Some Facts about String Interleaving
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Let p and q be (finite or infinite) strings. For any string x, the *projection* of x on p, $x.p$, is the subsequence of x consisting only of the symbols from p; for empty string ϵ, $x.\epsilon = \epsilon$. We call the symbols of p its *alphabet*. String x is an interleaving of p and q, where p and q have disjoint alphabets, if

$x.p = p$ and $x.q = q$ and $x.(p \cup q) = x$

The first two conditions imply that all symbols of p and q are in x and from the last condition, no other symbol is in x. The only interleaving of ϵ and q is q.

Let $p \mathbin{+\!\!\!+} q$ be the set of all interleavings of p and q. We generalize interleaving to sets of strings. Let P and Q be sets of strings with disjoint alphabets, i.e., no symbol appears in both a string of P and of Q. Write $x.P$ for the projection of x on the alphabet of P. For the empty set ϕ, $x.\phi = \epsilon$. Define the interleaving of P and Q, $P \mathbin{+\!\!\!+} Q$, by

$$x \in (P \mathbin{+\!\!\!+} Q) \;\equiv\; x.P \in P \;\wedge\; x.Q \in Q \;\wedge\; x.(P \cup Q) = x \qquad \text{(D)}$$

We will prove several properties of $\mathbin{+\!\!\!+}$. First, we note a few simple facts. Henceforth, we write $x.P.Q$ for $(x.P).Q$.

$P = \phi \;\Rightarrow\; P \mathbin{+\!\!\!+} Q = \phi$
$P = \{\epsilon\} \;\Rightarrow\; P \mathbin{+\!\!\!+} Q = Q$
$x.P.P = x.P$
$x.(P \cup Q).P = x.P$
$x.(P \mathbin{+\!\!\!+} Q) = x.(P \cup Q)$, since alphabets of $P \mathbin{+\!\!\!+} Q$ and $P \cup Q$ are equal.

Properties of Interleaving

Below, P, Q and R are sets of strings with disjoint alphabets.

- (Commutativity) $P \mathbin{+\!\!\!+} Q = Q \mathbin{+\!\!\!+} P$: follows from (D) that

  $x \in P \mathbin{+\!\!\!+} Q \;\equiv\; x \in Q \mathbin{+\!\!\!+} P$

- (Associativity) $(P \mathbin{+\!\!\!+} Q) \mathbin{+\!\!\!+} R = P \mathbin{+\!\!\!+} (Q \mathbin{+\!\!\!+} R)$:

$$
\begin{array}{ll}
 & x \in (P \mathbin{+\!\!\!+} Q) \mathbin{+\!\!\!+} R \\
\equiv & \{\text{from (D)}\} \\
 & x.(P \mathbin{+\!\!\!+} Q) \in (P \mathbin{+\!\!\!+} Q) \;\wedge\; x.R \in R \;\wedge\; x.((P \cup Q) \mathbin{+\!\!\!+} R) = x \\
\equiv & \{x.(P \mathbin{+\!\!\!+} Q) = x.(P \cup Q)\} \\
 & x.(P \cup Q) \in (P \mathbin{+\!\!\!+} Q) \;\wedge\; x.R \in R \;\wedge\; x.(P \cup Q \cup R) = x \\
\equiv & \{\text{from (D)}\} \\
 & \quad x.(P \cup Q).P \in P \\
 & \wedge\; x.(P \cup Q).Q \in Q \\
 & \wedge\; x.R \in R \\
 & \wedge\; x.(P \cup Q \cup R) = x \\
\equiv & \{x.(P \cup Q).P = x.P,\; x.(P \cup Q).Q = x.Q\} \\
 & x.P \in P \;\wedge\; x.Q \in Q \;\wedge\; x.R \in R \;\wedge\; x.(P \cup Q \cup R) = x
\end{array}
$$

Summarizing,

$$
\begin{array}{ll}
& x \in (P \mathbin{+\!\!+} Q) \mathbin{+\!\!+} R \\
\equiv & x.P \in P \ \wedge \ x.Q \in Q \ \wedge \ x.R \in R \ \wedge \ x.(P \cup Q \cup R) = x
\end{array} \tag{1}
$$

Now,

$$
\begin{array}{ll}
& x \in P \mathbin{+\!\!+} (Q \mathbin{+\!\!+} R) \\
\equiv & \{\text{Commutativity of } \mathbin{+\!\!+} \} \\
& x \in (Q \mathbin{+\!\!+} R) \mathbin{+\!\!+} P \\
\equiv & \{\text{replace } P, Q \text{ and } R \text{ in (1) by } Q, R \text{ and } P, \text{ respectively}\} \\
& x.Q \in Q \ \wedge \ x.R \in R \ \wedge \ x.P \in P \ \wedge \ x.(Q \cup R \cup P) = x \\
\equiv & \{\text{rewrite}\} \\
& x.P \in P \ \wedge \ x.Q \in Q \ \wedge \ x.R \in R \ \wedge \ x.(P \cup Q \cup R) = x \\
\equiv & \{\text{from (1)}\} \\
& x \in (P \mathbin{+\!\!+} Q) \mathbin{+\!\!+} R
\end{array}
$$

- (Distributivity over $\cup$) $(P \cup Q) \mathbin{+\!\!+} R = (P \mathbin{+\!\!+} R) \ \cup \ (Q \mathbin{+\!\!+} R)$:

$$
\begin{array}{ll}
& x \in (P \cup Q) \mathbin{+\!\!+} R \\
\equiv & \{\text{definition of } \mathbin{+\!\!+} \} \\
& x.(P \cup Q) \in (P \cup Q) \ \wedge \ x.R \in R \ \wedge \ x.((P \cup Q) \mathbin{+\!\!+} R) = x \\
\equiv & \{\text{set theory on first term; } x.(P \mathbin{+\!\!+} Q) = x.(P \cup Q) \text{ on last term}\} \\
& \quad (x.(P \cup Q) \in P \ \vee \ x.(P \cup Q) \in Q) \\
& \wedge \ x.R \in R \ \wedge \ x.(P \cup Q \cup R) = x \\
\equiv & \{P \text{ and } Q \text{ have disjoint alphabets:} \\
& \quad x.(P \cup Q) \in P \ \equiv \ x.P \in P \wedge x.Q = \epsilon\} \\
& \quad ((x.P \in P \wedge x.Q = \epsilon) \ \vee \ (x.Q \in Q \wedge x.P = \epsilon)) \\
& \wedge \ x.R \in R \ \wedge \ x.(P \cup Q \cup R) = x \\
\equiv & \{x.Q = \epsilon \ \wedge \ x.(P \cup Q \cup R) = x \ \equiv \ x.(P \cup R) = x \ \} \\
& \quad (x.P \in P \ \wedge \ x.R \in R \ \wedge \ x.(P \cup R) = x) \\
& \vee \ (x.Q \in Q \ \wedge \ x.R \in R \ \wedge \ x.(Q \cup R) = x) \\
\equiv & \{\text{from (D)}\} \\
& x \in (P \mathbin{+\!\!+} R) \ \vee \ x \in (Q \mathbin{+\!\!+} R) \\
\equiv & \{\text{set theory}\} \\
& x \in ((P \mathbin{+\!\!+} R) \ \cup \ (Q \mathbin{+\!\!+} R))
\end{array}
$$

On disjoint alphabets Our results hold even when the alphabets are not disjoint. To see this, consider the interleavings of strings $p = 01$ and $q = 02$. First, replace the common symbol, 0, by distinct symbols in both strings, to get $p' = 0'1$ and $q' = 0''2$ with distinct alphabets. Any interleaving of p' and q', say, $0'0''21$ can be mapped back to 0021 which is an interleaving of p and q. Identical sets of strings remain identical after the mapping of symbols. Therefore, all properties of $\mathbin{+\!\!+}$ proved under the assumption of disjoint alphabets also hold if the alphabets are non-disjoint.