'+1} p(t) \left(1 - \sum_{i'=0}^{k-2} B(p(t), i', j')\right) \pm o(1)$$

Now we substitute $\mu_j = \frac{(j+1)\lambda_{j+1}}{\sum_i i\lambda_i}$, and, noting that $\sum_i i\lambda_i = \frac{U(0)}{n}(1 \pm o(1))$, we proceed

$$\frac{E[H(t)]}{U(0)} = \sum_{j'=k-1}^{\infty} \mu_{j'} p(t) \left(1 - \sum_{i'=0}^{k-2} B(p(t), i', j')\right) \pm o(1)$$
$$= p(t) \left(1 - \sum_{i'=0}^{k-2} \sum_{j'=i'}^{\infty} \mu_{j'} B(p(t), i', j')\right) \pm o(1) \text{ (see Appendix for details)}$$

Since $p(t) = (U(t)/U(O))^{1/2}$, we write $U(t)/U(0) = p(t)^2$. Note that if $H(t) \ge U(t)$, then the k-core of G has been found, and if $H(t) < U(t)$ for all t then there is no k-core. Due to corollary 2.5, $H(t)/U(0) = E[H(t)/U(0)] \pm o(1)$ w.e.h.p. Therefore, if there exists a value t such that

$$p(t) \left(1 - \sum_{i'=0}^{k-2} \sum_{j'=i'}^{\infty} \mu_{j'} B(p(t), i', j')\right) - p(t)^2 = \Theta(1) \tag{9}$$

then $G(D)$ has a k-core containing at least $U(t)$ endpoints w.e.h.p., and if

$$p(t)\left(1-\sum_{i'=0}^{k-2}\sum_{j'=i'}^{\infty}\mu_{j'}B(p(t),i',j')\right)-p(t)^2=-\Theta(1). \tag{10}$$

for all $t \leq (1-\epsilon)U(0)/2$, then the k-core of $G(D)$ contains less than $U(0)(1-\epsilon)$ endpoints w.e.h.p.

Now, let X be a random variable with distribution $\{\mu_i\}$, and recall that $g(q) = E[q^X]$ is the probability generating function. We set $q(t) = 1 - p(t)$, and the left hand side of equations 9 and 10 becomes

$$\sum_{i=0}^{k-2}\sum_{j=i}^{\infty}\mu_j B(q(t),j,i)-q(t)=\sum_{i=0}^{k-2}\sum_{j=i}^{\infty}\mu_j\binom{j}{i}q(t)^{j-i}(1-q(t))^i-q(t)=f_{k-2}(q(t))-q(t),$$

(using the identity $f_{k-2}(q(t)) = \sum_{i=0}^{k-2}\sum_{j=i}^{\infty}\mu_j\binom{j}{i}q(t)^{j-i}(1-q(t))^i$ derived in proof of Theorem 2.2).

Since ϵ can be made arbitrarily small, the conditions established in equations 9 and 10 are equivalent to the statement of Theorem 2.1. ∎

3 Verifying the k-core Conditions.

Theorem 2.1 gives conditions under which a giant k-core is asymptotically present in a random graph with fixed degree sequence. Essentially, a k-core is present if the function f_{k-2} has a non-critical fixed point in the interval $[0, 1)$. However, determining whether such a fixed point exists for arbitrary degree distributions can be a non-trivial task. In this section we solve the problem in certain special cases and we discuss methods for finding such fixed points in other situations.

3.1 The Giant 2-core.

Theorem 3.1 *Let $\mathcal{D}$ be a sparse smooth asymptotic degree sequence with maximum degree $o(n^{1/3})$ and with residual degree distribution $\{\mu_i\}$. Then*

1. If $\sum_i i\mu_i > 1$ then there exists a constant C such that the 2-core of $G(\mathcal{D})$ contains $\geq Cn$ vertices w.e.h.p.
2. If $\sum_i i\mu_i \leq 1$ then for every $C > 0$, the 2-core of $G(\mathcal{D})$ contains less Cn vertices w.e.h.p.

Proof. The conditions necessary for an asymptotic degree sequence to produce a giant 2-core w.e.h.p. according to theorem 2.1 are that $f_0(q) = g(q)$ must have a fixed point in the interval $[0, 1)$. This is equivalent to the condition that a $\{\mu_i\}$ branching process survives with positive probability. Since g has all positive derivatives in $[0, 1)$, and since $g(1) = 1$ and $g(0) \geq 0$, it is well known that g has such a fixed point if and only if $g'(1) = \sum_i i\mu_i > 1$. ∎

From the results in [8, 9] for the a.a.s. presence of a giant component we can obtain the following.

Corollary 3.2 *A random graph G with a sparse smooth degree sequence has a giant 2-core a.a.s. if and only if it has a giant component a.a.s.*

3.2 Distributions with All Convergent Moments.

For $k > 2$ the conditions necessary for a giant k-core are less easily verified, since it is not necessarily true that f_{k-2} will have all positive derivatives. Here we consider the case where all of the moments of the distribution $\{\mu_i\}$ are convergent. Let X be a random variable with distribution $\{\mu_i\}$. By assumption, $g^{(i)}(1) = E[X(X-1)\cdots(X-i+1)] = \nu_i$, the i'th factorial moment of the distribution $\{\mu_i\}$, is finite for all i. This allows us to write

$$g(q) = \sum_{i=0}^{\infty} \frac{(q-1)^i}{i!} g^{(i)}(1) = \sum_{i=0}^{\infty} \frac{(q-1)^i}{i!} \nu_i.$$

We can now express f_r as a power series in $(q-1)$ (see appendix for details)

$$f_r(q) = \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q) = 1 + (-1)^r \sum_{j=r+1}^{\infty} \frac{(q-1)^j}{j!} \nu_j \binom{j-1}{r}.$$

We write $p = 1-q$, and note that and finding a fixed point $f_r(1-p) = 1-p$ is equivalent to solving

$$1 - p = 1 + (-1)^r \sum_{j=r+1}^{\infty} \frac{(-p)^j}{j!} \nu_j \binom{j-1}{r}.$$

In order to ascertain the presence of a giant k-core w.e.h.p., we must find a point where $f_{k-2}(q) < q$, or

$$p + (-1)^k \sum_{j=k-1}^{\infty} p^j \frac{(-1)^j}{j!} \nu_j \binom{j-1}{k-2} < 0. \tag{11}$$

3.2.1 Application to $\mathcal{G}_{n,m}$.

As shown by Molloy and Reed [8], the Erdos-Renyi random graph model $\mathcal{G}_{n,m}$ produces a random graph with a Poisson degree distribution, and thus results derived for random graphs with a specified Poisson distribution are valid for $\mathcal{G}_{n,m}$. Since a Poisson distribution has all convergent moments, we can re-derive some of the results of Pittel, Spencer, and Wormald [10] regarding the k-core of $\mathcal{G}_{n,m}$.

Consider a random graph whose limiting degree distribution is a Poisson distribution with expected value r, so $\lambda_i = \frac{r^i e^{-r}}{i!}$. We have

$$\mu_i = \frac{(i+1)\lambda_{i+1}}{\sum_i \lambda_i} = \lambda_i,$$

i.e., the residual degree distribution is identical to the limiting degree distribution.

Now, the factorial moments of a Poisson distribution are $\nu_j = r^j$. Using equation 11, if

$$p + (-1)^k \sum_{j=k-1}^{\infty} (pr)^j \frac{(-1)^j}{j!} \binom{j-1}{k-2} < 0$$

has a solution, then $\mathcal{G}_{n,m}$ with expected degree r has a giant k-core w.e.h.p. Let

$$C_k(x) = (-1)^{k+1} \sum_{j=k-1}^{\infty} x^j \frac{(-1)^j}{j!} \binom{j-1}{k-2},$$

and let $x = pr$. Then we must solve $x/r - C_k(x) < 0$ or $x/C_k(x) < r$. Thus the giant k-core threshold for the average degree r in $\mathcal{G}_{n,m}$ occurs at $\min \frac{x}{C_k(x)}$, which is essentially the same condition as in [10].

3.3 Power Law Graphs

Several massive graphs that occur in the real-world, including the web graph, have degree sequences that obey a power law [6], thus there has been considerable interest in understanding the properties of massive power law graphs. One approach to studying such graphs, introduced by Aiello, Chung, and Lu [1], is to generate random graphs with power law degree sequences.

A degree sequence obeys a power law if the number of vertices of degree i is proportional to $i^{-\beta}$ for some β. If $\beta \leq 2$, this degree sequence is not sparse, but for $\beta > 2$, this power law graph can be characterized by a smooth sparse asymptotic degree sequence with

$$\lambda_i = \frac{1}{\zeta(\beta)} \frac{1}{i^\beta},$$

where $\zeta(\beta) = \sum_{i=1}^{\infty} i^{-\beta}$ is the Riemann Zeta function. The corresponding residual endpoint distribution is

$$\mu_i = \frac{1}{\zeta(\beta-1)} \frac{1}{(i+1)^{\beta-1}}.$$

Since the number of vertices of degree i is approximately $n\lambda_i$, and $\lambda_i = \Theta(i^{-\beta})$, it might be natural to consider the largest degree in a random power law graph to be $\Theta(n^{1/\beta})$. This is the assumption made by [1]. The configuration model and Theorem 2.1 require that the maximum degree of a degree sequence be $o(n^{1/3})$, and hence for $\beta < 3$, this power-law graph model would violate the CM maximum degree requirement. However, we may extend our results to the maximum degree bound in the power law model of [1] by the following mechanism. There are $O(n^{1-\delta})$ edges incident on vertices of degree greater than $n^{1/3-\epsilon}$ for some $\delta > 0$ when $\beta > 2$. Consider exposing the edges of such a graph in the configuration model by first placing the edges of these high degree vertices in any (adversarial) manner while respecting the degree constraints on the other vertices. Then apply the configuration model mechanism to the graph represented by the residual degrees of the remaining vertices. Since only $O(n^{1-\delta})$ endpoints are affected, the resulting degree distribution is indistinguishable from the starting power-law distribution with respect to smoothness, hence the result in [7] continues to hold for this residual graph, which implies (from the result in Theorem 3.3 below) that the overall graph continues to have a k-core w.e.h.p.

As in Section 3.1, the a.a.s existence of a giant 2-core in a random power-law graph is equivalent to the a.a.s. existence of a giant component, which appears if $\beta < 3.47875\ldots$ [1]. For $k \geq 3$, we have the following theorem.

Theorem 3.3 *Let $k \geq 3$ be an integer constant.*

1. *For $\beta \geq 3$, a random power law graph does not have a giant k-core w.e.h.p.*
2. *For $2 < \beta < 3$, a random power law graph has a giant k-core w.e.h.p.*

Proof. (Sketch.) Since $f_r(q) = \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q)$, we derive

$$\begin{aligned} f'_r(q) &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i+1)}(q) - \sum_{i=1}^{r} \frac{(1-q)^{(i-1)}}{(i-1)!} g^{(i)}(q) \\ &= \frac{1}{r!} \frac{g^{(r+1)}(q)}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i} \quad \text{(see appendix for details)} \end{aligned}$$

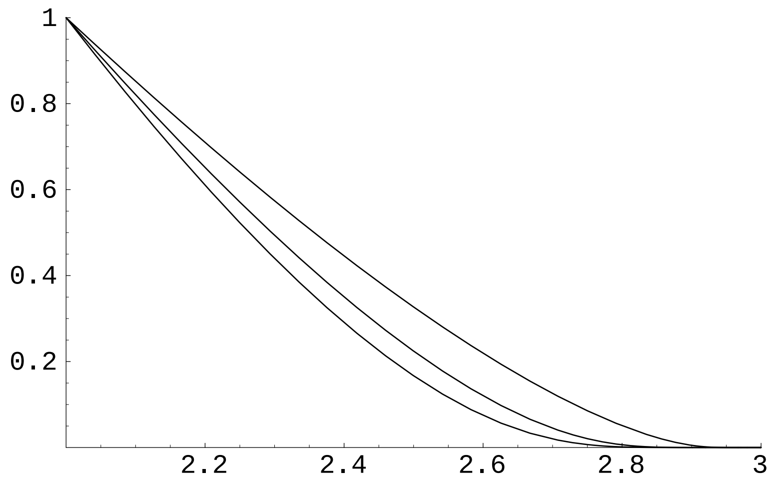

Figure 1: *Plot of the size of k-core as a function of β, for $2 < \beta < 3$ and for $k = 3, 4, 5$:* Here the y axis is the asymptotic value of $p(t)$ at which the k-core algorithm terminates, and the x-axis is β. Since the 3-core is the largest, the top curve corresponds to the 3-core, and the bottom curve corresponds to $k = 5$.

We show that if $\beta \geq 3$, then $f'_r(q) \leq \frac{1}{r\zeta(\beta-1)} < 1$ for all $0 \leq q \leq 1$, and since $f_r(1) = 1$, it follows that $f_r(q) > q$ for all q in the interval $[0, 1)$. For $\beta < 3$, we show that $\lim_{q \to 1} f'_r(q) = \infty$, and thus $f_r(1-\delta) < 1 - \delta$ for δ sufficiently small. (See appendix for details).

To determine the presence of a giant k-core, we examine the function f_{k-2}. Hence, for $k \geq 3$, a power law graph with $2 < \beta < 3$ w.e.h.p. has a giant k-core, and a power law graph with $\beta \geq 3$ w.e.h.p. does not have a giant k-core. $\blacksquare$

For $2 < \beta < 3$, we can also calculate the size of the k-core of a random power law graph. Figure 1 plots the asymptotic value of $p(t)$ at which the k-core algorithm terminates as a function of β, for $k = 3$, 4, and 5. The value of $p(t)$ measures the fraction of initially k-heavy vertices which belong to the k-core.

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APPENDIX

A Details of Section 2.1: Random Graphs and Branching Processes.

Intuitively, it is useful to note that a random graph with degree sequence D locally behaves like a branching process [4] based on the residual distribution $\{\mu_i\}$. So see this, choose a single endpoint s, and define random elements t and v, where t is the endpoint matched to s, and v is the vertex to which t belongs. Let X_s be the random variable given by $X_s = i$ if the degree of v is i. Since each vertex of degree j has j chances of matching to s, then X_s asymptotically satisfies

$$P(X_s = i) = \frac{i\lambda_i}{\sum_i \lambda_i}.$$

Now, define the *residual degree* of s to be a the random variable $Y_s = X_s - 1$. The residual degree of s counts the number of endpoints belonging to v other than t. For any endpoint s, the residual degree of s asymptotically satisfies

$$P(Y_s = i) = \frac{(i+1)\lambda_{i+1}}{\sum_i \lambda_i} = \mu_i,$$

where the μ_i are as defined in equation 5.

The residual degrees of the endpoints belonging to v other than t are identically distributed and almost independent from Y_s. So the sum of the residual degrees of all of the endpoints of v other than t is a random variable given by the sum of Y_s almost i.i.d. copies of Y_s. As we grow progressively larger neighborhoods of our initial endpoint s, the conditional residual degree distributions will eventually change substantially. Nevertheless, it is heuristically useful to imagine (or hope) that, at least locally, the residual degree distributions of all endpoints are i.i.d. random variables with distribution $\{\mu_i\}$. This allows us to treat the sizes of the neighborhoods of s as a branching process based on the residual degree distribution.

We now return to the question of a giant k-core in the random graph $G(D)$. Pittel, Spencer, and Wormald [10] noted, using the graph model $\mathcal{G}_{n,m}$, that a giant k-core in a random graph relates to the probability of finding an infinite $(k-1)$-ary subtree of a branching tree. The following informal argument is taken from [10].

Choose any vertex v in $G(D)$, and let us attempt to determine whether or not v is in the k-core of $G(D)$. Clearly, v must have degree at least k to be part of the k-core. Furthermore, v must have at least k endpoints each of whom have residual degree at least $k-1$, and these $k-1$ neighbors must in turn have k endpoints of residual degree at least $k-1$ and so on. If we assume that residual degrees are i.i.d. random variables, then in order for v to be in the k-core of $G(D)$, v must have k endpoints which generate branching trees containing a complete $(k-1)$-ary tree.

Of course, this argument is incomplete. As pointed out in [10], it is not clear that the link to the branching process can be made entirely rigorous; in particular, we have only argued that producing a complete $(k-1)$-ary branching tree with positive probability is necessary for a giant k-core. We are not aware of an equally simple argument that this condition is sufficient. Further, the assumption that the residual degrees are i.i.d. random variables is not accurate. Thus, the branching process argument should be treated as an intuitive explanation or perhaps as a guess at the true solution.

Nevertheless, the following theorem answers the question of when a branching process generates a infinite complete k-ary tree and may be of independent interest. (This result is tangential to our main result, and is possibly known in the branching process literature.) In this theorem (only), the probability distribution $\{\mu_i\}$ is an arbitrary one, and necessarily the distribution defined in equation 5; here μ_i is the probability that any given node in the branching tree has exactly i children..

Theorem A.1 *(Theorem 2.2) Let q_r be the smallest fixed point of the function f_r in the interval $[0,1]$. Then the probability that a branching process based on the probability distribution $\{\mu_i\}$ generates a genealogy tree which contains an infinite perfect $(r+1)$-ary tree is $1-q_r$.*

Proof. The argument is similar to the classical result regarding the survival probability of a branching process [4]. Let X be a random variable with distribution $\{\mu_i\}$. For any $0 \le q \le 1$, let Z_q be a random variable with distribution

$$P(Z_q = i) = \sum_{j=i}^{\infty} \mu_j \cdot \binom{j}{i} q^{j-i}(1-q)^i.$$

Note that

$$\begin{aligned} P(Z_q \le r) &= \sum_{i=0}^{r}\sum_{j=i}^{\infty} \mu_j \cdot \binom{j}{i} q^{j-i}(1-q)^i \\ &= \sum_{i=0}^{r} E\left[\binom{X}{i} q^{X-i}(1-q)^i\right] \\ &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} E\left[\frac{X!}{(X-i)!} q^{X-i}\right] \\ &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q) \\ &= f_r(q). \end{aligned}$$

Now, consider the genealogy tree of a branching process based on X. First, we calculate the probability that the root of the genealogy tree has at least $r+1$ children. Since Z_0 has the same distribution as X, then $P(X > r) = 1 - f_r(0)$. In order to produce an $(r+1)$-ary tree of depth 2, then the root must have at least $r+1$ children, each of whom produce $r+1$ grandchildren. Each child has probability $1 - f_r(0)$ of producing at least $(r+1)$ grandchildren, thus the number of such children is a random variable with distribution $Z_{f_r(0)}$. Accordingly, the probability of producing an $(r+1)$-ary tree of depth 2 is $1 - f_r(f_r(0))$.

In general, producing an $(r+1)$-ary tree of depth d is equivalent to having at least $(r+1)$ children who produce $(r+1)$-ary trees of depth $d-1$. Thus, we inductively conclude that the probability of an $(r+1)$-ary tree of depth d is $1 - f_r^{[d]}(0)$, where $f_r^{[d]}$ is the d'th iterate of f_r. Since $f_r(q)' \geq 0$ in the interval $[0,1]$ and $f_r(1) = 1$, then $f_r^{[d]}(0)$ approaches the lowest fixed point of f_r as $d \to \infty$. ∎

B Details of Section 2.3: Proof of Theorem 2.1.

We prove Theorem 2.1 as follows. First, we calculate the expected values of the $X_i(t)$ and $H(t)$, and then we prove that $H(t)$ concentrates around its expectation w.e.h.p. Then, if there exists a t such that $E[H(t)] - E[U(t)]$ is sufficiently large, $H(t) > U(t)$ must occur w.e.h.p., and the k-core will have been found by time t (also w.e.h.p.). Finally, we show that these conditions are equivalent to the conditions we derived informally using the branching process argument.

First, we define

$$p(t) = \left(\frac{U(t)}{U(0)}\right)^{1/2}.$$

Intuitively, $p(t)$ might be considered the approximate probability that an endpoint remains unexposed at time t. However, this is not quite correct, since once such an endpoint becomes k-light, it is much more likely to become exposed. The meaning or $p(t)$ is more clearly expressed in the following lemma.

Lemma B.1 *(Lemma 2.3) For $i \geq k$, and for $t = O(U(t))$,*

$$E\left[\frac{X_i(t)}{n}\right] = \sum_{j=i}^{\infty} \lambda_j \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1). \qquad \textit{(equation 7)}$$

Proof. Let $P_{i,j}(t)$ denote the probability that a vertex of original degree j has exactly i unexposed endpoints at time t. Since vertices of original degree j are initially indistinguishable, $P_{i,j}$ is well defined, and

$$E[X_i(t)] = \sum_{j=i}^{\infty} X_j(0) P_{i,j}(t).$$

We will show that

$$P_{i,j}(t) = \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1) \qquad \text{(equation 8)}$$

for every j. Then,

$$\begin{aligned}
E\left[\frac{X_i(t)}{n}\right] &= \sum_{j=i}^{\infty} \frac{X_j(0)}{n} \left(\binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1)\right) \\
&= \sum_{j=i}^{h} \frac{X_j(0)}{n} \left(\binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1)\right) + \sum_{j=h}^{\infty} \frac{X_j(0)}{n} \left(\binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1)\right) \\
&= \sum_{j=i}^{h} (\lambda_j \pm o(1)) \left(\binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1)\right) + O\left(\sum_{j=h}^{\infty} \frac{X_j(0)}{n}\right).
\end{aligned}$$

Since h is constant, we have a constant number of $o(1)$ terms in the left summation, which can be consolidated. Then, using the smoothness condition, we have

$$E\left[\frac{X_i(t)}{n}\right] = \sum_{j=i}^{h} \lambda_j \binom{j}{i} p(t)^i (1-p(t))^{j-i} \pm o(1) + O\left(\sum_{j=h}^{\infty} \lambda_j\right).$$

If we let h grow arbitrarily large, the $O\left(\sum_{j=h}^{\infty} \lambda_j\right)$ term becomes arbitrarily small. Thus, by first choosing h sufficiently large, and then choosing n sufficiently large,

$$\left| E\left[\frac{X_i(t)}{n}\right] - \sum_{j=i}^{\infty} \lambda_j \binom{j}{i} p(t)^i (1-p(t))^{j-i} \right|$$

can be made arbitrarily small, yielding equation 7.

To derive equation 8, choose a vertex v of original degree j, choose any i of v's endpoints, and assume that these i endpoints are unexposed at time t. We shall calculate the probability that they remain unexposed at time $t+1$.

First, suppose $t < t_k$, so we are in part 1 if the algorithm. Then, at time t, a single light endpoint is picked and matched to an unexposed endpoint chosen uniformly at random from the $U(t)-1$ remaining unexposed endpoints. Thus the probability that our set of i endpoints remain unexposed is $\frac{U(t)-1-i}{U(t)-1}$.

Next, suppose $t \geq t_k$. Let $r = \left\lfloor \frac{H(t)}{U(t)-1} \right\rfloor$ and $s = \frac{H(t)}{U(t)-1} - r$, and therefore $r + s = \frac{H(t)}{U(t)-1}$. Recall that in part 2 of the algorithm, we pick r unexposed k-heavy endpoints uniformly at random (with replacement) and designate them as exposed, and with probability s we expose an additional such endpoint. Also, note that since $H(t) \leq U(0)$ and $U(t) = \Theta(U(0))$, then $r = O(1)$.

Now, in order for every one of our given set of i endpoints to remain unexposed, none of the r uniformly chosen endpoints, nor the additional endpoint chosen with probability s can be a member of our set. We sample with replacement, so the probability of this occurring is

$$\left(\frac{H(t)-i}{H(t)}\right)^r \left(\frac{H(t)-si}{H(t)}\right).$$

If $H(t) < U(t)$ then $r = 0$, and this is simply

$$\frac{H(t)/s - i}{H(t)/s} = \frac{U(t)-1-i}{U(t)-1}.$$

Otherwise, since r and i are $O(1)$ we calculate

$$\begin{aligned}
\left(1 - \frac{i}{H(t)}\right)^r \left(1 - \frac{si}{H(t)}\right) &= 1 - \frac{(r+s)i}{H(t)} \pm O\left(\frac{1}{H(t)^2}\right) \\
&= \left(1 - \frac{(r+s)i}{H(t)}\right)\left(1 \pm O\left(\frac{1}{H(t)^2}\right)\right) \\
&= \left(\frac{U(t)-1-i}{U(t)-1}\right)\left(1 \pm O\left(\frac{1}{H(t)^2}\right)\right).
\end{aligned}$$

Since $H(t) \geq U(t)$, then $O(H(t)^{-2}) = O(U(t)^{-2})$, and this probability can be written

$$\left(\frac{U(t)-1-i}{U(t)-1}\right)\left(1 \pm O\left(\frac{1}{U(t)^2}\right)\right).$$

Assuming $t = O(U(t))$, the probability that the i endpoints remain unexposed through t steps, written as a function of $U(0)$ and t, is

$$
\begin{aligned}
f_i(U(0), t) &= \prod_{j=0}^{t-1} \left(\frac{U(j) - 1 - i}{U(j) - 1} \right) \left(1 \pm O\left(\frac{1}{U(j)^2} \right) \right) \\
&= \frac{(U(0) - 1 - i)(U(1) - 1 - i) \cdots (U(t-1) - i)}{(U(0) - 1)(U(1) - 1) \cdots (U(t-1) - 1)} \left(1 \pm O\left(\frac{1}{U(t)^2} \right) \right)^t \\
&= \frac{(U(0) - 1 - i)(U(0) - 3 - i) \cdots (U(0) - 2t + 1 - i)}{(U(0) - 1)(U(0) - 3) \cdots (U(0) - 2t + 1)} \left(1 \pm O\left(\frac{t}{U(t)^2} \right) \right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
f_i(U(0), t) \cdot f_i(U(0) - 1, t) &= \frac{(U(0) - i)(U(0) - 1 - i)(U(0) - 2 - i)(U(0) - 3 - i) \cdots (U(0) - 2t - i)}{(U(0))(U(0) - 1)(U(0) - 2)(U(0) - 3) \cdots (U(0) - 2t)} \cdot \\
&\quad (1 \pm o(1)) \\
&= \frac{(U(0) - 2t - 1) \cdots (U(0) - 2t - i)}{(U(0))(U(0) - 1) \cdots (U(0) - i + 1)} \cdot (1 \pm o(1)) \\
&= \left(\frac{U(t)}{U(0)} \right)^i \pm o(1) \text{ (since } i \text{ is constant and } U(0) \text{ grows with } n).
\end{aligned}
$$

Therefore

$$
f_i(U(0), t) = \left(\frac{U(t)}{U(0)} \right)^{i/2} \pm o(1) = p(t)^i \pm o(1).
$$

Using inclusion-exclusion,

$$
P_{i,j} = \sum_{i \le i' \le j} (-1)^{i' - i} \binom{j}{i'} p(t)^{i'} \pm o(1) = \binom{j}{i} p(t)^i (1 - p(t))^{j-i} \pm o(1).
$$

This establishes equation 8 and hence the lemma is proved. ∎

Lemma B.2 *Fix $\epsilon > 0$. For $t < \frac{U(0)}{2}(1 - \epsilon)$,*

$$
P\left(|H(t) - E[H(t)]| > \Theta(n^{2/3}) \right) < e^{-\Theta(n^{1/3})}.
$$

Proof. For t fixed we define the random process

$$
Y(t') = E[H(t) | G(t')]
$$

for $0 \le t' \le t$. Y is a martingale by definition, and clearly $Y(t) = H(t)$.

We now establish bounded differences for Y. Note that

$$
E[H(t) | G(t')] = \sum_{i \ge k} f(i, t') i X_i(t')
$$

where $f(i, t')$ is the probability that an endpoint with unexposed degree i at time t' remains both heavy and unexposed at time t. We claim that $|f(i, t')i - f(i-1, t')(i-1)| = O(1)$.

By linearity, $f(i, t')i$ is the expected number of k-heavy endpoints at time t produced by a vertex of unexposed degree i at time t'. There are two ways an endpoint can fail to be heavy and unexposed; accordingly, let $a(i,t')$ denote the expected number of endpoints from a degree i vertex which become exposed before time t, and let $b(i,t')$ denote the probability that a vertex of degree i at time t' becomes k-light before time t. Then

$$f(i,t')i = i - a(i,t') - kb(i,t').$$

Clearly, $|a(i-1,t') - a(i,t')| \leq 1$, so

$$\begin{aligned} |f(i,t')i - f(i-1,t')(i-1)i| &\leq i-(i-1) + |a(i-1,t') - a(i,t')| + k\,|b(i-1,t') - b(i,t')| \\ &= i-(i-1)+1+k, \end{aligned}$$

and since k is constant, this is $O(1)$.

Now, $O(1)$ vertices change unexposed degree at time t', thus

$$Y(t'+1) - Y(t') = \sum_{i\geq k}(f(i,t') - f(i,t'+1))(iX_i(t')) \pm O(1),$$

always. Since $E[Y(t'+1) - Y(t')] = 0$ it follows that

$$\left|\sum_{i\geq k}(f(i,t') - f(i,t'+1))(X_i(t'))\right| = O(1).$$

Hence $|Y(t'+1) - Y(t')| = O(1)$.

By Azuma's inequality,

$$P\left(|Y(t) - E[Y(t)]| > \Theta(n^{2/3})\right) < e^{-\Theta(n^{4/3}/t)}.$$

Since $t = O(n)$ and $Y(t) = H(t)$, the proof is complete. ∎

Corollary B.3 *2.5 For any $\epsilon > 0$, with probability $1 - e^{-O(n^{1/3})}$,*

$$|H(t) - E[H(t)]| = o(1)$$

for all $1 \leq t \leq U(0)/2(1-\epsilon)$.

Proof. This follows immediately since $U(0) = O(n)$, and $O(n)e^{-\Theta(n^{1/3})} = e^{-O(n^{1/3})}$. ∎

Proof of Theorem 2.1. Note that by Lemma 2.3

$$\begin{aligned} E[H(t)] &= \sum_{i=k}^{\infty} E[iX_i(t)] \\ \frac{E[H(t)]}{U(0)} &= \frac{n}{U(0)}\sum_{i=k}^{\infty} i \sum_{j=i}^{\infty}\lambda_j\binom{j}{i}p(t)^i(1-p(t))^{j-i} \pm o(1) \\ &= \frac{n}{U(0)}\sum_{i=k}^{\infty}\sum_{j=i}^{\infty} j\lambda_j \frac{(j-1)!}{(i-1)!(j-i)!}p(t)^i(1-p(t))^{j-i} \pm o(1) \\ &= \frac{n}{U(0)}\sum_{j=k}^{\infty}\sum_{i=k}^{j} j\lambda_j\binom{j-1}{i-1}p(t)^i(1-p(t))^{j-i} \pm o(1). \end{aligned}$$

Letting $B(p,i,j) = \binom{j}{i} p^i (1-p)^{j-i}$, we continue

$$\begin{aligned}
\frac{E[H(t)]}{U(0)} &= \frac{n}{U(0)} \sum_{j=k}^{\infty} j\lambda_j p(t) \sum_{i=k}^{j} B(p(t), i-1, j-1) \pm o(1) \\
&= \frac{n}{U(0)} \sum_{j'=k-1}^{\infty} (j'+1)\lambda_{j'+1} p(t) \sum_{i'=k-1}^{j'} B(p(t), i, j') \pm o(1) \\
&= \frac{n}{U(0)} \sum_{j'=k-1}^{\infty} (j'+1)\lambda_{j'+1} p(t) \left(1 - \sum_{i'=0}^{k-2} B(p(t), i', j')\right) \pm o(1),
\end{aligned}$$

with the last substitution resulting from the fact that $\sum_{i'=0}^{j'} B(p(t), i', j') = 1$.

Now we substitute $\mu_j = \frac{(j+1)\lambda_{j+1}}{\sum_i i\lambda_i}$, and, noting that $\sum_i i\lambda_i = \frac{U(0)}{n}(1 \pm o(1))$, we proceed

$$\begin{aligned}
\frac{E[H(t)]}{U(0)} &= \sum_{j'=k-1}^{\infty} \mu_{j'} p(t) \left(1 - \sum_{i'=0}^{k-2} B(p(t), i', j')\right) \pm o(1) \\
&= p(t) \left(\sum_{j'=k-1}^{\infty} \mu_{j'} - \sum_{j'=k-1}^{\infty} \sum_{i'=0}^{k-2} \mu_{j'} B(p(t), i', j') \pm o(1) \right) \\
&= p(t) \left(\left(1 - \sum_{j'=0}^{k-2} \mu_{j'}\right) - \sum_{j'=k-1}^{\infty} \sum_{i'=0}^{k-2} \mu_{j'} B(p(t), i', j') \right) \pm o(1) \\
&= p(t) \left(1 - \sum_{j'=0}^{k-2} \sum_{i'=0}^{j'} B(p(t), i', j')\mu_{j'} - \sum_{j'=k-1}^{\infty} \sum_{i'=0}^{k-2} \mu_{j'} B(p(t), i', j') \right) \pm o(1) \\
&= p(t) \left(1 - \sum_{i'=0}^{k-2} \sum_{j'=i'}^{\infty} \mu_{j'} B(p(t), i', j') \right) \pm o(1).
\end{aligned}$$

Since $p(t) = (U(t)/U(O))^{1/2}$, we write $U(t)/U(0) = p(t)^2$. Note that if $H(t) \geq U(t)$, then the k-core of G has been found, and if $H(t) < U(t)$ for all t then there is no k-core. Due to corollary 2.5, $H(t)/U(0) = E[H(t)/U(0)] \pm o(1)$ w.e.h.p. Therefore, if there exists a value t such that

$$E[H(t)/U(0)] - p(t)^2 = \Theta(1),$$

then $G(D)$ has a k-core containing at least $U(t)$ endpoints w.e.h.p., and if

$$E[H(t)/U(0)] - p(t)^2 = -\Theta(1)$$

for all $t \leq (1-\epsilon)U(0)/2$, then the k-core of $G(D)$ contains less than $U(0)(1-\epsilon)$ endpoints w.e.h.p.

These first condition can be written

$$p(t) \left(1 - \sum_{i'=0}^{k-2} \sum_{j'=i'}^{\infty} \mu_{j'} B(p(t), i', j') \right) - p(t)^2 = \Theta(1) \qquad \text{(equation 9)}$$

$$p(t) \left(1 - \sum_{i'=0}^{k-2} \sum_{j'=i'}^{\infty} \mu_{j'} B(p(t), i', j') \right) - p(t)^2 = -\Theta(1). \qquad \text{(equation 10)}$$

Next, set $q(t) = 1 - p(t)$, and the left hand side of equations 9 and 10 becomes

$$\begin{aligned}\sum_{i=0}^{k-2}\sum_{j=i}^{\infty}\mu_j B(q(t),j,i) - q(t) &= \sum_{i=0}^{k-2}\sum_{j=i}^{\infty}\mu_j \binom{j}{i} q(t)^{j-i}(1-q(t))^i - q(t) \\ &= f_{k-2}(q(t)) - q(t),\end{aligned}$$

using the identity

$$f_{k-2}(q(t)) = \sum_{i=0}^{k-2}\sum_{j=i}^{\infty}\mu_j \binom{j}{i} q(t)^{j-i}(1-q(t))^i$$

which is derived in the proof of theorem 2.2.

Since ϵ can be made arbitrarily small, the conditions established in equations 9 and 10 are equivalent to the statement of Theorem 2.1. ∎

C Details of Section 3: Verifying the k-core Conditions.

Theorem 2.1 gives conditions under which a giant k-core is asymptotically present in a random graph with fixed degree sequence. Essentially, a k-core is present if the function f_{k-2} has a non-critical fixed point in the interval $[0, 1)$. However, determining whether such a fixed point exists for arbitrary degree distributions can be a non-trivial task. In this section we solve the problem in certain special cases and we discuss methods for finding such fixed points in other situations.

C.1 The Giant 2-core.

Theorem C.1 *(Theorem 3.1) Let $\mathcal{D}$ be a sparse smooth asymptotic degree sequence with maximum degree $o(n^{1/3})$ and with residual degree distribution $\{\mu_i\}$. Then*

1. *If $\sum_i i\mu_i > 1$ then there exists a constant C such that the 2-core of $G(\mathcal{D})$ contains at least Cn vertices w.e.h.p.*
2. *If $\sum_i i\mu_i \leq 1$ then for every $C > 0$, the 2-core of $G(\mathcal{D})$ contains less Cn vertices w.e.h.p.*

Proof. The conditions necessary for an asymptotic degree sequence to produce a giant 2-core w.e.h.p. according to theorem 2.1 are that $f_0(q) = g(q)$ must have a fixed point in the interval $[0, 1)$. This is equivalent to the condition that a $\{\mu_i\}$ branching process survives with positive probability. Since g has all positive derivatives in $[0, 1)$, and since $g(1) = 1$ and $g(0) \geq 0$, it is well known that g has such a fixed point if and only if $g'(1) = \sum_i i\mu_i > 1$. ∎

The condition for the a.a.s. giant component derived in [8] is $\sum_i i(i-2)\lambda_i > 0$. We derive

$$\begin{aligned}\sum_{i=1}^{\infty} i(i-2)\lambda_i &= \sum_{i'=0}^{\infty}(i'+1)(i'-1)\lambda_{i'+1} \\ &= \frac{\sum_{i'=0}^{\infty} i'\mu_{i'} - \sum_{i'=0}^{\infty}\mu_{i'}}{\sum_{i=1}^{\infty} i\lambda_i},\end{aligned}$$

and since $\sum_{i'}\mu_{i'} = 1$, then $\sum_i i(i-2)\lambda_i > 0$ is equivalent to $\sum_{i'} i'\mu_{i'} - 1 > 0$, or $\sum_{i'} i'\mu_{i'} > 1$

Thus, in view of the results in [8, 9], we obtain the following corollary.

Corollary C.2 *A random graph G with a sparse smooth degree sequence has a giant 2-core a.a.s. if and only if it has a giant component a.a.s.*

For $k > 2$ the conditions necessary for a giant k-core are less easily verified, since it is not necessarily true that f_{k-2} will have all positive derivatives. In the following we consider some special cases where we can derive explicit results for $k > 2$.

C.2 Distributions with all convergent moments.

For $k > 2$ the conditions necessary for a giant k-core are less easily verified, since it is not necessarily true that f_{k-2} will have all positive derivatives. Here we consider the case where all of the moments of the distribution $\{\mu_i\}$ are convergent. Let X be a random variable with distribution $\{\mu_i\}$. By assumption, $g^{(i)}(1) = E[X(X-1)\cdots(X-i+1)] = \nu_i$, the i'th factorial moment of the distribution $\{\mu_i\}$, is finite for all i. This allows us to write

$$\begin{aligned} g(q) &= \sum_{i=0}^{\infty} \frac{(q-1)^i}{i!} g^{(i)}(1) \\ &= \sum_{i=0}^{\infty} \frac{(q-1)^i}{i!} \nu_i. \end{aligned}$$

We can now express f_r as a power series in $(q-1)$

$$\begin{aligned} f_r(q) &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q) \\ &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} \sum_{j=i}^{\infty} \frac{j!}{j-i!} \frac{(q-1)^{j-i}}{j!} \nu_j \\ &= \sum_{j=0}^{\infty} \frac{(q-1)^j}{j!} \nu_j \sum_{i=0}^{r} (-1)^i \binom{j}{i} \\ &= 1 + \sum_{j=r+1}^{\infty} \frac{(q-1)^j}{j!} \nu_j (-1)^r \binom{j-1}{r}, \end{aligned}$$

where the last step uses the binomial identity $\sum_{i=0}^{r}(-1)^i\binom{j}{i} = (-1)^r\binom{j-1}{r}$ for $j > 0$.

Now we write $p = 1-q$, and note that and finding a fixed point $f_r(1-p) = 1-p$ is equivalent to solving

$$\begin{aligned} 1-p &= 1 + (-1)^r \sum_{j=r+1}^{\infty} \frac{(-p)^j}{j!} \nu_j \binom{j-1}{r} \\ 0 &= p + (-1)^r \sum_{j=r+1}^{\infty} p^j \frac{(-1)^j}{j!} \nu_j \binom{j-1}{r}. \end{aligned}$$

In order to ascertain the presence of a giant k-core, we must find a point where $f_{k-2}(q) < q$, or

$$p + (-1)^k \sum_{j=k-1}^{\infty} p^j \frac{(-1)^j}{j!} \nu_j \binom{j-1}{k-2} < 0. \qquad \text{(equation 11)}$$

C.2.1 Application to $\mathcal{G}_{n,m}$.

As shown by Molloy and Reed [8], the Erdos-Renyi random graph model $\mathcal{G}_{n,m}$ produces a random graph with a Poisson degree distribution, and thus results derived for random graphs whose limiting degree distribution is a Poisson distribution are valid for $\mathcal{G}_{n,m}$. Since a Poisson distribution has all convergent moments, we can re-derive some of the results of Pittel, Spencer, and Wormald [10] regarding the k-core of $\mathcal{G}_{n,m}$.

Consider a random graph whose limiting degree distribution is a Poisson distribution with expected value r, so $\lambda_i = \frac{r^i e^{-r}}{i!}$. Since

$$i\lambda_i = r\frac{r^{i-1}e^{-r}}{(i-1)!},$$

and $\sum i\lambda_i = r$, then

$$\mu_i = \frac{(i+1)\lambda_{i+1}}{\sum_i \lambda_i} = \lambda_i,$$

the residual degree distribution is identical to the limiting degree distribution.

Now, the factorial moments of a Poisson distribution are $\nu_j = r^j$. Using equation 11 from the previous discussion, if

$$p + (-1)^k \sum_{j=k-1}^{\infty} (pr)^j \frac{(-1)^j}{j!}\binom{j-1}{k-2} < 0$$

has a solution, then $\mathcal{G}_{n,m}$ with expected degree r has a giant k-core w.e.h.p. Let

$$C_k(x) = (-1)^{k+1} \sum_{j=k-1}^{\infty} x^j \frac{(-1)^j}{j!}\binom{j-1}{k-2},$$

and let $x = pr$. Then we must solve

$$\begin{aligned} x/r - C_k(x) &< 0 \\ x/C_k(x) &< r. \end{aligned}$$

Thus the giant k-core threshold for $\mathcal{G}_{n,m}$ occurs at

$$\min \frac{x}{C_k(x)}.$$

C.3 Power Law Graphs

Several massive graphs that occur in the real-world, including the web graph, have degree sequences that obey a power law [6], thus there has been c onsiderable interest in understanding the properties of massive power law graphs. One approach to studying such graphs, introduced by Aiello, Chung, and Lu [1], is to generate random graphs with power law degree sequences.

A degree sequence obeys a power law if the number of vertices of degree i is proportional to $i^{-\beta}$ for some β. If $\beta \leq 2$, this degree sequence is not sparse, but for $\beta > 2$, this power law graph can be characterized by a smooth sparse asymptotic degree sequence with

$$\lambda_i = \frac{1}{\zeta(\beta)}\frac{1}{i^\beta},$$

where $\zeta(\beta) = \sum_{i=1}^{\infty} i^{-\beta}$ is the Riemann Zeta function. The corresponding residual endpoint distribution is

$$\mu_i = \frac{1}{\zeta(\beta-1)}\frac{1}{(i+1)^{\beta-1}}.$$

Since the number of vertices of degree i is approximately $n\lambda_i$, and $\lambda_i = \Theta(i^{-\beta})$, it might be natural to consider the largest degree in a random power law graph to be $\Theta(n^{1/\beta})$. This is the assumption made by [1]. The configuration model and Theorem 2.1 require that the maximum degree of a degree sequence be $o(n^{1/3})$, and hence for $\beta < 3$, this power-law graph model would violate the CM maximum degree requirement. However, we may extend our results to the maximum degree bound in the power law model of [1] by the following mechanism. There are $O(n^{1-\delta})$ edges incident on vertices of degree greater than $n^{1/3-\epsilon}$ for some $\delta > 0$ when $\beta > 2$. Consider exposing the edges of such a graph in the configuration model by first placing the edges of these high degree vertices in any (adversarial) manner while respecting the degree constraints on the other vertices. Then apply the configuration model mechanism to the graph represented by the residual degrees of the remaining vertices. Since only $O(n^{1-\delta})$ endpoints are affected, the resulting degree distribution is indistinguishable from the starting power-law distribution with respect to smoothness, hence the result in [7] continues to hold for this residual graph, which implies (from the result in Theorem 3.3 below) that the overall graph continues to have a k-core w.e.h.p.

As in Section 3.1, the a.a.s existence of a giant 2-core in a random power-law graph is equivalent to the a.a.s. existence of a giant component, which appears if $\beta < 3.47875\ldots$ [1]. For $k \geq 3$, we have the following theorem.

Theorem C.3 *(Theorem 3.3) Let $k \geq 3$ be an integer constant.*

1. *For $\beta \geq 3$, a random power law graph does not have a giant k-core w.e.h.p.*
2. *For $2 < \beta < 3$, a random power law graph has a giant k-core w.e.h.p.*

Proof. Since $f_r(q) = \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q)$, we derive

$$\begin{aligned} f'_r(q) &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i+1)}(q) - \sum_{i=1}^{r} \frac{(1-q)^{(i-1)}}{(i-1)!} g^{(i)}(q) \\ &= \frac{(1-q)^r}{r!} g^{(r+1)}(q). \\ &= \frac{1}{r!} \frac{g^{(r+1)}(q)}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i}. \end{aligned}$$

For a power law graph, the probability generating function of the endpoint distribution is given by

$$g(q) = \frac{1}{\zeta(\beta-1)} \sum_{i=0}^{\infty} \frac{q^i}{(i+1)^{\beta-1}},$$

and thus

$$g^{(r+1)}(q) = \frac{1}{\zeta(\beta-1)} \sum_{i=0}^{\infty} \frac{i(i-1)(i-2)\cdots(i-r)q^{i-r-1}}{(i+1)^{\beta-1}}.$$

Thus, for power law graphs,

$$f'_r(q) = \frac{1}{r!} \frac{g^{(r+1)}(q)}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i}$$

$$= \frac{1}{r!\zeta(\beta-1)} \frac{\sum_{i=0}^{\infty} \frac{i(i-1)(i-2)\cdots(i-r)q^{i-r-1}}{(i+1)^{\beta-1}}}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i}$$

$$= \frac{1}{r!\zeta(\beta-1)} \frac{\sum_{i=r+1}^{\infty} \frac{i(i-1)(i-2)\cdots(i-r)q^{i-r-1}}{(i+1)^{\beta-1}}}{\sum_{i=r+1}^{\infty} \binom{i-2}{r-1} q^{i-r-1}}.$$

Now, let $a_i = \frac{i(i-1)(i-2)\cdots(i-r)}{(i+1)^{\beta-1}}$ and let $b_i = r!\binom{i-2}{r-1}$, so

$$f'_r(q) = \frac{1}{\zeta(\beta-1)} \frac{\sum_{i=r+1}^{\infty} a_i q^{i-r-1}}{\sum_{i=r+1}^{\infty} b_i q^{i-r-1}},$$

and note that

$$\frac{a_i}{b_i} = \frac{\frac{i(i-1)(i-2)\cdots(i-r)}{(i+1)^{\beta-1}}}{r!\binom{i-2}{r-1}} = \frac{i(i-1)}{r(i+1)^{\beta-1}}.$$

In particular, if $\beta \geq 3$, then $a_i > b_1$ for all i, and if $\beta < 3$, then $\lim_{i\to\infty} \frac{a_i}{b_i} = \infty$.

Hence,

$$\lim_{q\to 1} f'_r(q) = \begin{cases} \infty & \text{if } \beta < 3 \\ \frac{1}{r\zeta(\beta-1)} & \text{if } \beta = 3 \\ 0 & \text{if } \beta > 3. \end{cases} \tag{12}$$

It follows that if $\beta \geq 3$, then $f'_r(q) \leq \frac{1}{r\zeta(\beta-1)} < 1$ for all $0 \leq q \leq 1$, and since $f_r(1) = 1$, then $f_r(q) > q$ for all q in the interval $[0,1)$. For $\beta < 3$, the limit in equation 12 implies that $\lim_{q\to 1} f'_r(q) = \infty$, and thus $f_r(1-\delta) < 1-\delta$ for δ sufficiently small.

To determine the presence of a giant k-core, we examine the function f_{k-2}. Hence, for $k > 3$, a power law graph with $2 \leq \beta < 3$ a.a.s. has a giant k-core, and a power law graph with $\beta > 3$ a.a.s. does not have a giant k-core. ∎